Corollary 9.1.3 has other remarkable consequences beyond Corollary 9.1.5. For one of them, let Γ be the hypergraph on $2^\omega \times 2^\omega$ of arity three consisting of those sets of cardinality three whose projections into both coordinate axes have cardinality two. In ZFC+CH, the chromatic number of Γ is countable, as follows from the work of Paul Erdős and Péter Komjáth: Countable decompositions of $\mathbb{R}^2$ and $\mathbb{R}^3$, Discrete and Computational Geometry, 5:325–331, 1990. We get additional information in the choiceless context.

Proposition 0.1. *(ZF) If the chromatic number of* Γ *is countable then there is a countable-to-one map from* 2^ω *to* ω_1*.*

Proof. Let $c\colon 2^\omega \times 2^\omega \to \omega$ be a Γ-coloring. For $x \in 2^\omega$ write M_x for the model of all sets hereditarily ordinally definable from x and c. Note that $c \restriction M_x \in M_x$ holds.

Case 1. There is a real x such that $2^\omega \cap M_x$ is uncountable. In this case, we show that $2^\omega \subset M_x$ and $M_x \models$CH, which will prove the proposition.

To show that $2^\omega \subset M_x$ holds, suppose towards contradiction that it does not, and pick $z \in 2^\omega \setminus M_x$. By a counting argument, there are distinct points $y_0, y_1 \in 2^\omega \cap M_x$ such that $c(y_0, z) = c(y_1, z)$, the common value being some $n \in \omega$. Then z is not the unique point such that $c(y_1, z) = n$–otherwise it would be definable from c and y_1, and therefore from c and x, contradicting the choice of z. Let $u \in 2^\omega$ be a point different from y_1 such that $c(y_1, u) = n$. Then $\{\langle y_0, z\rangle, \langle y_1, z\rangle, \langle y_1, u\rangle\}$ is a monochromatic Γ-hyperedge of color n, a contradiction.

To show that $M_x \models$CH, suppose towards a contradiction that it fails. Work in M_x; observe that it is a model of AC. Let N_0 be an elementary submodel of some large structure containing $c \restriction M_x$ such that N_0 has cardinality $\aleph_1$; let $x_0 \in X \setminus N_0$. Let N_1 be an elementary submodel of a large structure containing $c \restriction M_x$, x_0, and N_0, such that N_1 is countable. Let $x_1 \in 2^\omega \cap N_1 \setminus N_0$. Let $n = c(x_0, x_1)$. By the elementarity of N_0, there must be $u \in N_0$ such that $c(u, x_1) = n$. By the elementarity of N_1, there must be $v \in N_0 \cap N_1$ such that $c(x_0, v)$. Note that $u \neq x_0$ and $v \neq x_1$ holds. Clearly, $\{\langle x_0, x_1\rangle, \langle u, x_1\rangle, \langle x_0, v\rangle\}$ is a monochromatic Γ-hyperedge of color n, a contradiction.

Case 2. Case 1 fails. Let $\pi\colon 2^\omega \to \omega_1$ be the map defined by $\pi(x) = \omega_1^{M_x}$. The case assumption shows that the range of this map is indeed a subset of ω_1. We will show that π is in fact countable-to-one, proving the proposition in this case as well. Suppose towards contradiction that it is not, and let $\alpha \in \omega_1$ be an ordinal such that the set $\{x \in 2^\omega \colon \pi(x) = \alpha\}$ is uncountable. By the case assumption, there have to be points x_0, x_1 in this set such that $x_1 \notin M_{x_0}$. We will reach a contradiction by a split into cases.

Suppose first that $x_0 \notin M_{x_1}$. Set $L_0 = \{\langle x_0, y\rangle \colon y \in 2^\omega\}$ and $L_1 = \{\langle y, x_1\rangle \colon y \in 2^\omega\}$. Let $n = c(x_0, x_1)$. Then $\langle x_0, x_1\rangle$ is not the only point on L_0 which gets color n–otherwise x_1 would be definable from x_0. Let $\langle x_0, x_2\rangle \in L_0$

be a different point which gets color n. By the same argument, $\langle x_0, x_1\rangle$ is not the only point on L_1 which gets color n–otherwise x_0 would be definable from x_1. Let $\langle x_3, x_1\rangle \in L_1$ be a different point which gets color n. Then $\{\langle x_0, x_1\rangle, \langle x_0, x_2\rangle, \langle x_3, x_1\rangle\}$ is a monochromatic Γ-hyperedge of color n. A contradiction.

Assume now that $x_0 \in M_{x_1}$. The set $2^\omega \cap M_{x_0}$ then belongs to M_{x_1} and must be uncountable there because the two models have the same ω_1. By a counting argument in M_{x_1}, there must be distinct points $y_0, y_1 \in 2^\omega \cap M_{x_0}$ such that $\langle y_0, x_1\rangle$ and $\langle y_1, x_1\rangle$ get the same c-color, say n. Now, x_1 cannot be the only point such that $\langle y_1, x_1\rangle$ gets the color n–otherwise x_1 would be definable from y_1 and then also from x_0. So, pick a point $z \in 2^\omega$ such that $c(y_1, z) = n$ and note that the set $\{\langle y_0, x_1\rangle, \langle y_1, x_1\rangle, \langle y_1, z\rangle\}$ is a c-monochromatic Γ-hyperedge of color n. This is a final contradiction. $\square$

Corollary 0.2. *In cofinally balanced extensions of the Solovay model, the chromatic number of* Γ *is uncountable.*

Proof. Countable chromatic number of Γ yields a countable-to-one map $h\colon 2^\omega \to \omega_1$. The image of h must be uncountable–otherwise 2^ω would be a countable union of countable sets, an impossibility by Theorem 9.1.1 and DC in the Solovay model. Now, consider the map k from ω_1 such that $k(\alpha)$ is the set of those elements of 2^ω which get mapped to the α-th element of the range of h. This is an injection of ω_1 into $=^+$-classes, contradicting Corollary 9.1.3. $\square$