SUPPLEMENT TO CH. IV §9. BOUNDED SPHERICAL FUNCTIONS

SIGURDUR HELGASON

The notation in the book will be kept. Although the asymptotic properties of the spherical function ψ_λ (p. 467) has been studied by several authors (see e.g. Rader [1976]) the boundedness question (the analogy of Theorem 8.1) does not seem to have been settled. The following result us a step in this direction.

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Theorem. *Assume the group G is complex. Then the spherical function ψ_λ on G_0 is bounded if and only if λ is real, i.e. $\lambda \in \mathfrak{a}^*$.*

In this context, consider Ch. IV, Theorem 3.3.

For $\lambda \in \mathfrak{a}_c^*$ let $\lambda = \xi + i\eta$ with $\xi, \eta \in \mathfrak{a}^*$. It remains to prove that if $\lambda_0 = \xi_0 + i\eta_0$ with $\eta_0 \neq 0$ then ψ_{λ_0} is unbounded. With $i\lambda_0 = i\xi_0 - \eta_0$ we may by the W-invariance of ψ_λ in λ assume that $-A_{\eta_0} \in \overline{\mathfrak{a}^+}$ (the closure of $\mathfrak{a}^+$.)

Let $U \subset W$ be the subgroup fixing λ_0 and $V \subset W$ the subgroup fixing η_0. Then $U \subset V$ and

$$\psi_{s\xi_0 + i\eta_0} = \psi_{\xi_0 + i\eta_0} \text{ for } s \in V. \tag{1}$$

Using Theorem 5.35 in Ch. II we have

$$\psi_\lambda(\exp H) = c_0 \frac{\sum_{s\in W} \epsilon(s) e^{i\langle sA_\lambda, H\rangle}}{\pi(H)\,\pi(A_\lambda)} \qquad \langle H \in \mathfrak{a}\rangle, \tag{2}$$

where c_0 is a constant, $\langle\ ,\ \rangle$ the Killing form, $\epsilon(s) = \det s$ and π the product of the positive roots (Ch. IV, Theorem 5.7). Let π' denote the product of the positive roots $\alpha_1, \ldots, \alpha_r$ vanishing at λ_0 and π'' the product of the remaining positive roots. For $\lambda = \lambda_0$ we want to divide the factor $\pi'(\lambda_0)$ into the numerator of (2). We do this by multiplying (2) by $\pi'(\lambda)$, then applying the differential operator $\partial(\pi')$

in the variable λ and finally setting $\lambda = \lambda_0$. The theorem then follows from the following lemma.

Lemma. *Let* $\eta_0 \neq 0$. *Then the function*

$$\zeta_\lambda(H) = \frac{\sum_{s \in W} \epsilon(s) e^{i\langle sA_\lambda, H\rangle}}{\pi(A_\lambda)}$$

is for the case $\lambda = \lambda_0$ *unbounded on* $\mathfrak{a}^+$.

Proof. We have

$$\pi'(\lambda)\zeta_\lambda(H) = \frac{1}{\pi''(\lambda)} \sum_{s \in W} \epsilon(s) e^{i\langle sA_\lambda, H\rangle}.$$

Applying $\partial(\pi') = \partial(\alpha_1)\ldots\partial(\alpha_r)$ in λ and putting $\lambda = \lambda_0$ we see that

$$c\,\zeta_{\lambda_0}(H) = \sum_{s \in W} P_s(H) e^{i\langle sA_{\lambda_0}, H\rangle}. \tag{3}$$

Here c is a constant $\neq 0$ and P_s the polynomial

$$P_s(H) = \left[\partial(\pi')_\lambda \left(\epsilon(s)\frac{1}{\pi''(\lambda)} e^{is\lambda(H)}\right)\right]_{\lambda=\lambda_0} e^{-is\lambda_0(H)}$$

whose highest degree term is a constant times

$$\frac{1}{\pi''(\lambda_0)}(s\pi')(H). \tag{4}$$

We break the sum (3) into two parts, sum over V and sum over $W \backslash V$. For the first we consider Σ_V as $\Sigma_{V/U}\,\Sigma_U$. Then (3) can be written

$$c\,\zeta_{\lambda_0}(H) = e^{-\eta_0(H)}\left[\sum_{V/U} e^{is\xi_0(H)} \sum_{\sigma \in U} P_{s\sigma}(H)\right] + \sum_{W\backslash V} P_s(H) e^{is\lambda_0(H)}. \tag{5}$$

We put here $H' = -A_{\eta_0}$, let $H_0 \in \mathfrak{a}^+$ be arbitrary and set $H = tH_0 (t > 0)$. Then the second term in (5) equals

$$\sum_{s \notin V} P_s(tH_0) e^{is\xi_0(tH_0)}\, e^{\langle sH', tH_0\rangle}. \tag{6}$$

By a standard property of $\mathfrak{a}^+$ we have

$$\langle H', H_0\rangle > \langle H', sH_0\rangle \text{ for } s \neq e. \tag{7}$$

Consider (5) with $H = tH_0$. Assume the expression in the bracket has absolute value with $\limsup_{t\to+\infty} \neq 0$. Considering (7) the first term in (5) would have exponential growth larger than that of each term in (6).

Thus

$$\lim_{t\to+\infty} \zeta_{\lambda_0}(tH_0) = \infty$$

implying the lemma in this case.

We must still consider the possibility that the quantity in the bracket in (5) (with $H = tH_0$) has absolute value with $\limsup_{t\to\infty} = 0$. For this we use the following elementary result of Harish–Chandra (see Ch. I, Exercise D5)

Let $k_1 \dots k_n \in \mathbb{R}$ be different and $p_1, \dots, p_n$ polynomials. If

$$\limsup_{t\to+\infty} \left| \sum_1^n e^{ik_r t} p_r(t) \right| = a < \infty \tag{8}$$

then each p_n is constant. If $a = 0$ then each $p_r = 0$.

Note that in the sum

$$\sum_{V/U} e^{is\xi_0(tH_0)} \sum_{\sigma\in U} P_{s\sigma}(tH_0) \tag{9}$$

all the terms $s\xi_0$ are different ($s_1, s_2 \in V$ with $s_1\xi_0 = s_2\xi_0$ implies $s_2^{-1}s_1 \in U$). Thus we can choose $H_0 \in \mathfrak{a}^+$ such that all $s\xi_0(H_0)$ are different. If the bracket in (5) with $H = tH_0$ has absolute value with $\limsup_{t\to+\infty} = 0$ then by (8) the whole sum (9) is identically 0.

Hence it just remains to prove that the last term in (5) is unbounded. For this we restrict H_0 further. Since the angle between the two vectors in $\mathfrak{a}^+$ is $< \frac{\pi}{2}$ we can choose $H_0 \in \mathfrak{a}^+$ and an $s \notin V$ such that

$$\langle H', sH_0\rangle > 0. \tag{10}$$

Let $s_1, \dots, s_r$ be the elements in $W\backslash V$ maximizing this quantity. Writing it as $\langle s^*H', H_0\rangle$ the sum (6) equals

$$e^{\langle s^* H',\ tH_0 \rangle} \cdot \sum_j P_{s_j}(tH_0) e^{i s_j \xi_0(tH)} + \cdots \tag{11}$$

where the omitted terms have lower exponential growth. Because of (4) the P_{s_j} are not constant so using (8) we conclude

$$\lim_{t \to +\infty} \zeta_{\lambda_0}(tH_0) = \infty.$$

This proves the lemma and the theorem since the exponential growth of ζ_{λ_0} outweights the growth of $\pi(H)$.

□