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A foundation for PROPs, algebras, and modules. (English) Zbl 06449852

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Associative algebras, Lie algebras, Poisson algebras and their variants play a key role in algebra, topology, category theory, differential and algebraic geometry, mathematical physics. They are all algebras over an operad, a combinatorial structure which provides a unified framework to describe various sorts of algebras. Operads can be described by a particular class of graphs, called trees, and associative grafting laws on such trees. They present various nice features such as a good homotopy theory (in the sense of model categories), a good theory of minimal models (Koszul duality) describing various sorts of homotopy algebras, and provide a unified way of dealing with cohomology and deformation theories. One should also mention the prominent role of the little disks operads in the study of iterated loop spaces, deformation quantization (of Poisson manifolds and more recently in derived algebraic geometry), Goodwillie-Weiss calculus for embedding spaces (also deeply connected to knot theory), or factorization homology of manifolds. However, more general algebraic structures appeared in various places in mathematics, for which one has to consider additional structures such as compatible coproducts or traces which cannot be handled by the operad theory. Let us mention for instance Hopf algebras in representation theory, Frobenius algebras in algebraic topology, Lie bialgebras in quantum group theory, their unimodular version in the Batalin-Vilkovisky formalism in mathematical physics see [{\it S. A. Merkulov}, Commun. Math. Phys. 295, No. 3, 585--638 (2010; Zbl 1228.17019)], involutive Lie bialgebras in string topology, moduli spaces of stable curves with marked points in algebraic geometry$\dots$. More general objects called props were developed to handle structures with both products and coproducts, determined by directed cycle-free graphs (combinatorially much harder to deal with than trees) with two kind of compositions (horizontal concatenation and vertical grafting). Their connected version called properads were introduced as intermediate objects between operads and props, sufficient to encode various sorts of bialgebra but "small" enough to generalize several operadic methods such as Koszul duality theory. The wheeled cousins of operads, properads and props were introduced later, allowing graphs with wheels in order to encode traces on their algebras. The underlying construction seems to consist in picking up a certain collection of graphs, a "pasting scheme", and attach to it a collection of objects with structure maps satisfying the appropriate relations. This leads to two questions: first, how to make precise the notion of pasting scheme, second, the general procedure to attach to a given pasting scheme a given kind of prop. The main purpose of the present research monograph is to build a unified framework to handle all the aforementioned structures and their main properties, as well as their algebras and modules, filling an important gap in the understanding of the notion of algebraic structure in full generality. The base category underlying these constructions can be any complete and cocomplete closed monoidal symmetric category (say, for instance, topological spaces, simplicial sets, or chain complexes). Part I lays down the graph theoretic foundations necessary to a rigorous definition of pasting schemes and their main properties. The six first chapters introduce new notions and operations to reach the following goals. First, to have a purely graph-theoretic and combinatorial description of these various kinds of graphs and operations on it (for instance, bypassing the use of geometric realization). Second, to define properly the notion of graph substitution (removing a vertex of a given graph to replace it by another graph with matching inputs and outputs) at the required level of generality, and to handle compatibilities between graph substitution and graph operations well known by operad users (relabelling, grafting, partial grafting, contractions$\dots$). Actually, it turns out that any operation compatible with graph substitution can be described itself as a graph substitution, a crucial property which places graph substitution at the heart of this whole machinery. Concrete examples of such graphs and operations are given, helping the reader to digest these preliminary requirements and get an intuition of what is going on. Pasting schemes can then be defined as graph groupoids (graphs with their isomorphisms) which are stable under graph substitution (see Chapter 8). The author develop moreover a kind of presentation by generators and relations for pasting schemes, called strong generating sets. Strong generators are a particular collection of graphs in a given graph groupoid which generate the whole groupoid under iterated graph substitutions, and satisfy an analogue of Reidemeister theorem in knot theory: such iterated substitutions can be replaced by finite sequences of special kinds of substitutions called relaxed moves. The goal of this presentation by generators and relations is to describe the algebraic structure that will be associated to a given pasting scheme in Part II (operads, props$\dots$) as a collection of objects indexed by inputs, outputs, colors, equipped with composition maps and relations between these maps. The last chapter of Part I then focuses on free-forgetful adjunctions between pasting schemes. Any smaller pasting scheme which embeds into a bigger one induces a free-forgetful adjunction, which gives a free-forgetful adjunction at the level of the associated algebraic structures, for instance, the adjunction between operads and props: there is a prop associated to any operad (which encodes the same category of algebras), and conversely each prop has an underlying operad (keep only the operations with one single output). The authors introduce the notion of well-matched pasting schemes to characterize the situations were this free construction can be built "nicely". It gives in particular a conceptual explanation behind the following phenomenon. The free functors from operads to props, from properads to props, from half-props to props, are exact, that is, they commute with the homology functor (when the base category is the one of chain complexes over a field of characteristic zero). The free functors from dioperads to props or from props to wheeled props does not commute with homology. This fact was noticed several years ago by Markl-Voronov, who called the first sort of free functors polynomial. It finally turns out that this polynomial property is a particular manifestation of the well-matching property of the two underlying pasting schemes. Part II is devoted to the applications of the graph theoretic structures of Part I, namely generalized props, their algebras and their modules. In Chapter 11, given a pasting scheme $\mathcal{G}$, the notion of $\mathcal{G}$-prop is defined in two equivalent ways: first, as an algebra over a monad built out of $\mathcal{G}$, second, via a presentation of this structure as a collection of objects with composition maps satisfying some relations (commutative diagrams), using the theory of strong generating sets of Chapter 7 (this gives the so-called "biased definition" of generalized props). The authors give a complete definition of the relevant structures in various cases (operads, dioperads, half-props, properads, props, their wheeled versions$\dots$). Chapter 12 studies the free-forgetful adjunctions between these structures, applying the results of Chapter 9, and the conditions under which functors and adjunctions at the level of the base symmetric monoidal categories lift at the level of $\mathcal{G}$-props for a fixed $\mathcal{G}$. Algebras over a $\mathcal{G}$-prop P are then defined in Chapter 13 as $\mathcal{G}$-prop morphisms $P \to End_A$, where A is the object of the base category underlying the algebra, and End_A its associated endomorphism $\mathcal{G}$-prop. Let us note that one needs the fact that the base category is closed to define endomorphism objects. Properties of these categories of algebras under a change of base category or a change of $\mathcal{G}$-prop are explored. Chapter 14 presents an alternate construction of $\mathcal{G}$-props as algebras over a colored operad built out of $\mathcal{G}$, or as enriched muticategorical functors from a small enriched multicategory to the base category. Th two last chapters then deal with a notion of module over a $\mathcal{G}$-prop which is completely new at this level of generality. This unified approach to algebraic structures is carefully worked out, very detailed, and the authors took care of connecting their framework to the various previous works in this area. All the constructions throughout the book are applied each time to relevant examples which allow the reader to grasp these concepts and link them with their various incarnations. To conclude, the reviewer would like to emphasize a possible continuation of this work with a more homotopical flavour. The recent work of {\it M. Batanin} and {\it C. Berger} ["Homotopy theory for algebras over polynomial monads", \url{arXiv:1305.0086}] equip various categories of algebras over polynomial monads with a model category structure. It turns out that this formalism of polynomial monads allows recover various structures studied in the present book. It would be very interesting to rephrase this model category structure in the framework of $\mathcal{G}$-props, hence developing an explicit homotopy theory of $\mathcal{G}$-props generalizing the already known homotopy theory of operads, properads and props. Inside this homotopy theory one could then get some explicit cofibrant resolutions or even notions of cofibrant minimal models, which will allow to handle homotopy algebras over a $\mathcal{G}$-prop (algebra over a cofibrant resolution of this $\mathcal{G}$-prop) generalizing homotopy algebras and homotopy bialgebras already mentioned at the beginning of this review. This would be the first step to develop a unified treatment of deformation theory and obstruction theory of algebras over $\mathcal{G}$-props with potential applications in the various situations where such structures occur. Reviewer: Sinan Yalin (København)

MSC:

18D50 Operads
18-02 Research monographs (category theory)
18B40 Groupoids, semigroupoids, semigroups, groups (viewed as categories)
55U40 Topological categories, foundations of homotopy theory
81T45 Topological field theories

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