Paul Larson, Jindrich Zapletal: Geometric set theory

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The following theorem provides a desirable strengthening of the conclusion of Theorem 4.2.9: the intersection models of generic coherent sequences satisfy DC, the axiom of dependent choices. The improvement does not have any impact on other results or proofs in the book. Recall that a sequence $\langle M_n \colon n \in \omega\rangle$ of transitive models of ZFC is *coherent* if for every ordinal α and every number n, the sequence $\langle M_m \cap V_\alpha \colon m \geq n\rangle$ belongs to M_n.

Theorem 0.1. *Suppose that M is a transitive model of ZFC and $\langle M_n \colon n \in \omega\rangle$ is a coherent sequence of models of ZFC such that all models on it are generic extensions of M. Then $M_\omega = \bigcap_n M_n$ is a model of DC.*

Proof. Let $\langle M_n \colon n \in \omega\rangle$ be a coherent sequence of models of ZFC, and each of them is a generic extension of some common model M of ZFC. Let $\langle P, \leq\rangle$ be a partial ordering in M_ω such that for every $p_0 \in P$ there is $p_1 \in P$ such that $p_1 < p_0$; we need to produce a strictly decreasing sequence $\langle p_n \colon n \in \omega\rangle$ in M_ω. To this end, choose a cardinal κ such that M_0 is a generic extension of M by a poset in V_κ, and $P \in V_\kappa$ holds. Let $\prec$ be a well-ordering of the set $V_\kappa \cap M$ in the model M. By recursion on $n \in \omega$ construct objects p_n, Q_n, G_n in the following way. $p_0 \in P$ is arbitrary, $Q_0 \in V_\kappa \cap M$ is an arbitrary poset such that M_0 is a generic extension of M via Q_0, and $G_0 \subset Q_0$ is an arbitrary filter generic over M such that $M_0 = M[G_0]$. Now, suppose that p_n, Q_n, G_n have been found in such a way that $p_n \in P$, $Q_n \in V_\kappa \cap M$, and $G_n \subset Q_n$ is a filter generic over M such that $M_n = M[G_n]$. Observe that M_{n+1} is an intermediate model of ZFC between M and M_n, and therefore it is a forcing extension of M by a poset in $V_\kappa \cap M$. Let $Q_{n+1} \in V_\kappa \cap M$ be the $\prec$-least poset such that there is a filter $H \subset Q_{n+1}$ generic over M_{n+1} such that $M_{n+1} = M[H]$, or equivalently $V_\kappa \cap M_{n+1} = V_\kappa \cap M[H]$. Let τ_{n+1} be the $\prec$-least Q_n-name in $V_\kappa \cap M$ such that τ_{n+1}/G_n is such a filter and let $G_{n+1} = \tau_{n+1}/G_n$. Finally, let $\sigma_{n+1} \in V_\kappa \cap M$ be the $\prec$-least Q_{n+1}-name such that $\sigma_{n+1}/G_{n+1} \in P$ is an element strictly below p_n, and let $p_{n+1} = \sigma_{n+1}/G_{n+1}$.

It is not difficult to see that for each number $n \in \omega$, the infinite sequence $\langle p_m, Q_m, G_m \colon m \geq n\rangle$ belongs to M_n, since the recursive process used to define it can be recovered from p_n, Q_n, G_n, the sequence $\langle M_m \cap V_\kappa \colon m \geq n\rangle$, and the well-ordering $\prec$. It follows that the sequence $\langle p_m \colon m \in \omega\rangle$ belongs to every model M_n, and therefore it belongs to the intersection model M_ω. Thus, the sequence $\langle p_m \colon m \in \omega\rangle$ is the desired strictly decreasing sequence in P in the intersection model. □