Paul Larson, Jindrich Zapletal: Geometric set theory
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The results of Section 14.2 can be handled in much easier way using the following powerful general theorem.

Theorem 0.1. *In cofinally balanced extensions of the choiceless Solovay model W, every σ-ideal on a Polish space σ-generated by closed sets is closed under increasing unions.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is cofinally balanced below κ. Let W be the choiceless Solovay model derived from κ. Work in W. Let X be a Polish space, let ι be a name for a σ-ideal on X σ-generated by closed sets, let τ be a P-name for a collection of sets in W, each a subset of X in the σ-ideal ι, which is linearly ordered by inclusion. Towards a contradiction, assume that $p \in P$ is a condition which forces $\bigcup \tau \notin \iota$. The names τ and ι are definable from some real parameter $z \in 2^\omega$ and some additional ground model parameters. Let $V[K]$ be an intermediate extension which contains z and the condition p, and such that the poset P is balanced in $V[K]$. Work in $V[K]$.

Let $\langle Q, \sigma \rangle$ be a balanced pair in P such that Q is a poset of cardinality less than κ and $Q \Vdash \sigma \leq \check{p}$. Note that for each closed set $C \subset X$ coded in the model $V[K]$, $Q \times \mathrm{Coll}(\omega, < \kappa)$ forces σ to decide the membership of C in the σ-ideal ι by Proposition 5.2.4. Let I be the collection of those closed sets for which the decision is affirmative. Note also that $Q \times \mathrm{Coll}(\omega, < \kappa)$ forces that $\bigcup \tau$ is not covered by the closed sets in I, since there are only countably many such sets and ι is a σ-ideal. Thus, there must be a poset R of cardinality less than κ and $Q \times R$-names

- η for a condition in P stronger than σ;
- ν for an element of X which does not belong to any closed set in the ideal I;
- μ for an element of 2^ω, together with a formula ϕ with ground model parameters and two free variables;
- θ for a function with domain ω and range consisting of closed sets,

such that $Q \times R \times \mathrm{Coll}(\omega, < \kappa)$ forces the following. Writing χ for the object in the choiceless Solovay model defined by $\phi(\mu)$, then χ is a P-name and in the poset P, η forces $\mathrm{rng}(\theta) \subset \iota$ and $\chi \subseteq \bigcup \mathrm{rng}(\theta)$, and $\nu \in \chi \in \tau$.

Move back to the model W. Let $H_0, H_1 \subset Q \times R$ be filters mutually generic over the model $V[K]$. Write $p_0 = \eta/H_0$, $x_0 = \nu/H_0$, χ_0 for the P-name defined by $\phi(\mu/H_0)$, $z_0 = \theta/H_0$, and similarly for subscript 1. The conditions p_0, p_1 are compatible by the balance assumption on the pair $\langle Q, \sigma \rangle$ and their common lower bound forces $\chi_0, \chi_1 \in \tau$. We will show that it also forces these two sets not to be linearly ordered by inclusion, as x_0 belongs only to the former and x_1 only to the latter. This will be a contradiction to the initial assumption on the name τ.

Now, $p_0 \Vdash \check{x}_0 \in \chi_0$ by the initial choices. To prove that p_1 forces $\check{x}_0 \notin \chi_1$, it will be enough to show $x_0 \notin z_1(n)$ for any $n \in \omega$. To this end, move back to the model $V[K]$ and consider the product forcing $(Q \times R) \times (Q \times R)$. Suppose that $q_0, q_1 \in Q$ and $r_0, r_1 \in R$ are conditions. It will be enough to find a basic open set $O \subset X$ and conditions $q_0' \leq q_0, q_1' \leq q_1$, $r_0' \leq r_0$ and $r_1' \leq r_1$ such that $\langle q_0', r_0' \rangle \Vdash \nu \in O$ and $\langle q_1', r_1' \rangle \Vdash O \cap \theta(\check{n}) = 0$. To find these objects, let B be the set of all basic open subsets $O \subset X$ such that some condition below $\langle q_1, r_1 \rangle$ forces $O \cap \theta(\check{n}) = 0$. Since the condition p_1 forces the set $\theta(\check{n})$ to be closed and in the ideal ι, it must be the case that the set $X \setminus \bigcup B$, which is in $V[K]$ and is forced to be a subset of $\theta(\check{n})$, belongs to I. By the choice of the name ν, there must be a condition $q_0' \leq q_0$ and $r_0' \leq r_0$ and an open set $O \in B$ such that $\langle q_0', r_0' \rangle \Vdash \nu \in O$. By the definition of the set B, there must be conditions $q_1' \leq q_1$ and $r_1' \leq r_1$ such that $\langle q_1', r_1' \rangle \Vdash \dot{O} \cap \theta(\check{n}) = 0$. This concludes the argument.

The proof of $p_1 \Vdash \check{x}_1 \in \chi_1$ and $p_0 \Vdash \check{x}_1 \notin \chi_1$ is symmetric. The theorem has been proved. □

Corollary 0.2. *In cofinally balanced extensions of the choiceless Solovay model W, every σ-ideal I on a Polish space, σ-generated by closed sets, is closed under well-ordered unions.*

Proof. This is to say that the union of any transfinite sequence of sets in the ideal I belongs to I. This is proved by transfinite induction, applying the conclusion of the theorem to the unions of initial segments of the sequence. □

Corollary 0.3. (Theorem 14.2.1) *Suppose that $\leq$ is an F_σ-preorder on a Polish space X such that every countable set has a common upper bound. Then in cofinally balanced extensions of the choiceless Solovay model, $\leq$ contains no unbounded linearly ordered subsets.*

Proof. Consider the σ-ideal I on X σ-generated by all bounded subsets of X. The complexity of the preorder X shows that I is σ-generated by an analytic collection of closed sets. It follows that the σ-ideal is closed under arbitrary increasing unions in all cofinally balanced extensions of the choiceless Solovay model. In particular, $\leq$ cannot contain any unbounded linearly ordered subsets. □

The conclusion of the theorem cannot be easily generalized to other σ-ideals. Consider the following simple delimitative result.

Proposition 0.4. (ZF) *If there is a nonprincipal ultrafilter on ω then the non-dominating ideal on ω^ω is not closed under increasing unions.*

Here, the non-dominating ideal on ω^ω is generated by sets $B_y = \{x \in \omega^\omega : \exists^\infty n\ y(n) \geq x(n)\}$ as y ranges over all elements of ω^ω. It is easily seen to be a σ-ideal, and it is not σ-generated by closed sets.

Proof. Let U be a non-principal ultrafilter on ω. Consider the modulo U equality $=_U$ and the modulo U linear ordering $\leq_U$ on ω^ω. For each $y \in \omega^\omega$, let $C_y = \{x \in \omega^\omega : x \leq_U y\}$. This is clearly a set in the non-dominating ideal, as it is a subset of B_y. It is clear that the set C_y depends only on the $=_U$-class of y, and $y_0 \leq_U y_1$ implies $C_{y_0} \subseteq C_{y_1}$. Thus, the whole space ω^ω is exhausted as the increasing union of sets C_y for $y \in \omega^\omega$, all of which belong to the non-dominating ideal. □