Symmetric Differential Operators

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Abstract. We extend the known class of symmetric differential expressions and characterize the boundary conditions which determine the associated symmetric and self-adjoint differential operators in Hilbert space.

1. Introduction

In this paper we characterize the *symmetric* operator realizations S in the Hilbert space $L^2(J,w)$ of the equation

$$My = \lambda wy \ \ on \ \ J=(a,b), \ \ -\infty \leq a < b \leq \infty, \ \lambda \in \mathbb{C} \tag{1.1}$$

where w is a weight function ($w > 0$ a.e. on J), and M is a symmetric ordinary differential expression of any order $n > 1$ and any deficiency index. These operators S satisfy

$$S_{\min} \subset S \subset S^* \subset S_{\max}. \tag{1.2}$$

Real and complex $n \times n$ anti–diagonal matrices C:

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_{1n} \\ 0 & 0 & \cdots & c_{2,n-1} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & c_{n-1,2} & \cdots & 0 & 0 \\ c_{n1} & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{1.3}$$

which satisfy

$$C^{-1} = C^* = (-1)^{n+1}C \tag{1.4}$$

play a major role in both: (1) the construction of the *symmetric operators* M and (2) the characterization of the *symmetric boundary conditions. The self-adjoint operators are a special case.*

For $n = 2k$, this construction of M as well as the characterization of *self-adjoint boundary condition* is well known for the special matrix

$$C = E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n.$$

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This E_n construction has a long history dating back to the work of Glazman and Naimark in the middle of the 20th century, see the 2015 survey paper of Zettl and Sun [**18**].

Recently Bao, et. al. [**1**], [**2**], constructed symmetric expressions and characterized the *self-adjoint boundary conditions* using matrices C of the form (1.3) for even order expressions $n = 2k$, $k \geq 1$.

2. C-Symmetric Differential Expressions, Maximal and Minimal Operators, the Lagrange Identity

In this section we construct general as well as symmetric quasi-differential expressions M, define the maximal and minimal operators $S_{\max}$, $S_{\min}$, prove the Lagrange Identity and the Patching Lemma. These differential expressions M are constructed using matrix functions Q and constant matrices C satisfying (1.3) and (1.4).

Notation 1. *Let $M_{n,m}(X)$ denote the n by m matrices with elements from the set X; this is abbreviated to $M_n(X)$ when $n = m$. The elements of the set X may be the real or complex numbers or functions with certain specified properties.*

Next we define matrices Q which are used below to construct general quasi-differential expressions $M = M_Q$. The symmetric expressions M are a subset of these.

Definition 1. *Let C be given by (1.3). For $n > 1, J = (a, b)$, let*

$$Z_n(J) := \{Q = (q_{rs})_{r,s=1}^n \in M_n(L_{loc}(J))$$
$$q_{r,r+1} \neq 0 \quad a.e.\ on\ J,\ q_{r,r+1}^{-1} \in L_{loc}(J),\ 1 \leq r \leq n-1,$$
$$q_{rs} = 0 \quad a.e.\ on\ J,\ 2 \leq r+1 < s \leq n;\ q_{rs} \in L_{loc}(J),\ s \neq r+1,\ 1 \leq r \leq n-1\}.$$

For $Q \in Z_n(J)$ we define

$$V_0 := \{y : J \to \mathbb{C},\ y \text{ is measurable}\}$$

and

$$y^{[0]} = y \qquad (y \in V_0).$$

Inductively, for $r = 1, \ldots, n$, we define

$$V_r = \{y \in V_{r-1} : y^{[r-1]} \in (AC_{loc}(J))\},$$
$$y^{[r]} = q_{r,r+1}^{-1}\{y^{[r-1]\prime} - \sum_{s=1}^{r} q_{rs} y^{[s-1]}\} \qquad (y \in V_r),$$

where $q_{n,n+1} := c_{n1}$ is the element in the $(n, 1)$ position of the matrix C. Finally we set

$$M\,y = M_Q\,y = i^n\,y^{[n]} \qquad (y \in V_n). \tag{2.1}$$

The expression $M = M_Q$ is called the quasi-differential expression associated with, or generated by, Q and the constant c_{n1}. For V_n we also use the notation $D(M)$. The function $y^{[r]}$ $(0 \leq r \leq n)$ is called the r-th quasi-derivative of y. Since the quasi-derivative depends on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$. Note that only $y^{[n]}$ depends on c_{n1}.

If q_{rs} are real valued functions, $1 \leq r, s \leq n$, we use the notation $Q \in Z_n(J, \mathbb{R})$.

Next we introduce some notation and properties of $Z_n(J)$.

DEFINITION 2. *Let $Q \in Z_n(J)$, $J = (a,b)$. The expression $M = M_Q$ is said to be regular (R) at a or we say a is a regular endpoint, if for some c, $a < c < b$, we have*

$$q_{r,r+1}^{-1} \in L(a,c), \quad r = 1, \cdots, n-1;$$
$$q_{rs} \in L(a,c), \quad 1 \le r, s \le n, \quad s \ne r+1.$$

Similarly the endpoint b is regular if for some c, $a < c < b$, we have

$$q_{r,r+1}^{-1} \in L(c,b), \quad r = 1, \cdots, n-1;$$
$$q_{rs} \in L(c,b), \quad 1 \le r,\ s \le n, \quad s \ne r+1.$$

Note that from the definition of $Q \in Z_n(J)$ it follows that if the above hold for some $c \in J$, then they hold for any $c \in J$. We say that M is regular on J, or just M is regular, if M is regular at both endpoints. An endpoint is singular if it is not regular. In equation (1.1) with $w > 0$ on J; if both endpoints are regular we assume that $w \in L(J)$, if a is singular $w \in L_{loc}(a,c)$, if b is singular $w \in L_{loc}(c,b)$, $a < c < b$. In all cases we say that w is a weight function.

REMARK 1. *In much of the literature when an endpoint of J is infinite this endpoint and the associated problem is automatically classified as singular. Note that in the above definition $a = -\infty$ is allowed; similarly for $b = \infty$. For any interval $J = (a,b)$ observe that M is regular on any compact subinterval of J.*

REMARK 2. *$a = -\infty$ is allowed; similarly for $b = \infty$. For any interval $J = (a,b)$ observe that M is regular on any compact subinterval of J.*

Next we briefly discuss the 'connection' between the first order systems and scalar equations of order n. The first-order vector system $Y' = QY + F$ and the quasi-differential equation $c_{n1} y_Q^{[n]} = f$ are equivalent in the following sense:

PROPOSITION 1. *Let $Q \in Z_n(J)$ and $f \in L_{loc}(J)$. Let $M = M_Q$,*

$$F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

(i) If $Y \in (AC_{loc}(J))^n$ is a solution of

$$Y' = QY + F,$$

then there is a unique $y \in D(M)$ such that

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}$$

and

$$c_{n1} y^{[n]} = f;$$

(ii) If $y \in D(M)$ is a solution of (2.1), then

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix} \in (AC_{loc}(J))^n$$

and

$$Y' = QY + F.$$

Proof. This follows from a straightforward computation. □

The next two lemmas will be used in the proof of the Lagrange Identity below.

Lemma 1. *Let $Q, P \in Z_n(J)$. Let F, G be n by m matrix functions on J. Let the constant matrix $K \in M_n(\mathbb{C})$. If*

$$Y' = QY + F, \;\; Z' = PZ + G \;\; \text{on } J,$$

Then

$$(Z^*KY)' = Z^*(P^*K + KQ)\,Y + G^*KY + Z^*KF.$$

Proof. By computation we obtain

$$\begin{aligned}(Z^*KY)' &= Z^{*\prime}KY + Z^*KY' = (Z^*P^* + G^*)KY + Z^*K(QY + F) \\ &= Z^*(P^*K + KQ)\,Y + G^*KY + Z^*KF.\end{aligned}$$

□

Lemma 2. *Assume $Q, P \in Z_n(J)$, $K \in M_n(\mathbb{C})$ is invertible and $Q = -K^{-1}P^*K$. If $Y' = QY + F$ and $Z' = PZ + G$ on J, where F, G are n by m matrices on J, then*

$$(Z^*KY)' = G^*KY + Z^*KF.$$

Proof. Since $Q = -K^{-1}P^*K$ implies that $P^*K + KQ = 0$ the conclusion follows from the previous lemma. □

Remark 3. *Note that in Lemma 2 we have assumed only that $K \in M_n(\mathbb{C})$ is invertible.*

From the existence and uniqueness theorem it follows that the initial value problems associated with $Y' = QY + F$ have a unique solution:

Corollary 1. *For each $F \in (L_{loc}(J))^n$, each α in J and each $K \in \mathbb{C}^n$ there is a unique $Y \in (AC_{loc}(J))^n$ such that*

$$Y' = QY + F \;\; \text{on } J \;\; \text{and} \;\; Y(\alpha) = K.$$

For each $f \in L_{loc}(J)$, each $\alpha \in J$ and $c_0, \ldots, c_{n-1} \in \mathbb{C}$ there is a unique $y \in D(Q)$ such that

$$c_{n1}y^{[n]} = f \quad \text{and} \quad y^{[r]}(\alpha) = c_r \quad (r = 0, \ldots, n-1).$$

If $f \in L(J)$, J is bounded and all components of Q are in $L(J)$, then $y \in AC(J)$.

DEFINITION 3. *Let $Q \in Z_n(J)$, C satisfy (1.3), (1.4), and let*

$$Q^+ = -C^{-1}Q^*C. \tag{2.2}$$

We call this Q^+ the C-adjoint matrix of Q, and it follows directly from the definition of $Z_n(J)$ that $Q^+ = (q_{ij}^+) \in Z_n(J)$; $M^+ = M_{Q^+}$ is called the C-adjoint expression of $M = M_Q$. Note that for $P, Q \in Z_n(J)$ we have

$$(P^+)^+ = P, \quad (P+Q)^+ = P^+ + Q^+, \quad (PQ)^+ = -Q^+P^+, \quad (cP)^+ = \bar{c}P^+, \ c \in \mathbb{C}.$$

Next we define the maximal and minimal operators and establish their basic properties. In a sense the maximal operator is the largest operator on which the differential expressions M can 'operate' and map the result into $L^2(J, w)$.

DEFINITION 4. *Let $Q \in Z_n(J)$, let w be a weight function and let $H = L^2(J, w)$. The maximal operator $S_{\max} = S_{\max}(Q, J)$ with domain $D_{\max} = D_{\max}(Q, J)$ is defined by:*

$$D_{\max}(Q, J) = \{y \in H : y \in D(Q), \ w^{-1}M_Q\, y \in H\},$$
$$S_{\max}(Q, J)\, y = w^{-1}M_Q\, y, \quad y \in D_{\max}(Q, J).$$

We use the next theorem to define the minimal operator.

THEOREM 1. *Let $Q \in Z_n(J)$, let C satisfy (1.3) (1.4), and let*

$$Q^+ = -C^{-1}Q^*C.$$

Then

(1) *$D_{\max}(Q, J)$ is dense in H. $S_{\min}(Q, J)$ is a closed operator in H with dense domain and we have*

$$S_{\min}(Q, J) = S^*_{\max}(Q^+, J), \quad S^*_{\min}(Q, J) = S_{\max}(Q^+, J).$$

(2) *If $Q^+ = Q$, then $S_{\min}(Q, J)$ is a closed symmetric operator in H with dense domain $D_{\min}(Q, J)$ and*

$$S^*_{\min}(Q, J) = S_{\max}(Q, J), \quad S_{\min}(Q, J) = S^*_{\max}(Q, J).$$

PROOF. The method of Naimark [**10**, Chapter V] can be adapted to prove this theorem with minor modifications. See also [**6**], [**9**]. □

DEFINITION 5. *Let $Q \in Z_n(J)$ and let C satisfy (1.3) (1.4). If*

$$Q = Q^+ = -C^{-1}Q^*C$$

we call Q a C-symmetric matrix and the corresponding differential expression $M = M_Q$ is called a C-Symmetric differential expression.

Next we prove the Lagrange Identity which is fundamental in the study of boundary value problems and the associated operators. The proofs of this identity given in Weidmann [**14**] and Zettl [**17**] are long and cumbersome using integration by parts and mathematical induction. Here we use an elegant method of Everitt and Neuman [**5**] which makes direct use of the system formulation of scalar equations discussed above for the construction of the differential expression $M = M_Q$ and its domain $D(M)$.

THEOREM 2 (Lagrange Identity). *Let $Q \in Z_n(J)$, C satisfy (1.3), (1.4), and let $P = -C^{-1}Q^*C$. Then $P \in Z_n(J)$ and for any $y \in D(M_Q)$, $z \in D(M_P)$ we have*

$$\overline{z}\, M_Q\, y - y\, \overline{M_P\, z} = [y, z]',$$

where

$$[y,z] = (-1)^{n+1} i^n \sum_{r=0}^{n-1} c_{n-r,r+1} \overline{z}_P^{[n-r-1]} y_Q^{[r]} = (-1)^{n+1} i^n Z^* C Y. \tag{2.3}$$

PROOF. That $P \in Z_n(J)$ follows directly from the definition of $Z_n(J)$. Recall that $M_Q y = i^n y^{[n]}$ and $M_P z = i^n z^{[n]}$. Let $f = c_{n1} y_Q^{[n]}$, $g = c_{n1} z_P^{[n]}$ and let

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}, \quad Z = \begin{pmatrix} z^{[0]} \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix}.$$

Then

$$Y' = QY + F, \quad Z' = PZ + G.$$

By computation, we have

$$G^*CY + Z^*CF = -i^n(\overline{z} M_Q y - y \overline{M_P z}).$$

Moreover,

$$Z^*CY = \sum_{r=0}^{n-1} c_{n-r,r+1} \overline{z}_P^{[n-r-1]} y_Q^{[r]}.$$

From the above lemmas we have

$$(Z^*CY)' = G^*CY + Z^*CF.$$

Therefore (2.3) holds. □

For the special case $C = E_n$ we have the following Lagrange Identity.

COROLLARY 2. *Let $Q \in Z_n(J)$, $P = -E^{-1}Q^*E$. Then $P \in Z_n(J)$ and for any $y \in D(M_Q)$, $z \in D(M_P)$ we have*

$$\overline{z}\, M_Q\, y - y\, \overline{M_P\, z} = [y, z]',$$

where

$$[y,z] = i^n \sum_{r=0}^{n-1} (-1)^{n+1-r} \overline{z}_P^{[n-r-1]} y_Q^{[r]} = -i^n Z^* E Y. \tag{2.4}$$

PROOF. See the proof of Theorem 3.3.1 in [**16**]. □

REMARK 4. *For the known special case $C = E_n$, $q_{n,n+1} = 1$. For the general case when C satisfies (1.3), (1.4) we define $q_{n,n+1} = c_{n1}$.*

COROLLARY 3. *If $My = \lambda wy$ and $Mz = \overline{\lambda} wz$ on some interval $(\alpha, \beta) \in (a, b)$, then $[y, z]$ is constant on (α, β). In particular, if λ is real and $My = \lambda wy$, $Mz = \lambda wz$ on some interval $(\alpha, \beta) \in (a, b)$, then $[y, z]$ is constant on (α, β).*

PROOF. This follows directly from the Lagrange Identity. □

REMARK 5. *We comment on the difference between this Lagrange Identity and the classical Lagrange Identity as found, for example, in the well known books by Coddington and Levinson* [**3**] *and Dunford and Schwartz* [**4**]. *The fundamental differences are: (i) the matrix C is a constant matrix where in the classical case it is a complicated nonconstant function depending on the coefficients, (ii) we assume only that the coefficients are locally Lebesgue integrable in contrast to* [**3**], [**4**] *where strong smoothness assumptions are required. The price paid for this generalization and simplification is the use of quasi-derivatives $y^{[r]}$ in place of the classical derivatives $y^{(r)}$. These quasi-derivatives $y^{[r]}$ depend on the coefficients.*

For the rest of this paper we assume that

$$Q = Q^{+} = -C^{-1}Q^{*}C,$$

i.e. we study the C-Symmetric differential expressions M_Q. Let $M = M_Q$, then note that $M = M^{+} = M_{Q^{+}}$.

LEMMA 3. *For any y, z in $D(M)$ we have*

$$\int_{c_1}^{c} \{\overline{z}My - y\overline{Mz}\} = [y,z](c) - [y,z](c_1), \tag{2.5}$$

for any $c, c_1 \in J = (a,b)$.

PROOF. This follows from the Lagrange Identity and integration. □

LEMMA 4. *For any y, z in $D(M)$ the limits*

$$\lim_{t\to b^-}[y,z](t), \quad \lim_{t\to a^+}[y,z](t) \tag{2.6}$$

exist and are finite, and

$$\int_{a}^{b} \{\overline{z}My - y\overline{Mz}\} = [y,z](b) - [y,z](a).$$

PROOF. This follows from (2.5) by taking limits as $c_1 \to a,\ c \to b$. That the limits exist and are finite can be seen from the definition of $D(M)$. □

REMARK 6. *These finite limits play a critical role in the characterization of singular symmetric and self-adjoint operators studied below; they 'play the role' of the quasi-derivatives at regular endpoints.*

LEMMA 5. *Since M is regular at $c \in (a,b)$, for any $y \in D(M)$ the limits*

$$y^{[r]}(c) = \lim_{t\to c} y^{[r]}(c)$$

exist and are finite, $r = 0, \cdots, n-1$. This also holds for $c = a$ if a is a regular endpoint and for $c = b$ if b is a regular endpoint. At each endpoint the limit is the appropriate one sided limit.

PROOF. See [**10**] or [**15**]. Although our result is more general than those stated in these references the same method of proof can be used here. □

NOTATION 2. *Below we will also use minimal and maximal domain functions and their restrictions on subintervals (α,β) of $J = (a,b)$, particularly for $(\alpha,\beta) = (a,c)$ and $(\alpha,\beta) = (c,b)$ with $c \in (a,b)$. Since $Q \in Z_n(J)$ implies that $Q \in Z_n((\alpha,\beta))$ the above Definitions and Theorems can be applied in the Hilbert space $L^2((\alpha,\beta),w)$ with J replaced by (α,β). When we use the notation $D_{\max}(\alpha,\beta)$,*

$D_{\min}(\alpha,\beta)$ it is understood that we use the above definitions and theorems with J replaced by (α,β), Q and w replaced by their restrictions to (α,β) and $L^2(J,w)$ replaced by $L^2((\alpha,\beta),w)$. Below, Q, J, (α,β), as well as the Hilbert space, may be omitted when these are clear from the context.

For $Q\in Z_n(J)$, $M=M_Q$, if we consider the equation $My=\lambda wy$ and say that w is a 'weight' function, we mean that $w>0$ on (a,c) and in $L^1(a,c)$ if a is regular and in $L_{loc}(a,c)$ if a is singular. Similarly for the interval (c,b).

The next lemma is known as the Naimark Patching Lemma or just the Patching Lemma.

Lemma 6 (Naimark Patching Lemma). *Let $Q\in Z_n(J)$ and assume that both endpoints are regular. Let $\alpha_0,\dots,\alpha_{n-1}$, $\beta_0,\dots,\beta_{n-1}\in\mathbb{C}$. Then there is a function $y\in D_{\max}$ such that*

$$y^{[r]}(a)=\alpha_r,\quad y^{[r]}(b)=\beta_r\quad (r=0,\dots,n-1).$$

Proof. The proof in Naimark [**10**] can readily be adapted to prove this more general Patching Lemma. □

Corollary 4. *Let $a<c<d<b$ and $\alpha_0,\dots,\alpha_{n-1},\beta_0,\dots,\beta_{n-1}\in\mathbb{C}$. Then there is a $y\in D_{\max}$ such that y has compact support in J and satisfies :*

$$y^{[r]}(c)=\alpha_r,\quad y^{[r]}(d)=\beta_r\quad (r=0,\dots,n-1).$$

Proof. The Patching Lemma gives a function y_1 on $[c,d]$ with the desired properties. Let c_1,d_1 with $a<c_1<c<d<d_1<b$. Then use the Patching Lemma again to find y_2 on (c_1,c) and y_3 on (d,d_1) such that

$$y_2^{[r]}(c_1)=0,\quad y_2^{[r]}(c)=\alpha_r,\quad y_3^{[r]}(d)=\beta_r,\quad y_3^{[r]}(d_1)=0\qquad (r=0,\dots,n-1).$$

Now set

$$y(x):=\begin{cases} y_1(x) & for\quad x\in[c,d]\\ y_2(x) & for\quad x\in(c_1,c)\\ y_3(x) & for\quad x\in(d,d_1)\\ 0 & for\quad x\in J\setminus(c_1,d_1).\end{cases}$$

Clearly y has compact support in J. Since the quasi-derivatives at c_1,c,d,d_1 coincide on both sides, $y\in D_{\max}$ follows. □

Corollary 5. *Let $a_1<\dots<a_k\in J$, where a_1 and a_k can also be regular endpoints. Let $\alpha_{jr}\in\mathbb{C}$ $(j=1,\dots,k;\ r=0,\dots,n-1)$. Then there is a $y\in D_{\max}$ such that*

$$y^{[r]}(a_j)=\alpha_{jr}\quad (j=1,\dots,k;\ r=0,\dots,n-1).$$

Proof. This follows from repeated applications of the previous Corollary. □

The next theorem gives a useful characterization of the minimal domain $D(S_{\min})$.

Theorem 3. *Let $Q=Q^+\in Z_n(J)$. Then*

$$D(S_{\min})=\{y\in D_{\max}\ :\ [y,z](a)=0=[y,z](b),\ for\ all\ z\in D_{\max}\}.$$

Proof. This follows from the Lagrange Identity and the Naimark Patching Lemma. □

3. Symmetric Operators

In this section we characterize the symmetric operators S in $L^2(J,w)$ satisfying (1.2) in terms of the two point boundary conditions of the interval $J=(a,b)$ for regular or singular endpoints a,b. The self-adjoint characterization is a special case.

The von Neumann formula for the domain of the adjoint of a symmetric operator in Hilbert space is fundamental for the study of self-adjoint and symmetric operators.

THEOREM 4 (von Neumann). *Let T be a closed densely defined symmetric operator on a complex Hilbert space H, and let N_+ and N_- be the deficiency spaces of T. Then we have*

$$D(T^*) = D(T) \dotplus N_+ \dotplus N_- \tag{3.1}$$

An operator S is a closed symmetric extension of T if and only if there exist closed subspaces F_+ of N_+ and F_- of N_- and an isometric mapping V of F_+ onto F_- such that

$$D(S) = D(T) + \{g + Vg : \ g \in F_+\}. \tag{3.2}$$

Furthermore, S is self-adjoint if and only if $F_+ = N_+$ and $F_- = N_-$.

PROOF. For the definition of deficiency spaces and a proof of this theorem see Dunford and Schwartz [**4**], Naimark [**10**], or Weidmann [**15**]. □

The von Neumann formula can be applied to get the following decomposition of $D(S_{\max})$.

THEOREM 5. *Suppose $Q \in Z_n(J)$ satisfies $Q = Q^+ = -C^{-1}Q^*C$, $M = M_Q$ and let $S_{\min} = S_{\min}(Q)$. Assume the deficiency index of M is d, i.e. $d^+ = d^- = d$. Then*

$$D(S_{\max}) = D(S_{\min}) \dotplus N_\lambda \dotplus N_{\overline{\lambda}}, \quad \mathrm{Im}(\lambda) \neq 0, \tag{3.3}$$

where

$$N_\lambda = \{y \in D(S_{\max}) : M_Q\, y = \lambda w\, y, \ \ \mathrm{Im}(\lambda) \neq 0\}. \tag{3.4}$$

Since the deficiency index is d the equation $My = M_Q\, y = \lambda w\, y$ has exactly d linearly independent solutions on $J=(a,b)$ for every λ with $\mathrm{Im}(\lambda) \neq 0$. Thus it is clear from (3.3) that $D_{\max}$ is a $2d$ dimensional extension of $D_{\min}$. Therefore $S_{\min}$ has self-adjoint extensions and every self-adjoint extension is a d dimensional extension. Furthermore, every d dimensional symmetric extension of $S_{\min}$ is self-adjoint. Moreover, every symmetric extension of $S_{\min}$ is an m dimensional extension with

$$0 \le m \le d$$

and an $l = 2d - m$ dimensional restriction of $S_{\max}$ with

$$d \le l \le 2d.$$

PROOF. This decomposition of $D_{\max}$ is well known [**15**], [**18**], [**10**] and the furthermore and moreover statements follow from the von Neumann Theorem. □

Note that the deficiency spaces N_λ and $N_{\overline{\lambda}}$ in the von Neumann formula (3.3) consist of solutions of the equation $My = \lambda wy$, $\mathrm{Im}(\lambda) \neq 0$, on the whole interval $J=(a,b)$. These solutions may be very different near the two endpoints a,b of the interval J. To take this different behavior at the endpoints into account we develop the following different decomposition of $D_{\max}$.

Notation 3. *Let $a < c < b$. Consider equation $M_Q y = \lambda w y$. Note that if $Q \in Z_n(J)$, then it follows that $Q \in Z_n(a,c)$, $Q \in Z_n(c,b)$ and we can study equation (2.1) on (a,c) and on (c,b) as well as on $J = (a,b)$. Note that from the definition of $Z_n(a,c)$ and $Q \in Z_n(c,b)$ it follows that c is a regular endpoint for both intervals. Also the minimal and maximal operators are defined for these two subintervals and we can study the operator theory generated by M in the Hilbert spaces $L^2((a,c),w)$ and $L^2((c,b),w)$. Below we will use the notation $S_{\min}(I)$, $S_{\max}(I)$ for the minimal and maximal operators on the interval I for $I = (a,c)$, $I = (c,b)$, $I = (a,b) = J$. The interval I may be omitted from this notation when it is clear from the context. So we make the following definition.*

Definition 6. *Let $a < c < b$. Let d_a^+, d_b^+ denote the dimension of the solution space of $My = i\,wy$ lying in $L^2((a,c),w)$ and $L^2((c,b),w)$, respectively, and let d_a^-, d_b^- denote the dimension of the solution space of $My = -i\,wy$ lying in $L^2((a,c),w)$ and $L^2((c,b),w)$, respectively. Then d_a^+ and d_a^- are called the positive deficiency index and the negative deficiency index of $S_{\min}(a,c)$, respectively. Similarly for d_b^+ and d_b^-. Also d^+, d^- denote the deficiency indices of $S_{\min}(a,b)$; these are the dimensions of the solution spaces of $My = iwy$, $My = -iwy$ lying in $L^2((a,b),w)$. If $d_a^+ = d_a^-$, then the common value is denoted by d_a and is called the deficiency index of $S_{\min}(a,c)$, or the deficiency index at a. Similarly for d_b. Note that d_a, d_b are independent of c. If $d^+ = d^-$, then we denote the common value by d and call it the deficiency index of $S_{\min}(a,b)$ or just of $S_{\min}$. Below when we speak of 'the deficiency index' d or d_a, d_b it is automatically assumed that $d^+ = d^- = d$, $d_a^+ = d_a^- = d_a$, $d_b^+ = d_b^- = d_b$.*

Remark 7. *It is well known that $S_{\min}(I)$ has a self-adjoint extension if and only if $d^+(I) = d^-(I) = d(I)$. In this case $My = \lambda wy$ has exactly $d(I)$ linearly independent solutions in $L^2(I,w)$ for any λ with $\mathrm{Im}(\lambda) \neq 0$.*

The relationships between d_a, d_b and d are well known and summarized in the next lemma along with some additional information.

Lemma 7. *For $d_a^+, d_b^+, d_a^-, d_b^-, d^+, d^-, d_a, d_b$ defined above, we have*

1. *$d^+ = d_a^+ + d_b^+ - n$, $d^- = d_a^- + d_b^- - n$;*
2. *if $d_a^+ = d_a^- = d_a, d_b^+ = d_b^- = d_b$, then $[\frac{n+1}{2}] \le d_a, d_b \le n$;*
3. *$m_a = 2d_a - n \le d_a$, $m_b = 2d_b - n \le d_b$, $m_a + m_b = 2d$.*
4. *the minimal operator $S_{\min}$ has self-adjoint extensions in H if and only if $d^+ = d^-$, in this case we let $d = d^+ = d^-$. If $d = 0$ then $S_{\min}$ is self-adjoint with no proper self-adjoint extension. In all other cases $S_{\min}$ has an uncountable number of symmetric extensions, i.e. there are an uncountable number of symmetric operators S in H satisfying*

$$S_{\min} \subset S \subset S^* \subset S_{\max}.$$

Proof. This is well known, e.g. see the book [**14**]. □

What are the domains which characterize these symmetric operators S ? The next theorem gives a decomposition of $D_{\max}$ when both endpoints are singular which will help answer this question.

Theorem 6. *Let $Q \in Z_n(J)$, $J = (a,b)$, $-\infty \le a < b \le \infty$, let C satisfy (1.3) and (1.4) , and let w be a weight function. Assume that Q is C-Symmetric, i.e. $Q = Q^+ = -C^{-1}Q^*C$ and let $My = M_Q y = \lambda wy$ be the corresponding symmetric* *differential equation. Let $a < c < b$. Then $Q \in Z_n((a,c))$, $Q \in Z_n((c,b))$ and Q is C-Symmetric in each interval. Let d_a, d_b and d be the deficiency indices of $My = \lambda wy$ on (a,c), (c,b) and (a,b), respectively; let $m_a = 2d_a - n$, $m_b = 2d_b - n$. Fix $\lambda_a \in \mathbb{C}$, $\lambda_b \in \mathbb{C}$ with $\mathrm{Im}(\lambda_a) \neq 0 \neq \mathrm{Im}(\lambda_b)$. Then*

a:

(1) *there exist linearly independent solutions $u_1, \cdots, u_{m_a}$ of $My = \lambda_a wy$ on (a,c) such that the $m_a \times m_a$ matrix*

$$F_{m_a} = \begin{pmatrix} [u_1,u_1] & \cdots & [u_{m_a},u_1] \\ \cdots & \cdots & \cdots \\ [u_1,u_{m_a}] & \ldots & [u_{m_a},u_{m_a}] \end{pmatrix}(a)$$

is nonsingular;

(2) *$u_1, \cdots, u_{m_a}$ can be extended to (a,b) such that the extended functions, still denoted by $u_1, \cdots, u_{m_a}$, are in $D_{\max}(a,b)$ and are identically 0 near b;*

(3) *$u_1, \cdots, u_{m_a}$ are linearly independent modulo $D_{\min}$;*

b:

(1) *there exist linearly independent solutions $v_1, \cdots, v_{m_b}$ of $My = \lambda_b wy$ on (c,b) such that the $m_b \times m_b$ matrix*

$$F_{m_b} = \begin{pmatrix} [v_1,v_1] & \cdots & [v_{m_b},v_1] \\ \cdots & \cdots & \cdots \\ [v_1,v_{m_b}] & \ldots & [v_{m_b},v_{m_b}] \end{pmatrix}(b)$$

is nonsingular;

(2) *$v_1, \cdots, v_{m_b}$ can be extended to (a,b) such that the extended functions, still denoted by $v_1, \cdots, v_{m_b}$, are in $D_{\max}(a,b)$ and are identically 0 near a;*

(3) *$v_1, \cdots, v_{m_b}$ are linearly independent modulo $D_{\min}$;*

c:

The maximal domain has the following representation:

$$D_{\max}(a,b) = D_{\min}(a,b) \dot{+} span\{u_1, \cdots, u_{m_a}\} \dot{+} span\{v_1, \cdots, v_{m_b}\}. \tag{3.5}$$

PROOF. Use Theorem 4.4.4 from the book [**16**] with E replaced by C. □

REMARK 8. *Note that F_{m_a} and F_{m_b} are nonsingular and $F_{m_a}^* = -F_{m_a}$, $(F_{m_a}^{-1})^* = -F_{m_a}^{-1}$, $F_{m_b}^* = -F_{m_b}$, $(F_{m_b}^{-1})^* = -F_{m_b}^{-1}$, $m_a + m_b = 2d$.*

REMARK 9. *Note that the decomposition (3.5) is very different from the decomposition which is based on the abstract von Neumann formula (3.1) where the deficiency spaces N_+ and N_- consist of solutions which are defined on the whole interval (a,b). As mentioned above, the behavior of solutions may be very different near the two endpoints. The decomposition (3.5) makes no restriction on the behavior of the solutions near the endpoints a, b. In particular, there is no restriction on their asymptotic or oscillatory behavior.*

At a singular endpoint the quasi-derivatives $y^{[i]}$ are, in general, not defined at that endpoint. From the Lagrange Identity it follows that for any $y, z \in D_{\max}$ the Lagrange brackets $[y,z]$ are well defined at each singular endpoint. These brackets can be used to replace the quasi-derivatives as we will see below.

We start with the introduction of boundary matrices and boundary conditions.

Definition 7. *For any $y \in D_{\max}$ define*

$$Y_{a,b} = \begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix}, \; Y(a) = \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_{m_a}](a) \end{pmatrix}, \; Y(b) = \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix} \tag{3.6}$$

and recall that the Lagrange brackets $[y,u_j](a)$ and $[y,v_j](b)$ exist as finite limits.

Definition 8. *A matrix $U \in M_{l,2d}$ with rank l, $0 \leq l \leq 2d$, $2d = m_a + m_b$, is called a boundary condition matrix. And for $y \in D_{\max}$ and $Y_{a,b}$ given by (3.6) the equation*

$$U\,Y_{a,b} = 0 \tag{3.7}$$

is called a boundary condition. The null space of U is denoted by $\mathcal{N}(U)$, $\mathcal{R}(U)$ denotes its range and U^* is the conjugate transpose of U.

Note that any boundary condition (3.7) can be reduced by elementary matrix operations to the case that the rank of U is the number of its rows.

Definition 9. *Suppose $U \in M_{l,2d}$ is a boundary condition matrix. Define an operator $S(U)$ in $L^2(J,w)$ by*

$$\begin{aligned} D(S(U)) &= \{y \in D_{\max} : U\,Y_{a,b} = 0\}, \\ S(U)y &= S_{\max}y \;\; for \;\; y \in D(S(U)). \end{aligned} \tag{3.8}$$

Remark 10. *From (3.8) for any boundary condition matrix U, $D(S(U))$ is a linear submanifold of $D_{\max}$ and we have*

$$S_{\min} \subset S(U) \subset S_{\max}. \tag{3.9}$$

Consequently, since $S_{\max}$ is a closed finite dimensional extension of $S_{\min}$, it follows that every operator $S(U)$ is a closed finite dimensional extension of $S_{\min}$. For which matrices U is $S(U)$ a symmetric operator in $L^2(J,w)$? This is the question answered next. Let

$$P = \begin{pmatrix} F_{m_a}^{-1} & 0 \\ 0 & -F_{m_b}^{-1} \end{pmatrix} \tag{3.10}$$

and note that

$$P^* = -P, \;(P^{-1})^* = -P^{-1}. \tag{3.11}$$

The next theorem determines for which boundary matrices $U = (A,B)$ the operators $S(U)$ are not symmetric, which ones are symmetric and which ones are self-adjoint.

Theorem 7. *Let $M = M_Q$, $Q \in Z_n(J)$, $J = (a,b)$, $-\infty \leq a < b \leq \infty$, $Q = Q^+ = -C^{-1}Q^*C$, let C satisfy (1.3), let w be a weight function. Let d_a, d_b and d be the deficiency indices of $My = \lambda wy$ on (a,c), (c,b) and (a,b), respectively. Recall that $d = d_a + d_b - n$. Let $m_a = 2d_a - n$, $m_b = 2d_b - n$, let $Y_{a,b}$ be defined by (3.7). Assume $U \in M_{l,2d}$ has rank l, $0 \leq l \leq m_a + m_b = 2d$ and let $U = (A : B)$ with $A \in M_{l,m_a}$ consisting of the first m_a columns of U in the same order as they are in U and $B \in M_{l,m_b}$ consisting of the next m_b columns of U in the same order as they are in U. Define the operator $S(U)$ in $L^2(J,w)$ by (3.8) and let*

$$G = K(A,B) = UP^{-1}U^* = AF_{m_a}A^* - BF_{m_b}B^*, \; and \; r = rank\, G.$$

Then we have:

(1) *If $l < d$, then $S(U)$ is not symmetric.*
(2) *If $l = d$, then $S(U)$ is self-adjoint (and hence also symmetric) if and only if $r = 0$.*
(3) *Let $l = d + s$, $0 < s \leq d$. Then $S(U)$ is symmetric if and only if $r = 2s$.*

PROOF. The proof of Theorem 7 will be given in the next section. □

Based on Theorem 7, when the deficiency index d is maximal i.e. $d = n$, we have the following direct Corollary.

COROLLARY 6. *Suppose the hypotheses of Theorem 7 hold and, in addition, $d = n$, boundary conditions of the operator $S(U)$ have the four special cases. Then all three conclusions of this Theorem hold for the following boundary conditions:*

(1) *Both endpoints are regular.*

$$A\begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B\begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(2) *The endpoint a is singular and b is regular.*

$$A\begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_n](a) \end{pmatrix} + B\begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(3) *The endpoint a is regular and b is singular.*

$$A\begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B\begin{pmatrix} [y,u_1](b) \\ \vdots \\ [y,u_n](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(4) *Both endpoints are singular.*

$$A\begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_n](a) \end{pmatrix} + B\begin{pmatrix} [y,u_1](b) \\ \vdots \\ [y,u_n](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

REMARK 11. *Recall that at a regular endpoint the Lagrange brackets in (3.6) which involve that endpoint can be replaced by quasi-derivatives evaluated at that point.*

4. Symmetric Domains And Proofs

In this section we characterize the boundary conditions which determine the symmetric operators satisfying (1.2) and call these the symmetric boundary conditions. The self-adjoint operators and boundary conditions are a special case. This characterization is based on the maximal domain decomposition (3.5). The following singular patching theorem and two algebra lemmas are important for the proof of symmetric domains.

Theorem 8 (Singular Patching Theorem). *Let the hypotheses and notation of Theorem 6 hold. For any complex numbers* $\alpha_1, \alpha_2, \cdots, \alpha_{m_a}, \beta_1, \beta_2, \cdots, \beta_{m_b}$, *there exists* $y \in D_{\max}(a,b)$ *such that*

$$[y,u_1](a)=\alpha_1, \quad [y,u_2](a)=\alpha_2, \ \cdots, \quad [y,u_{m_a}](a)=\alpha_{m_a},$$
$$[y,v_1](b)=\beta_1, \quad [y,v_2](b)=\beta_2, \ \cdots, \quad [y,v_{m_b}](b)=\beta_{m_b}.$$

Proof. Consider the equation

$$\begin{pmatrix} [u_1,u_1](a) & \cdots & [u_{m_a},u_1](a) \\ \cdots & \cdots & \cdots \\ [u_1,u_{m_a}](a) & \ldots & [u_{m_a},u_{m_a}](a) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{m_a} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{m_a} \end{pmatrix}$$

i.e.

$$F_{m_a} \begin{pmatrix} c_1 \\ \vdots \\ c_{m_a} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{m_a} \end{pmatrix}. \tag{4.4.10}$$

Since F_{m_a} is nonsingular, equation (4.4.10) has a unique solution

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{m_a} \end{pmatrix} = F_{m_a}^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{m_a} \end{pmatrix}.$$

Similarly, by the fact that F_{m_b} is nonsingular, the following equation

$$\begin{pmatrix} [v_1,v_1](b) & \cdots & [v_{m_b},v_1](b) \\ \cdots & \cdots & \cdots \\ [v_1,v_{m_b}] & \cdots & [v_{m_b},v_{m_b}](b) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_{m_b} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{m_b} \end{pmatrix},$$

has a unique solution

$$\begin{pmatrix} h_1 \\ \vdots \\ h_{m_b} \end{pmatrix} = F_{m_b}^{-1} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{m_b} \end{pmatrix}.$$

Let

$$y = y_0 + c_1u_1 + \cdots + c_{m_a}u_{m_a} + h_1v_1 + \cdots + h_{m_b}v_{m_b},$$

where $y_0 \in D_{\min}$. By the decomposition (3.5), we have $y \in D_{\max}(a,b)$, and then

$$[y,u_1](a) = c_1[u_1,u_1](a) + c_2[u_2,u_1](a) + \cdots + c_{m_a}[u_{m_a},u_1](a) = \alpha_1,$$
$$[y,u_2](a) = c_1[u_1,u_2](a) + c_2[u_2,u_2](a) + \cdots + c_{m_a}[u_{m_a},u_2](a) = \alpha_2,$$
$$\cdots\cdots\cdots$$
$$[y,u_{m_a}](a) = c_1[u_1,u_{m_a}](a) + c_2[u_2,u_{m_a}](a) + \cdots + c_{m_a}[u_{m_a},u_{m_a}](a) = \alpha_{m_a}.$$

Similarly,

$$[y,v_1](b)=\beta_1, \quad [y,v_2](b)=\beta_2, \ \cdots, \quad [y,v_{m_b}](b)=\beta_{m_b}.$$

This completes the proof. □

Lemma 8. *If* S *is a subset of* $\mathbb{C}^n$, $n \in \mathbb{N}_2$, *then*

(1) $S^\perp$ *is a subspace of* $\mathbb{C}^n$.

(2) $(S^{\perp})^{\perp} =$ *span of* S.

(3) $(S^{\perp})^{\perp} = S$, *if* S *is a subspace.*

(4) $n = \dim S^{\perp} + \dim(S^{\perp})^{\perp}$.

(5) *Suppose* $A \in M_{l,m}$. *Then* $\mathcal{R}(A) = (\mathcal{N}(A^*))^{\perp}$ *i.e.* $Ax = y$ *has a solution (not necessarily unique) if and only if* $y^*z = 0$ *for all* $z \in \mathbb{C}^l$ *such that* $A^*z = 0$.

LEMMA 9. *Let* G *be any invertible* $p \times p$ *matrix and* F *an* $l \times p$ *matrix with rank* $F = l$. *Then the following assertions are equivalent:*

(i) $\mathcal{N}(F) \subset \mathcal{R}(GF^*)$;

(ii) $rank(FGF^*) \leq 2l - p$;

(iii) $rank(FGF^*) = 2l - p$;

(iv) $\mathcal{N}(F) = GF^*\big(\mathcal{N}(FGF^*)\big)$.

The next lemma 'connects' the Lagrange Identity with the boundary conditions.

LEMMA 10. *Assume that* $U \in M_{l,2d}$, *rank* $U = l$, $d \leq l \leq 2d$. *Let* $y, z \in D_{\max}$ *and define* $Y_{a,b}$, $Z_{a,b}$ *by*

$$Y_{a,b} = \begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix}, \ Z_{a,b} = \begin{pmatrix} Z(a) \\ Z(b) \end{pmatrix},$$

which are defined in (3.6). *Let*

$$P = \begin{pmatrix} F_{m_a}^{-1} & 0 \\ 0 & -F_{m_b}^{-1} \end{pmatrix} \tag{4.1}$$

and note that $P^* = -P$, $(P^{-1})^* = -P^{-1}$. *Then* $S(U)$ *is symmetric if and only if*

$$Z_{a,b}^* P Y_{a,b} = 0, \quad \text{for all } y, z \in D(S(U)). \tag{4.2}$$

PROOF. By the Lagrange Identity, for any $y, z \in D_{\max}$,

$$\int_a^b \{\overline{z} My - y\overline{Mz}\} = [y,z](b) - [y,z](a).$$

Therefore, it follows from the definition of $S(U)$ that $S(U)$ is symmetric if and only if for all $y, z \in D(S(U))$,

$$(S(U)y, z) - (y, S(U)z) = \int_a^b \{\overline{z} My - y\overline{Mz}\} = [y,z](b) - [y,z](a) = 0.$$

By the decomposition of $D_{\max}$ given by (3.5) functions $y, z \in D_{\max}$ can be represented as

$$y = y_0 + c_1 u_1 + \cdots + c_{m_a} u_{m_a} + h_1 v_1 + \cdots + h_{m_b} v_{m_b},$$

$$z = z_0 + \widehat{c}_1 u_1 + \cdots + \widehat{c}_{m_a} u_{m_a} + \widehat{h}_1 v_1 + \cdots + \widehat{h}_{m_b} v_{m_b},$$

where $y_0, z_0 \in D_{\min}$ and $c_j, \widehat{c}_j \in \mathbb{C}$, $j = 1, \cdots, m_a$; $h_j, \widehat{h}_j \in \mathbb{C}$, $j = 1, \cdots, m_b$.

Since

$$\begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix} = F_{m_b} \begin{pmatrix} h_1 \\ \vdots \\ h_{m_b} \end{pmatrix},$$

we have

$$\begin{pmatrix} h_1 \\ \vdots \\ h_{m_b} \end{pmatrix} = F_{m_b}^{-1} \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} \widehat{h}_1 \\ \vdots \\ \widehat{h}_{m_b} \end{pmatrix} = F_{m_b}^{-1} \begin{pmatrix} [z,v_1](b) \\ \vdots \\ [z,v_{m_b}](b) \end{pmatrix}.$$

Therefore

$$\begin{aligned} [y,z](b) =& (\overline{\widehat{h}}_1,\ \overline{\widehat{h}}_2,\ \cdots,\overline{\widehat{h}}_{m_b}) F_{m_b} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{m_b} \end{pmatrix} \\ =& -\left(\ \overline{[z,v_1](b)}, \quad \cdots, \quad \overline{[z,v_{m_b}](b)} \ \right) F_{m_b}^{-1} \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix}. \end{aligned}$$

Similarly,

$$[y,z](a) = -\left(\ \overline{[z,u_1](a)}, \quad \cdots, \quad \overline{[z,u_{m_a}](a)} \ \right) F_{m_a}^{-1} \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_{m_a}](a) \end{pmatrix}.$$

So
$[y,z](b) - [y,z](a) =$

$$\left(\overline{[z,u_1](a)},\cdots,\overline{[z,u_{m_a}](a)},\overline{[z,v_1](b)},\cdots,\overline{[z,v_{m_b}](b)}\right) \begin{pmatrix} F_{m_a}^{-1} & 0 \\ 0 & -F_{m_b}^{-1} \end{pmatrix} \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_{m_a}](a) \\ [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix}.$$

Hence, the operator $S(U)$ is symmetric if and only if

$$[y,z](b) - [y,z](a) = 0 \quad for\ all\ y,z \in D(S(U)),$$

i.e.

$$Z_{a,b}^{*} P Y_{a,b} = 0 \quad for\ all\ y,z \in D(S(U)).$$

□

LEMMA 11. *Each of the following statements is equivalent to (4.2):*

(1) *For all* $Y,Z \in \mathcal{N}(U)$, $Z^{*}PY = 0$*;*

(2) $\mathcal{N}(U) \perp P(\mathcal{N}(U))$;

(3) $P(\mathcal{N}(U)) \subset \mathcal{N}(U)^{\perp} = \mathcal{R}(U^*)$;

(4) $\mathcal{N}(U) \subset \mathcal{R}(P^{-1}U^*)$.

PROOF. Now we prove that the equivalence of (4.2) and (1). For all $Y, Z \in \mathcal{N}(U)$, let

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2d} \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2d} \end{pmatrix}.$$

Then $UY = 0$ and $UZ = 0$. By the Singular Patching, there exist $y, z \in D_{\max}$ such that

$$Y_{a,b} = \begin{pmatrix} [y, u_1](a) \\ \vdots \\ [y, u_{m_a}](a) \\ [y, v_1](b) \\ \vdots \\ [y, v_{m_b}](b) \end{pmatrix} = Y, \quad Z_{a,b} = \begin{pmatrix} [z, u_1](a) \\ \vdots \\ [z, u_{m_a}](a) \\ [z, v_1](b) \\ \vdots \\ [z, v_{m_b}](b) \end{pmatrix} = Z,$$

and $UY_{a,b} = UZ_{a,b} = 0$. Therefore $y, z \in D(S(U))$, and we have

$$Z^*PY = Z^*_{a,b}PY_{a,b} = 0.$$

Hence, for all $Y, Z \in \mathcal{N}(U)$, $Z^*PY = 0$.

On the other hand, for all $y, z \in D(S(U))$, $UY_{a,b} = UZ_{a,b} = 0$. This shows that $Y_{a,b}, Z_{a,b} \in \mathcal{N}(U)$. If (1) holds, then $Z^*_{a,b}PY_{a,b} = 0$. Therefore, when (1) holds, then for all $y, z \in D(S(U))$, $Z^*_{a,b}PY_{a,b} = 0$.

Statements (1) and (2) are essentially the same statements, just written differently. The equivalence of (2) and (3) follows from Lemma 8. Whereas the equivalence of (3) and (4) immediately follows from the fact that P is an invertible matrix. □

THEOREM 9. *Let U be an $l \times 2d$ matrix with $\operatorname{rank} U = l$, where $d \le l \le 2d$, $d = d_a + d_b - n$. Then the operator $S(U)$ is symmetric if and only if*

$$\mathcal{N}(U) \subset \mathcal{R}(P^{-1}U^*),$$

where P is defined by (4.1).

PROOF. This follows from the Singular Patching Theorem 8, Lemma 10 and Lemma 11. □

LEMMA 12. *Suppose $U \in M_{l,2d}$. Let $U = (A : B)$, where $A \in M_{l,m_a}$, $B \in M_{l,m_b}$, and recall that $m_a + m_b = 2d$. Assume that $\operatorname{rank} U = l$. Then the operator $S(U)$ is self-adjoint if and only if*

$$l = d \quad \text{and} \quad UP^{-1}U^* = 0, \text{ i.e. } AE_{m_a}A^* - BE_{m_b}B^* = 0.$$

PROOF. It follows from Theorem 5 and Theorem 9. □

Next we study matrices U such that $(S(U))^*$ is symmetric.

THEOREM 10. *Let $U \in M_{l,2d}$, $0 \le l \le 2d$ and assume that* rank $U = l$. *Then*

$$D((S(U))^*) = \{z \in D_{\max} : Z_{a,b} = \begin{pmatrix} [z,u_1](a) \\ \vdots \\ [z,u_{m_a}](a) \\ [z,v_1](b) \\ \vdots \\ [z,v_{m_b}](b) \end{pmatrix} \in \mathcal{R}(P^{-1}U^*)\}.$$

PROOF. Let $z \in D_{\max}$. Then $z \in D((S(U))^*)$ if and only if

$$(S_{\max}y, z) = (y, S_{\max}z), \quad for\ all \quad y \in D(S(U)).$$

This is equivalent to $Z^*_{a,b}PY_{a,b} = 0$ for all $y \in D(S(U))$. Therefore $z \in D((S(U))^*)$ if and only if $Y^*_{a,b}P^*Z_{a,b} = 0$, i.e. $P^*Z_{a,b} \in \mathcal{N}(U)^{\perp} = \mathcal{R}(U^*)$. Hence $Z_{a,b} \in \mathcal{R}(P^{-1}U^*)$. This completes the proof. □

LEMMA 13. *Let $U \in M_{l,2d}$ and assume* rank $U = l$ *and* $0 \le l \le d$. *Then the following statements are equivalent:*

(1) $(S(U))^*$ *is symmetric;*
(2) $\mathcal{N}(U) \supset \mathcal{R}(P^{-1}U^*)$;
(3) $UP^{-1}U^* = 0$.

PROOF. From Lemma 10 and Theorem 10, it follows that $(S(U))^*$ is symmetric if and only if

$$Z^*_{a,b}PY_{a,b} = 0, \quad for\ all\ y, z \in D((S(U))^*), \tag{6.3.3}$$

where $Y_{a,b}$, $Z_{a,b} \in \mathcal{R}(P^{-1}U^*)$ are defined as in (3.6). It follows from the Singular Patching Theorem 8 and Theorem 10 that (6.3.3) is equivalent to

$$Z^*PY = 0, \quad for\ all\ Y, Z \in \mathcal{R}(P^{-1}U^*). \tag{6.3.4}$$

Since P is invertible, (6.3.4) is equivalent to $\mathcal{R}(P^{-1}U^*) \perp \mathcal{R}(U^*)$. By Lemma 8, this is equivalent to $\mathcal{R}(P^{-1}U^*) \perp (\mathcal{N}(U))^{\perp}$. Therefore (1) and (2) are equivalent. The equivalence of (2) and (3) can be obtained immediately. □

LEMMA 14. *Let $U \in M_{l,2d}$ and assume that* rank $U = l$ *and* $d \le l \le 2d = m_a + m_b$. *Then the following statements are equivalent:*

(1) $S(U)$ *is a symmetric extension of the minimal operator* $S_{\min}$;

(2) $\mathcal{N}(U) \subset \mathcal{R}(P^{-1}U^*)$;

(3) *There exists a* $d \times 2d$ *matrix* $\widetilde{U}$ *satisfying* rank $\widetilde{U} = d$, $\mathcal{N}(U) \subset \mathcal{N}(\widetilde{U})$ *and* $\widetilde{U}P^{-1}\widetilde{U}^* = 0$;

(4) *There exists a* $d \times l$ *matrix* $\widetilde{V}$ *satisfying* rank $\widetilde{V} = d$ *and* $\widetilde{V}UP^{-1}U^*\widetilde{V}^* = 0$;

(5) $rank(UP^{-1}U^*) = 2l - (m_a + m_b) = 2(l-d)$;

(6) $rank(UP^{-1}U^*) \le 2l - (m_a + m_b) = 2(l-d)$;

(7) $\mathcal{N}(U) = P^{-1}U^*(\mathcal{N}(UP^{-1}U^*))$.

PROOF. The equivalence of (1) and (2) is given in Theorem 9.

(1) $\Rightarrow$ (3): Note that every symmetric extension of $S_{\min}$ is a restriction of a self-adjoint extension of $S_{\min}$. By (1), $S(U)$ is a symmetric extension of $S_{\min}$, and by Lemma 12, $S(\widetilde{U})$ is self-adjoint. Therefore (3) holds.

(3) $\Rightarrow$ (2): By Lemma 12 and condition (3), we obtain that $\mathcal{N}(U) \subset \mathcal{N}(\widetilde{U}) = \mathcal{R}(P^{-1}\widetilde{U}^*)$. It follows from $\mathcal{N}(U) \subset \mathcal{N}(\widetilde{U})$ that

$$\mathcal{R}(\widetilde{U}^*) = \mathcal{N}(\widetilde{U})^\perp \subset \mathcal{N}(U)^\perp = \mathcal{R}(U^*).$$

Thus $\mathcal{R}(P^{-1}\widetilde{U}^*) \subset \mathcal{R}(P^{-1}U^*)$, and then it follows that $\mathcal{N}(U) \subset \mathcal{R}(P^{-1}U^*)$. This shows that (2) holds.

(3) $\Rightarrow$ (4): Since $\mathcal{N}(U) \subset \mathcal{N}(\widetilde{U})$, we have $\mathcal{R}(U^*) \supset \mathcal{R}(\widetilde{U}^*)$. Therefore there exists a $d \times l$ matrix $\widetilde{V}$ such that $\widetilde{U}^* = U^*\widetilde{V}^*$, i.e. $\widetilde{U} = \widetilde{V}U$. From $\widetilde{U}P^{-1}\widetilde{U}^* = 0$, it follows that $\widetilde{V}UP^{-1}U^*\widetilde{V}^* = \widetilde{U}P^{-1}\widetilde{U}^* = 0$. Therefore $\operatorname{rank}\widetilde{V} = \operatorname{rank}(\widetilde{V}U) = \operatorname{rank}\widetilde{U} = d$.

(4) $\Rightarrow$ (3): Set $\widetilde{U} = \widetilde{V}U$. Then $\widetilde{U}P^{-1}\widetilde{U}^* = \widetilde{V}UP^{-1}U^*\widetilde{V}^* = 0$. It follows from $\operatorname{rank} U = l$ that $\operatorname{rank}\widetilde{U} = \operatorname{rank}(\widetilde{V}U) = \operatorname{rank}\widetilde{V} = d$. For any $Y \in \mathcal{N}(U)$, $\widetilde{U}Y = \widetilde{V}UY = 0$ which shows that $\mathcal{N}(U) \subset \mathcal{N}(\widetilde{U})$.

The equivalence of (2), (5), (6) and (7) can be obtained from the Linear Algebra Lemma 9. □

Based on the above lemmas and theorems we now obtain our main result: the characterization of symmetric operators $S(U)$ in the Hilbert space $L^2(J, w)$ determined by two-point boundary conditions. Although this result was stated above as Theorem 7 we restate it here for the benefit of the reader.

THEOREM 11. *Let $M = M_Q$, $Q \in Z_n(J)$, $J = (a, b)$, $-\infty \le a < b \le \infty$, be C-Symmetric. Let d_a, d_b and d be the deficiency indices of $My = \lambda wy$ on (a, c), (c, b) and (a, b), respectively. Recall that $d = d_a + d_b - n$. Let $m_a = 2d_a - n$, $m_b = 2d_b - n$, let $Y_{a,b}$ be defined by (3.6). Assume $U \in M_{l,2d}$ has rank l, $0 \le l \le m_a + m_b = 2d$ and let $U = (A : B)$ with $A \in M_{l,m_a}$ consisting of the first m_a columns of U in the same order as they are in U and $B \in M_{l,m_b}$ consisting of the next m_b columns of U in the same order as they are in U. Define the operator $S(U)$ in $L^2(J, w)$ by (3.8) and let*

$$G = K(A, B) = UP^{-1}U^* = AF_{m_a}A^* - BF_{m_b}B^*, \text{ and } r = rank\, G.$$

Then we have:

(1) *If $l < d$, then $S(U)$ is not symmetric.*
(2) *If $l = d$, then $S(U)$ is self-adjoint (and hence also symmetric) if and only if $r = 0$.*
(3) *Let $l = d + s$, $0 < s \le d$. Then $S(U)$ is symmetric if and only if $r = 2s$.*

PROOF. Part (1) follows from the von Neumann formula.

Part (2) is given by Lemma 12.

Part (3): $d < l \le 2d$. From Lemma 14 it follows that $S(U)$ is symmetric if and only if $rank\, C = rank\, UP^{-1}U^* = 2(l - d) = 2s$. □

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