Paul Larson, Jindrich Zapletal: Geometric set theory
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Theorem 9.1.1 can be proved as a corollary of a simpler and stronger result.

Theorem 0.1. *In cofinally balanced extensions of the choiceless Solovay model W, the class of sets in W is closed under arbitrary increasing unions.*

This is to say that in such extensions, if $A \subset W$ is a set linearly ordered by inclusion, then $\bigcup A \in W$ holds.

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is cofinally balanced below κ. Let W be the choiceless Solovay model derived from κ. Work in W. Let τ be a P-name for a collection of sets in W which is linearly ordered by inclusion. Towards a contradiction, suppose that $p \in P$ is a condition which forces $\bigcup \tau \notin W$. The name τ is definable from some real parameter $z \in 2^\omega$ and some additional ground model parameters. Let $V[K]$ be an intermediate extension which contains z and the condition p, and such that the poset P is balanced in $V[K]$. Work in $V[K]$.

Let $\langle Q, \sigma \rangle$ be a balanced pair in P such that Q is a poset of cardinality less than κ and $Q \Vdash \sigma \leq \check{p}$. Since $Q \times \mathrm{Coll}(\omega, < \kappa)$ forces that the condition σ does not decide the membership of all sets in the choiceless Solovay model in the set $\bigcup \tau$, there must be a poset R of cardinality less than κ and a $Q \times R$-names

- η^-, η^+ for conditions in P stronger than σ;
- ν, μ for elements of 2^ω and formulas ϕ, ψ with parameters in V and two free variables,

such that $Q \times R \times \mathrm{Coll}(\omega, < \kappa)$ forces the following. Writing x for the set in the choiceless Solovay model defined by $\phi(\nu)$, and y for the set defined by $\psi(\mu)$, then $x \in y$ and in the poset P, $\eta^- \Vdash \check{x} \notin \bigcup \tau$ and $\eta^+ \Vdash_P \check{y} \in \tau$.

Move back to the model W. Let $H_0, H_1 \subset Q \times R$ be filters mutually generic over the model $V[K]$. Write $p_0^- = \eta^-/H_0$, $p_0^+ = \eta^+/H_0$, x_0 for the set defined by $\phi(\nu/H_0)$ and y_0 for the set defined by $\psi(\mu/H_0)$, and similarly for subscript 1.

First, consider the conditions p_0^+ and p_1^+. The balance assumption on the pair $\langle Q, \sigma \rangle$ implies that they are compatible in P. Since they respectively force $\check{y}_0 \in \tau$ and $\check{y}_1 \in \tau$, it must be the case that the sets y_0, y_1 are comparable with respect to inclusion. For definiteness, assume that $y_0 \subseteq y_1$ holds.

Now, consider the conditions p_0^- and p_1^+. The former forces $x_0 \notin \bigcup \tau$. At the same time, the latter forces $x_0 \in \bigcup \tau$ since $x_0 \in y_0 \subseteq y_1$ and it forces $\check{y}_1 \in \tau$. The two conditions are compatible by the balance assumption on the pair $\langle Q, \sigma \rangle$ and they force opposite statements. This is a contradiction. ☐

Corollary 0.2. (Theorem 9.1.1) *In cofinally balanced extensions of the Solovay model W, every well-ordered sequence of elements of W belongs to W.*

Proof. This is proved by transfinite induction on the length of the sequence, applying the theorem at each step. □

Corollary 0.3. *Let E be a Borel equivalence relation on a Polish space X. In cofinally balanced extensions of the Solovay model, there is no linear ordering on an uncountable subset of the quotient space X/E with all proper initial segments countable.*

Proof. Let κ be an inaccessible cardinal. Let W be the choiceless Solovay model derived from κ. We first show that there is no such an ordering in W. Work in W. Suppose towards a contradiction that $\leq$ is such an ordering on an uncountable set $A \subset X/E$. Both A and $\leq$ must be defined from parameters in the ground model and an additional parameter $z \in 2^\omega$. Let $V[K] \subset W$ be an intermediate forcing extension containing z. Work in $V[K]$.

Since the equivalence relation E is Borel, it has only fewer than κ many virtual classes by Theorem 2.5.6. Thus, there must be a partial ordering Q of cardinality smaller than κ and a Q-name τ for an element of X^ω such that $Q \times \mathrm{Coll}(\omega, < \kappa)$ forces τ to enumerate representatives of all equivalence classes in some initial segment of $\leq$, containing also some class which is not a realization of a virtual E-class in $V[K]$.

Moving back to W, let $H_0, H_1 \subset Q$ be filters mutually generic over $V[K]$. The points $\tau/H_0, \tau/H_1 \in X^\omega$ enumerate representatives of all equivalence classes in some initial segments of $\leq$. For definiteness, assume that the former segment is a subset of the latter. Now, the range of τ/H_0 contains a point which realizes no virtual E-equivalence class over $V[K]$. This point has no E-equivalent in $V[K][H_1]$ by Proposition 2.1.7. This contradicts the assumption that τ/H_1 enumerates representatives of a longer initial segment than τ/H_0.

Finally, to prove the corollary, if $\leq$ is such a ordering in a cofinally balanced extension of W, use the theorem on the initial segments of $\leq$ to show that $\leq \in W$ must hold. However, such an option has just been disproved. □