Supplementary Notes to:
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These notes deal with two theorems in the book. The first is Theorem 4.11, Ch. I (originally from [1]) which is the inversion formula for the antipodal Radon transform. We adopt Rouvière's method of proof [2] of the noncompact analog of Theorem 4.11, resulting in a much simpler proof.

The second is the theorem stated without proof in Ch. IV, Exercises and Further Results, C6, p 489. The theorem describes the precise image of the Schwartz space $\mathcal{I}^2(G)$ under the spherical transform. The proof is a modification of the proof by Anker [1991], a proof which included also the generalizations to $\mathcal{I}^p(G)$ $(0 < p \leq 2)$ described on p. 489. Anker's proof was much simpler than the preceding ones and was accomplished by a skillful use and extension of the Paley–Wiener theorem (Ch. IV, Theorem 7.1) for the spherical transform. For the case $p = 2$ we shall here simplify the proof a bit further.

We shall use notation from the text (mainly Ch. IV) without repetition of definition.

1 Spherical Functions

Here we prove some estimates from Harish–Chandra [1958a] of the spherical function and its derivatives.

Theorem 1.1. *Let φ_λ denote the spherical function*

(1.1) $$\varphi_\lambda(g) = \int_K e^{(i\lambda-\rho)(H(gk))}\,dk \qquad \lambda = \mathrm{Re}\,\lambda + i\,\mathrm{Im}\,\lambda\,.$$

Then we have the following estimates:

(i) $e^{-\rho(H)} \leq \varphi_0(\exp H) \leq c(|H|+1)^d e^{-\rho(H)}\,, \quad H \in \mathfrak{a}^+,$
where c is a constant, $d = \mathrm{Card}\,(\Sigma_0^+)$.

(ii) $0 \leq \varphi_{-i\lambda}(H) \leq e^{\lambda(H)}\varphi_0(\exp H), \quad H \in \mathfrak{a}^+\,,\ \lambda \in \mathfrak{a}^*_+.$

(iii) *Given $D \in \mathbf{D}(G)$ there is a constant $c > 0$ such that*

$$|(D\varphi_\lambda)(g)| \leq c(|\lambda|+1)^{\deg D}\varphi_{i\,\mathrm{Im}\,\lambda}(g)\,.$$

(iv) *Given a polynomial $P \in S(\mathfrak{a}^*)$ there is a constant $c > 0$ such that*

$$\left|P\left(\frac{\partial}{\partial\lambda}\right)\varphi_\lambda(g)\right| \leq c(|g|+1)^{\deg P}\varphi_{i\,\mathrm{Im}\,\lambda}(g)\,,$$

where $|g| = |H|$ if $g = k_1 \exp H k_2, \quad H \in \mathfrak{a}.$

Proof:

(i) This part is Exercise IV, B1, and the proof is on p. 580. See also Harish–Chandra [1958a], p. 279 for the original proof.

(ii) We have

$$\varphi_{-i\lambda}(a) = \int_K e^{(\lambda-\rho)(H(ak))} ak \leq e^{\lambda(\log a)}\varphi_0(a) \quad a \in A^+ \tag{1.2}$$

by IV, Lemma 6.5.

(iii) According to Ch. IV, Lemma 4.4,

$$\varphi_\lambda(gh) = \int_K e^{(-i\lambda+\rho)(A(kg^{-1}))} e^{(i\lambda+\rho)(A(kh))}\, dk\,. \tag{1.3}$$

Let $X \in \mathfrak{g}$ and $\widetilde{X}$ the corresponding left-invariant vector field. Put $h = \exp tX$ in (1.3) and take $(d/dt)_0$. We have $A(k \exp tX) = \exp tAd(k)X$ so

$$(\widetilde{X}\varphi_\lambda)(g) = \int_K e^{(-i\lambda+\rho)(A(kg^{-1}))}((Ad(k)\widetilde{X})\eta_\lambda)(e)\, dk\,,$$

where $\eta_\lambda(h) = e^{(i\lambda+\rho)(A(h))}$. More generally, if $D \in \mathbf{D}(G)$,

$$(D\varphi_\lambda)(g) = \int_K e^{(-i\lambda+\rho)(A(kg^{-1}))}((Ad(k)D)\eta_\lambda)(e)\, dk\,. \tag{1.4}$$

Since η_k is left N-invariant and right K-invariant we have by Ch. II, Lemma 5.14,

$$(Ad(k)D\eta_\lambda)(e) = ((Ad(k)D)_{\mathfrak{a}}\bar{\eta}_\lambda)(e)\,, \tag{1.5}$$

the bar denoting restriction to A.

If $\ell = \deg D$ we fix a basis $D_1, \ldots, D_m$ of $\mathbf{D}_\ell(A)$, the space of elements in $\mathbf{D}(A)$ of degree $\leq \ell$. Then

$$(Ad(k)D)_{\mathfrak{a}} = \sum_1^m \eta_i(k) D_i\,, \quad \eta_i \in \mathcal{E}(K)\,.$$

Thus expression (1.5) reduces to

$$\sum_{i=1}^m \eta_i(k) D_i(i\lambda + \rho)$$

so the right hand side of (1.4) is majorized by

$$c\int_K e^{(\text{Im }\lambda+\rho)(A(kg^{-1}))}\, dk \quad (|\lambda|+1)^\ell \qquad (c = \text{const.})\,.$$

Since (by (1.3)) $\varphi_\mu(g^{-1}) = \varphi_{-\mu}(g)$, this proves (iii). Harish–Chandra's original proof is in his paper [1958a], p. 294.

For (iv) we observe from (1.1) that

$$P\left(\frac{\partial}{\partial\lambda}\right)\varphi_\lambda(g) = \int_K e^{(i\lambda-\rho)(H(gk))} P(iH(gk))\, dk$$

and now the result follows from IV §10, (14).

2 The Schwartz Spaces

The Schwartz space $\mathcal{I}^2(G)$ consists of the K-bi-invariant functions $f \in \mathcal{E}(G)$ for which the seminorm

$$\sigma_{D,q}(f) = \sup_g (|g|+1)^q \phi_0(g)^{-1} |Df(g)| \tag{2.1}$$

is finite for each $q \in \mathbf{Z}^+$, $D \in \mathbf{D}(G)$. With these seminorms, $\mathcal{I}^2(G)$ is a Fréchet space.

We consider the following transforms: The *spherical transform* $\mathcal{F} : f \to \widetilde{f}$ given by

$$\widetilde{f}(\lambda) = \int_G f(g)\varphi_{-\lambda}(g)\, dg \qquad f \quad K-\text{bi-invariant}, \tag{2.2}$$

the *Euclidean Fourier transform* $\mathcal{F}_0 : \varphi \to \varphi^*$ given by

$$\varphi^*(\lambda) = \int_A \varphi(a) e^{-i\lambda(\log a)}\, da\,, \tag{2.3}$$

and the *Abel transform* $f \to \mathcal{A}f$ given by

$$(\mathcal{A}f)(a) = e^{\rho(\log a)} \int_N f(an)\, dn\,, \quad f \quad K-\text{bi-invariant}. \tag{2.4}$$

We then have the commutative diagram

$$\begin{array}{ccc} & \mathcal{H}_W(\mathfrak{a}_c^*) & \\ {\scriptstyle F}\nearrow & & \searrow{\scriptstyle \mathcal{F}_0} \\ \mathcal{D}^\natural(G) & \xrightarrow[\mathcal{A}]{} & \mathcal{D}_W(A) \end{array}$$

from Ch. IV, Theorem 7.1 and Cor. 7.4.

The Schwartz space $\mathcal{S}(\mathfrak{a}^*)$ is topologized by the seminorms

$$\tau_{P,m}(h) = \sup_{\mathfrak{a}^*} (|\lambda|+1)^m \left| P\left(\frac{\partial}{\partial\lambda}\right) h(\lambda) \right| \tag{2.5}$$

$m \in \mathbf{Z}^+$, $P \in S(\mathfrak{a}^*)$. Since the Laplacian L on G/K has the property

$$L\varphi_\lambda = -(\langle\lambda,\lambda\rangle + |\rho|^2)\varphi_\lambda$$

it is sometimes convenient to use the seminorms

$$\overset{o}{\tau}_{P,m}(h) = \sup_{\mathfrak{a}^*} \left| P\left(\frac{\partial}{\partial\lambda}\right) (\langle\lambda,\lambda\rangle + |\rho|^2) h(\lambda) \right|, \tag{2.6}$$

which define the same topology on $\mathcal{S}(\mathfrak{a}^*)$.

We shall have use for the following simple result. If $f \in \mathcal{S}(\mathbf{R}^n)$ then

$$\left| \int_{\mathbf{R}^n} f(x)\, dx \right| \leq c_n \sup_x (|x|^{n+1} |f(x)|) \tag{2.7}$$

where c_n is a constant. In fact $|f(x)| \leq M(|x|+1)^{-n-1}$ so left hand side of (2.7) is bounded by constant multiple of M.

The aim is now to prove that the bijection $\mathcal{F} : \mathcal{D}^\natural(G) \to \mathcal{H}_W(\mathfrak{a}_c^*)$ given by the Paley–Wiener theorem is bicontinuous *for the topologies induced by* $\mathcal{I}^2(G)$ and $\mathcal{S}(\mathfrak{a}^*)$. Let $\mathcal{S}_W(\mathfrak{a}^*)$ denote the set of W-invariants in $\mathcal{S}(\mathfrak{a}^*)$.

Lemma 2.1. *The spherical transform* $f \to \widetilde{f}$ *given by (2) maps* $\mathcal{I}^2(G)$ *continuously into* $\mathcal{S}_W(\mathfrak{a}^*)$*.*

Proof:

To check the convergence of the integral

$$\widetilde{f}(\lambda) = \int_G f(g)\varphi_{-\lambda}(g)\,dg\,, \quad \lambda \in \mathfrak{a}^*\,, \tag{2.8}$$

we use Theorem 5.8 in Ch. I to reduce the integral to one over A^+. Here the density δ satisfies

$$\delta(\exp H) \leq ce^{2\rho(H)}\,, \quad H \in \mathfrak{a}^+\,, \tag{2.9}$$

for a constant c. Since $|\varphi_\lambda(g)| \leq \varphi_0(g)$ and, by (2.1), $|f(g)| \leq \text{const}\,(|g|+1)^{-q}\varphi_0(g)$ for each q the absolute convergence is clear from Theorem 1.1, (i). The smoothness in λ follows from Theorem 1.1 (iv). For the remaining statements we just have to prove that a seminorm $\tau = \overset{o}{\tau}_{P,m}$ on $\mathcal{S}_W(\mathfrak{a}^*)$, there exists a seminorm $\sigma = \sigma_{D,q}$ such that

$$\tau(\widetilde{f}) \leq c_1\sigma(f) \qquad f \in \mathcal{I}^2(G)\,, \tag{2.10}$$

where c_1 is a constant. We have

$$P\left(\frac{\partial}{\partial\lambda}\right)\left((\langle\lambda,\lambda\rangle)^m\widetilde{f}(\lambda)\right) = \int_G (-L)^m f(g) P\left(\frac{\partial}{\partial\lambda}\right)\varphi_{-\lambda}(g)\,dg\,.$$

Again we reduce the integral to A^+, use the estimates Theorem 1.1, (iv), (i), and combine with (2.9) and the estimate

$$|(Df)(a)| \leq (|a|+1)^{-q}\varphi_0(a) \leq c(|a|+1)^{-q+d}e^{-\rho(\log a)}\,.$$

Taking $D = (-L)^m$ and q large (2.10) follows.

We now come to Anker's principal lemma.

Lemma 2.2. *The inverse map* $\mathcal{F}^{-1} : \mathcal{H}(\mathfrak{a}_c^*) \to \mathcal{D}^\natural(G)$ *given by*

$$(\mathcal{F}^{-1}h)(g) = f(g) = \int_{\mathfrak{a}^*} h(\lambda)\varphi_\lambda(g)|c(\lambda)|^{-2}\,d\lambda$$

is continuous in the topologies induced from $\mathcal{S}_W(\mathfrak{a}^*)$ *and* $\mathcal{I}^2(G)$*.*

Proof: Given a seminorm $\sigma = \sigma_{D,q}$ on $\mathcal{I}^2(G)$ the problem is to find a seminorm $\tau = \tau_{P,m}$ on $\mathcal{S}_W(\mathfrak{a}^*)$ such that

$$\sigma(f) \leq \tau(h) \qquad \text{for } f \in \mathcal{D}^\natural(G)\,. \tag{2.11}$$

We have for $D \in \mathbf{D}(G)$

$$(Df)(g) = \int_{\mathfrak{a}^*} h(\lambda) D\varphi_\lambda(g)|c(\lambda)|^{-2}\,d\lambda \tag{2.12}$$

and wish to estimate

$$F(g) = (|g|+1)^q\varphi_0(g)(Df)(g) \tag{2.13}$$

because $\sigma_{D,q}(f) = \sup_g |F(g)$. From Theorem 1.1(iii) and Ch. IV, Prop. 7.2 we have for a suitable m_0

$$|(F(g))| \leq c_0(|g|+1)^q \int_{\mathfrak{a}^*} (|\lambda|+1)^{m_0}|h(\lambda)|\,d\lambda\,. \tag{2.14}$$

One would now like to remove the factor $(|g|+1)^q$ by replacing h with a suitable derivative. It seems hard to do this globally, that is on all of G. Following Anker we do this locally, that is by dividing G up into pieces on which this process works.

Consider the balls $B_j = \{H \in \mathfrak{a} : |H| \leq j\}$, $j \in \mathbf{Z}^+$, and put $G_j = K \exp B_j K$. Let $\omega \in \mathcal{C}^\infty(\mathbf{R})$ be an even function, $0 \leq \omega(x) \leq 1$, with the properties:

$$\omega(x) = 1 \text{ for } |x| \leq \frac{1}{2}\,,\ \omega \text{ has support in } (-1,1)\,.$$

We define $\omega_j \in \mathcal{D}_W(\mathfrak{a})$ for $j \geq 1$ by

$$\omega_j(H) = \begin{cases} 1 & \text{for } |H| \leq j-1 \\ \omega(|H|-j+1) & \text{for } |H| > j-1\,. \end{cases}$$

Then ω_j and each of its derivatives is bounded uniformly in j.

With $f \in \mathcal{D}^\natural(G)$ let $h = \mathcal{F}f$, $g(H) = \mathcal{A}f(\exp H)$. Thus $g \in \mathcal{D}_W(\mathfrak{a})$ and $h \in \mathcal{H}_W(\mathfrak{a}_c^*)$. We decompose $g = \omega_j g + (1-\omega_j)g$, put $g_j = (1-\omega_j)g$ and let f_j and h_j be the corresponding functions in $\mathcal{D}^\natural(G)$ and $\mathcal{H}(\mathfrak{a}_c^*)$, respectively. We know from Ch. IV, Theorem 7.1 and Cor. 7.4, that for each closed ball $B \subset \mathfrak{a}$ with center 0,

$$\text{supp}(f) \subset K \exp BK \Leftrightarrow \text{supp}(g) \subset B\,.$$

Since $g - g_j = 0$ outside B_j, $f - f_j = 0$ outside G_j. The constants below will depend on σ but neither on f nor on j. We now use (2.12) with f and h replaced by f_j and h_j, respectively. This does not change F outside G_j. Thus using (2.7) (and $j+2 \leq 3j$) (2.14) implies

$$\sup_{G_{j+1}-G_j} |F(g)| \leq c_1 j^q \tau_{1,m}(h_j)\,, \quad m = m_0 + \dim \mathfrak{a} + 1,.$$

We shall now prove, by Euclidean Fourier analysis, that given $q, m \in \mathbf{Z}^+$ there exists a seminorm

$$\tau^*_{d,t}(h) = \sum_{k=0}^{d} \sup_{\mathfrak{a}^*} (|\lambda|+1)^t |\nabla^k h(\lambda)| \quad (\nabla = \text{ gradient})$$

such that for all j,

$$j^q \tau_{1,m}(h_j) \leq c_2 \tau^*_{d,t}(h)\,.$$

This would prove (2.11). For this consider

$$h_j(\lambda) = \int_{\mathfrak{a}} g_j(H) e^{-i\lambda(H)}\, dH\,.$$

We now shift polynomial factors on h_j to derivatives of g_j. Using (2.7) and the fact that g_j vanishes on B_{j-1} we get with $p = q + \dim \mathfrak{a} + 1$,

$$\begin{aligned} j^q \tau_{1,m}(h_j) &\leq j^q c_3 \sum_{k=0}^{m} \int_{\mathfrak{a}} |\nabla^k g_j(H)|\, dH \\ &\leq c_4 \sum_{k=0}^{m} \sup_{\mathfrak{a}} (|H|+1)^p |\nabla^k g_j(H)| \\ &\leq c_5 \sum_{k=0}^{m} \sup_{\mathfrak{a}} (|H|+1)^p |\nabla^k g(H)|\,, \end{aligned}$$

the last inequality coming from calculating the derivatives of $g_j = (1-\omega_j)g$ by the product rule and recalling that each derivative of $1-\omega_j$ is uniformly bounded in j. On the other hand,

$$g(H) = c_6 \int_{\mathfrak{a}^*} h(\lambda) e^{i\lambda(H)}\, d\lambda\,,$$

so from the last inequality we derive

$$j^q \tau_{1,m}(h_j) \leq c_7 \sum_{\ell=0}^{p} \int_{\mathfrak{a}^*} (|\lambda|+1)^m |\nabla^\ell h(\lambda)|\, d\lambda$$

which again by (2.7) is dominated by a suitable $\tau^*_{p,m}(h)$. This proves the lemma.

It is well known that $\mathcal{D}(\mathbf{R}^n)$ is dense in $\mathcal{S}(\mathbf{R}^n)$. This implies for our situation that $\mathcal{H}_W(\mathfrak{a}^*_c)$ is dense in $\mathcal{S}_W(\mathfrak{a}^*)$.

Lemma 2.3. *$\mathcal{D}^\natural(G)$ is dense in $\mathcal{I}^2(G)$.*

Proof: We extend the function ω_j above to a K-bi-invariant smooth function ψ_j on G. Then $\psi_j \equiv 1$ on G_{j-1} and $\psi_j = 0$ outside G_j. Consider the seminorm $\sigma = \sigma_{D,q}$ in (2.1). Then

$$\begin{aligned} \sigma(\psi_j f - f) &\leq \sup_{|g|>j-1} (1+|g|)^q \varphi_0(g)^{-1} |D(\psi_j f - f)(g)| \\ &\leq \frac{1}{j}\big(\sigma_{D,q+1}(\psi_j f) + \sigma_{D,q+1}(f)\big)\,. \end{aligned}$$

Also

$$D(\psi_j f) = \sum_i D_i(\psi_j) E_i(f) \qquad D_i, E_i \in \mathcal{D}(G)$$

so our expression is majorized by

$$c\,\frac{1}{j} \sum_i \big(\sigma_{E_i,q+1}(f) + \sigma_{D,q+1}(f)\big)$$

where c is a constant. Here we used again the uniform boundedness of $D_i\psi_i$ in j. The last estimate shows $\sigma(\psi_j f - f) \to 0$ proving the lemma.

Theorem 2.4. *The spherical transform $\mathcal{F} : f \to \widetilde{f}$ given by*

$$\widetilde{f}(\lambda) = \int_G f(g) \varphi_{-\lambda}(g)\, dg$$

is a homeomorphism of $\mathcal{I}^2(G)$ onto $\mathcal{S}_W(\mathfrak{a}^*)$. The inverse $\mathcal{F}^{-1}$ is given by

$$(\mathcal{F}^{-1}h)(g) = \text{const} \int_{\mathfrak{a}^*} h(\lambda) \varphi_\lambda(g) |c(\lambda)|^{-2}\, d\lambda\,. \tag{2.15}$$

Proof: The spaces $\mathcal{I}^2(G)$ and $\mathcal{S}_W(\mathfrak{a}^*)$ are Fréchet spaces. Because of their completeness and the density in Lemma 2.3 the inverse $\mathcal{F}^{-1}$ extends to a linear homeomorphism of $\mathcal{S}_W(\mathfrak{a}^*)$ onto $\mathcal{I}^2(G)$. The inverse of this map must by Lemma 2.2 coincide with $\mathcal{F}$. Thus $\mathcal{F} : \mathcal{I}^2(G) \to \mathcal{S}_W(\mathfrak{a}^*)$ is surjective.

We must still prove that our "abstract" extension of $\mathcal{F}^{-1}$ to $\mathcal{S}_W(\mathfrak{a}^*)$ is given by (2.15). Let $h \in \mathcal{S}_W(\mathfrak{a}^*)$ and let $h_n \in \mathcal{H}_W(\mathfrak{a}^*_c)$ converge to h. Since $\varphi_\lambda(g)$ is bounded, and $|c(\lambda)|^{-2}$ bounded by a polynomial, we have $\tau_{1.m}(h_n - h) \to 0$. Thus the validity of (2.15) for h_n implies its validity for h.

Reference

Anker, J.-Ph. "The spherical transform of rapidly decreasing functions — a simple proof of a characterization due to Harish–Chandra, Helgason, Trombi and Varadarajan", J. Funct. Anal. 96 (1991), 331–349.

3 Proof of Theorem 4.11, Ch. I.

We start with formula (37):

$$(\hat{f})^\vee(x) = \Omega_X \int_0^L \frac{1}{A(r)} \left(\int_{S_r(x)} f(\omega)\, d\omega_r \right) A_1(r)\, dr\,. \tag{3.1}$$

We leave out the case $X = \mathbf{P}^n(\mathbf{R})$ already done in Theorem 4.7. Writing

$$\sigma(x) = \Omega_X A_1(r)/A(r) \qquad r = d(o,x) \tag{3.2}$$

we have

$$(\hat{f})^\vee = f \times \sigma \tag{3.3}$$

where $\times$ is the convolution on X as defined in Ch. II, §5. We shall find a polynomial $P(L)$ in the Laplacian satisfying $P(L)\sigma = \delta$ which by (14), Ch. II, §5 would yield

$$P(L)(\hat{f})^\vee = f\,.$$

Note the decisive simplification (compared to Lemmas 4.13–4.15) that σ only involves powers of $\sin(\lambda r)$ (not $\sin(2\lambda r)$).

Let $g_a(r) = \sin^a(\lambda r)$ and recall from Lemma 4.10 that

$$A(r) = \Omega_n \lambda^{-n+1} g_{n-1}(r) \cos^q(\lambda r)\,. \tag{3.4}$$

Writing the radial part L_r of L in the form

$$L_r f = A^{-1}(Af')' \tag{3.5}$$

and using the formulas

$$\begin{aligned} g_a(r) &= g_{a-2}(r) - g_{a-2}(r)\cos^2(\lambda r) \\ g_a'(r) &= \lambda\, a\, g_{a-1}(r) \cos(\lambda r) \end{aligned} \tag{3.6}$$

it is easy to derive

$$(L_r + \lambda^2 a(a+n+q-1))g_a = \lambda^2 a(a+n-2)\, g_{a-2}\,. \tag{3.7}$$

Lemma 2.5 *Define G_a on X by*

$$G_a(x) = g_a(d(o,x))\,.$$

If $a+n \geq 2$ G_a is a locally integrable function on X which as a distribution satisfies

$$(L + \lambda^2 a(a+n+q-1))G_a = \begin{cases} \lambda^2 a(a+n-2) G_{a-2} & \text{if } a+n > 2 \\ \lambda^{-n+2}\Omega_n\, a\, \delta & \text{if } a+n = 2 \end{cases}.$$

Proof:

The distribution $f \to G_a(f)$ is K-invariant so we may take f radial, $f(x) = F(d(o,x))$. Then

$$(LG_a)(f) = G_a(L_f) = \int_0^L g_a(r)(L_rF)(r)A(r)\,dr$$
$$= \int_0^L g_a(r)(AF')'(r)\,dr = \left[A(r)F'(r)g_a(r)\right]_0^L - \int_0^L A(r)F'(r)g_a'(r)\,dr\,.$$

The boundary term vanishes both for $r = 0$ and $r = L$ so the expression reduces to

$$-\int_0^L F'(r)A(r)g_a'(r)\,dr - \left[F(r)A(r)g_a'(r)\right]_0^L + \int_0^L F(r)(Ag_a')'(r)\,dr\,.$$

If $a + n > 2$ the boundary terms vanishes at $r = 0$ and $r = L$ so we obtain

$$(LG_a)(f) = \int_0^L F(r)(L_rg_a)(r)A(r)\,dr$$

so (3.7) implies the lemma in this case.

If $a + n = 2$ the boundary term at $r = 0$ contributes

$$f(o)\lim_{r\to 0} A(r)g_a'(r) = f(o)\,\lambda\,a\,\Omega_n\lambda^{-n+1}$$

but at $r = L$ the contribution is 0 because of (3.6) and $\lambda = \pi/2L$. The second part of the lemma thus follows from (3.7).

From the data listed (p. 168) about X we see that

$$\sigma(x) = C\sin^{-2\ell}(\lambda r) \qquad r = d(o,x)$$

where C is a constant and $\ell = 1, 2, 4$ in the respective cases $\mathbf{P}^n(\mathbf{C})$, $\mathbf{P}^n(\mathbf{H})$ and $\mathbf{P}^{16}(\mathbf{C}ay$, for which $q = 1, 3$ and 7. The antipodal manifolds have dimensions $2k = n-2\ell$ in these three cases. We apply the lemma k times to $\sigma = CG_{-2\ell}$ and end up with a multiple of δ. Putting $L_a = L+\lambda^2a(a+n+q-1)$ and

$$P_k(L) = L_{-2\ell}L_{-2\ell-2}\dots L_{2-n}$$

the lemma implies

$$cP_k(L)(\hat{f})^\vee = f$$

where c is a constant. By (40) we have

$$\lambda^2 = \frac{\pi^2}{4L^2} = \frac{1}{2p+8q}$$

and we find that $cP_k(L)$ coincides with the polynomials in (44)–(50). The constant c is calculated in the text by taking $f \equiv 1$. This proves Theorem 4.11(ii).

References

1. S. Helgason, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds. Acta Math. 113 (1965), 153–180.

2. F. Rouvière, "Inverting Radon transforms; the group-theoretic approach". Enseign. Math. 47 (2001), 205–252.