GeoGebra labs

to accompany

Geometry:

The Line and the Circle

Carroll Rykken

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Introduction

Geometry is a visual subject that lends itself to hands-on exploration. These laboratory projects offer guided explorations to accompany topics found in our book, *Geometry: The Line and the Circle*. Like lab projects in the physical sciences, these labs provide a road map for experimentation and observation, but our laboratory space requires only a computer program, GeoGebra, and a device to run it.

GeoGebra is dynamic mathematics software that can be used for geometry, algebra, graphing, statistics and calculus. We use GeoGebra to simulate working within synthetic geometry, that is, a non-coordinate geometry. The virtual compass and unmarked straightedge afford us a precision and fluidity that is unmatched by their physical counterparts. In particular, these tools allow for the dynamic manipulation of geometric figures while preserving their intrinsic properties and the relationships between them. An active sketch is impossible to achieve with paper and pencil, and thus, these GeoGebra labs offer a perfect way to illustrate and showcase geometric constructions and theorems.

Each lab explores topics in the textbook as specified in the following table. The GeoGebra files associated with the labs are packaged together as a zip file.

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Lab #	Lab Title	Companion File	Textbook Sections
0	Preparing the Lab Equipment		
1	An Introduction to Geometry with GeoGebra	Lab1Starter.ggb Lab2Starter.ggb	3.1
2	Six Constructions of Neutral Geometry	Lab2Starter.ggb	3.2, 3.3
3	New Built-in Tools and Creating Custom Tools	Lab3Starter.ggb	3.1–3.3, 7.5
4	Euclid's Proof of the Pythagorean Theorem	Lab3SquareTool.ggb	7.4–7.6
5	Constructions from Book III of *The Elements*	Lab5Starter.ggb	10.2–10.4
6	Construction of the Four "Centers" of a Triangle	Lab6Starter.ggb	11.2
7	Regular Polygons	Lab7Starter.ggb	11.3–11.5
8	The Poincaré Half-plane Model for Hyperbolic Geometry	Lab8Starter.ggb	12.2, 13.3
9	Isometries of the Plane: Reflections	Lab9Starter.ggb	15.2
10	Isometries of the Plane: Translations and Rotations	Lab10Starter.ggb	15.2, 15.3
11	Constructible Lengths and Hippasian Numbers	Lab11Starter.ggb	16.2

Textbook sections and files corresponding to each lab

LAB 0

Preparing the Lab Equipment

Before we can perform any experiments, we need to set up our laboratory equipment. This requires two pieces of equipment: GeoGebra and a computer. GeoGebra is found at www.geogebra.org. The site offers several different free applications that can be used online or offline. All of the material in these lab projects was produced using the most recent offline version, GeoGebra Classic 6, on a Windows pc. Thus, we recommend the installation of Classic 6 to ensure compatibility. For those who prefer the look of a classic desktop user-interface, the older GeoGebra Classic 5 may be used, but the location of menu items will vary from the descriptions given in these labs.

> **Other versions of GeoGebra...or, stick with 6?**
>
> GeoGebra offers a stand-alone GeoGebra Geometry app. It is a pared-down version of Classic 6 with an interface streamlined for geometry investigations and a radically different user-interface. There are also companion mobile apps available. While these are all excellent apps, they are not a viable option to complete these labs since they do not allow for the creation of custom tools. For these labs, stick with 6!

To install GeoGebra Classic 6 on a computer:

At www.geogebra.org, follow the App Downloads link listed under **Offline Apps**. Select the DOWNLOAD link given for the latest version of GeoGebra Classic. When downloading this file, you may choose to run the GeoGebra installer immediately or to download the file and run the installer after locating the executable installation file in your download folder. Follow any specific installation instructions given for your system. Once installed, start the GeoGebra program. If prompted to choose a type of session on the menu shown on the right-hand side of the screen, choose Geometry. Now you are ready to move on to the Introduction to GeoGebra in Lab 1.

Beyond the labs: the immediacy of online and mobile apps

Once you learn how useful GeoGebra is for developing and checking your geometric intuition, you will want to use it beyond these lab projects. We encourage you to use GeoGebra to follow along with Euclid's proofs, to provide insight into homework problems, or to export an image of a nice diagram. With this in mind, why not have GeoGebra available on your mobile device or bookmarked in your favorite browser for easy use anytime? Download the free GeoGebra app from your mobile's app store and try it on your device. Alternatively, use one of the online apps at www.geogebra.org. Just click the GeoGebra Classic link listed under Classic Apps to start the online version of Classic, or click the Geometry link listed under New Math Apps to try the geometry stand-alone app. Bookmark this site in your favorite browser for easy future access.

LAB 1

An Introduction to Geometry with GeoGebra

To accompany section 3.1 of *Geometry: The Line and the Circle*

Euclid's *Elements* begins with definitions, postulates and common notions. The definitions assign meaning to the words we use to describe geometric objects like line, circle and triangle. The postulates and common notions are statements that we agree to take as true statements, the former geometric in nature and the latter, algebraic. Three of the five postulates describe the two tools that we use to construct our geometric objects, the compass and the unmarked straightedge. With these tools and the ability to place a point on the page, Euclid sets about constructing objects and discovering relationships between these objects. With GeoGebra, we can perform the same Euclidean constructions with the aid of a digital straightedge and compass. In this lab, after familiarizing ourselves with these virtual tools, we produce the first two constructions of the *Elements*.

1. Getting to Know the Basics of GeoGebra

• First, save the file provided with this lab project, **Lab1Starter.ggb**, to your computer.

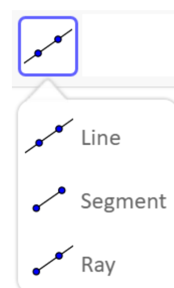
• Start GeoGebra. If prompted to choose between the different GeoGebra apps like Graphing, Geometry, CAS, and Probability, choose Geometry Geometry.

• The main window should be completely empty. If there is a grid or a set of axes showing then remove them by first clicking on in the upper-right corner to reveal a set of display options for the canvas on which we will be working. Hover over the leftmost icon to see *Show or hide the axes*. Click this icon to toggle the display of the axes. Hover over the next icon to see *Show or hide the grid*. Click on this to see the four grid options shown here: . Choose the leftmost icon to work on a blank canvas.

• Open **Lab1Starter.ggb**: GeoGebra's file menu options are found by clicking to access the main menu. Click **File** then **Open**. Clicking will open a file browser window where you can locate **Lab1Starter.ggb** then open it.

• Tools in GeoGebra Classic are selected with the icons at the top of the window. (Note: Tools utilized in these labs are identical in all versions of GeoGebra, but the location of a tool may vary between versions. These instructions are based on the offline version of GeoGebra Classic 6 running on a Windows machine. If you are using the stand-alone GeoGebra Geometry application, then the tools are displayed on the left.) Some of the tool icons displayed at the start of a GeoGebra session include **Move** , **Point** , **Line** , **Polygon** , **Circle with Center through Point** , **Distance or Length** , and **Move Graphics View** . Clicking on a tool icon may reveal other similar tool options. We will refer to such grouped icons as a toolbox.

For example, clicking on the **Line** tool reveals options for constructing line segments and rays. We will call this the **Line toolbox**. The **Point toolbox** is directly to its left and contains the point tools. Likewise, there is a **Circle toolbox**. We refer to the toolbox on the far right as the **General toolbox** since it contains display and editing options. Notice that each icon is a miniature visual identifier for the specific tool. Also, notice that clicking on any of the tool icons produces a help box with instructions for the use of the tool near the bottom of the window.



a. Click on the construct **Line** tool then click the mouse at any two points in the window to create a line through these points. This GeoGebra tool is the virtual equivalent of Euclid's straightedge tool as given in Postulate 2.

b. Click on the construct **Circle with Center through Point** tool to create a circle where the first point selected is the center of the circle, and the second point selected lies on the circle. This GeoGebra tool is the virtual equivalent of Euclid's compass tool as given in Postulate 3. While a physical compass produces a static circle, our digital tool produces a dynamic figure. Change tools to the **Move** tool by clicking on it. Then click and hold down the left mouse button while pointing at the center of the circle, and then drag the mouse around. The other way to change the radius of the circle is by selecting and dragging the other point. To move the circle without resizing it, select the **Move** tool then click on the circle to highlight it (making it appear thicker). Now click and hold the left mouse button on the circle and drag the circle around the window without changing its shape.

c. Using the **Move** tool, highlight the circle again. Whenever an object is highlighted, a menu of options to change the object's color or style is available in the upper-right corner of the screen. If these options are not visible, click on while the circle is highlighted to reveal this menu.

- The first option is the **Color and Transparency** option. Use this to pick another color for the circumference of the circle. To color the *inside* of the circle, move the slider below the color choices. Click-and-hold the black dot and drag it to the right to increase the opaqueness of the coloring. Choose your favorite color. More color options are available by clicking the plus symbol.
- The second option is the **Line Style** option. Select the dashed option and move the slider to see how the display of the circle changes.

d. Next, let's create a triangle. Click on the construct **Polygon** tool. Click on three points in the window then click on the first point to close the construction of the triangle. Have you noticed that a tool is active until a different tool is selected? Change tools to the **Move** tool to highlight the triangle, then color its interior with the **Paint** tool. Pick your second favorite color. Use the Move tool to select one vertex of the triangle. Drag the vertex around. Select one side of the triangle and then move that around. (Note how the side moves. How is it different from moving around a point?) With just the one side of the triangle highlighted, use the **Paint** tool to change the color of this side. Select the triangle interior and drag it around. Note how the entire triangle moves. If you do not want the triangle's interior to be shaded, you can highlight the triangle interior then move the slider bar under the **Paint** tool to 0.

e. **Labels for points and other objects:** GeoGebra provides labels for all objects of a diagram, and it keeps a list of these names behind the scenes.

Right-click on the center point of the circle to reveal a small window of options. When there is a check mark in the **Show Label** box, GeoGebra displays its automatically-generated label for this point. We can change the label for this point to Q by right-clicking on the point. Select **Rename** then enter Q in the input box. Object names may not have any spaces in them. GeoGebra positions the label automatically. To move the label, select it with the **Move** tool and drag it to a different location. GeoGebra limits the range of motion to ensure that the label remains close to its associated object.

f. **Labels and Captions**

Automatic Labels: GeoGebra has the option to display labels automatically for all new points of the diagram. Click ☰ to access the main menu options, then click Settings to access Global Settings. Change the Label option to Labeling: New Points Only then click Save Settings. This ensures that the name for any new point will be displayed for this and all future sessions until this option is changed. Click X in the upper right to close this menu, then add a few new points to your diagram to test this feature.

Toggle display of labels: There are several ways to toggle the display of a label for any object. One way is to right-click on the object and remove the check mark on the **Show Label** box. It is faster to use the **Show/Hide Label** AA tool when doing this for more than one object. Select this tool from the General Toolbox then click on some objects in the diagram to see how it works. Use this tool to display the label of every point in your sketch.

Displaying Captions: Sometimes we need to display a longer description for an object. In this case we can use a caption. Choose **Move** tool then use the mouse to right-click a point on the circle, then choose the **Settings** gear to open a large menu of options on the right side of the window where we can change the way this point is displayed. Under the **Basic** tab, enter *"This is a point on the circle."* in the **Caption:** input box. Leave the check mark on the **Show Label:** option, but click the drop-down menu arrow and choose **Caption** to have GeoGebra display the caption instead of the name. Tour the other options available here, including **Color** and **Style**. Hit X to close this options window when finished.

g. **Moving everything:** Sometimes we need to recenter our diagram within the display window. To do this, click the **Move Graphics View** tool, then click and drag anywhere in the graphics window to shift the entire diagram accordingly.

h. **Text boxes:** A text box is a helpful addition to a diagram as it allows for a lengthier explanation of some element of the diagram, and unlike a caption, it is not tethered to any object in the diagram. Select the **Text** ABC tool. To add a text box to the diagram, click the mouse somewhere within your sketch and the Text dialog box will appear. Type within the input field to comment on something that you have on the screen. Experiment with B for boldface type or the I for italics type.

The input box is Latex-capable for those familiar with the standard typesetting code used to produce mathematical notation. For others who would like to include special symbols, just click on **Advanced** then the $\alpha\beta\gamma$ tab to reveal a table of useful symbols. There's also a **Latex** tab with a menu of useful code. Try one or both of these options to type some symbols.

Click **OK** when you are finished typing. Move your text box around the graphics window. You may also use the Paint tool (accessed by) to color the background of the text box. Within the same menu containing the Paint tool, you will also see options to change the color, style or size of the typeface.

i. Use the **Ray** tool to create a ray. (Note: This tool is grouped in the Line Toolbox.) Notice that there is an **Undo** and **Redo** button in the upper-right corner should you need to revise any missteps.

Save this sketch containing a circle, a triangle, a line, a text box and a ray with the file name **Lab1p1YourName.ggb**.

Saving a sketch

The file menu is accessed by clicking to access the main menu options. Click **File** then **Save**. To save the file to your computer, scroll down to the bottom of this window and click **Continue without signing in**. Change GeoGebra's default file name to Lab1p1YourName.ggb. Clicking **Save** opens a window where you may choose the save location on your computer. Alternatively, you may create a GeoGebra account and follow the provided instructions.

2. EUCLIDEAN CONSTRUCTION: PROPOSITION I.1

Euclid's *Elements*, Book I, Proposition 1: *On a given finite straight line to construct an equilateral triangle.* Let's use GeoGebra to follow the steps of Euclid's construction of an equilateral triangle.

a. Open **Lab1Starter.ggb** using GeoGebra's file menu. If prompted for the type of session, choose Geometry Geometry. Using the **Segment** tool in the Line toolbox, construct a line segment somewhere near the middle of the screen. This GeoGebra tool is the virtual equivalent of Euclid's straightedge tool as given in Postulate 1. The endpoints should be labelled automatically as A and B. We will use segment AB as the base of the equilateral triangle of Proposition I.1. Using the **Circle with Center through Point** tool, click point A and then point B. These steps produce a circle with center A that passes though point B. Next, construct a second circle, but this time with center B that passes through point A.

b. In the Point toolbox, select the **Intersect** tool. Highlight one circle and then the other, making sure that these are the only highlighted elements in your sketch. Notice that two new points are created, C and D, at the intersection of these circles. Construct triangle △ABC using the **Polygon** tool. Select level 0 for the shading. Reposition the point labels as needed to ensure that they are unobstructed.

Explore the difference between points A and C

Use the **Move** tool to choose point A. As you click then drag point A, notice that all figures are resized dynamically. The same is true for point B, but not C. This is because A and B were freely chosen points at the start of the construction. Since we were free to place these points anywhere in the graphics window at the start of the construction, GeoGebra allows each of these points to be moved and adjusts all other elements which depend on their placement accordingly. This is why we refer to this sketch as *dynamic*. Since the position of C is prescribed by the positions of A and B, we are not free to move C. For GeoGebra, C is viewed as a *dependent object* since its existence relies on other objects. This is similar to the difference between an independent and a dependent variable in algebra. Notice that GeoGebra automatically assigns the color of blue to the free points A and B, and black for the dependent points C and D.

c. Triangle $\triangle ABC$ is the object that Euclid constructs in Proposition I.1. How can we verify that our △ABC is equilateral? Explore the menu options to discover a tool that will help you answer the question. Create a text box to explain your findings.

d. If you only wish to see the constructed equilateral triangle, you can remove the clutter by hiding any of the construction elements that were required to produce it. In this sketch, you could hide the two circles and point D. To do this, select the **Show/Hide Object** tool from the General toolbox. Select both circles and the unused point D. These objects will appear lighter. When you pick another tool, say **Move**, they will disappear. The objects are not removed completely from your sketch. If you ever need to "unhide" an object, simply select the Show/Hide Object tool to reveal all hidden objects, then select the objects you would like to display. As its name advertises, this tool simply toggles between showing and hiding any object.

e. Use the **Move** tool to choose point A. As you click then drag point A, notice that the triangle you've created remains equilateral regardless of where you move A. This is a *dynamic sketch* since moving the point does not affect the essential properties of the constructed triangle. All of our GeoGebra constructions will be dynamic as they will preserve the relationships between the constructed geometric objects regardless of how we reposition or resize an element of the sketch. Use a text box to label your triangle as: *Equilateral Triangle: Book I Proposition 1.*

Save this sketch of the equilateral triangle and the text box with your answer to #2c with the file name **Lab1p2YourName.ggb**.

3. Book I, Proposition 2: *To place at a given point (as an extremity) a straight line equal to a given straight line.* Here, we use GeoGebra to illustrate Euclid's proof of Proposition I.2.

   a. Open **Lab1Starter.ggb** using GeoGebra's file menu. Construct a point and label it A. Now construct a line segment, BC, above and to the right of A, as demonstrated in the diagram for Proposition I.2 in the text. Be sure to label its endpoints as B and C.

   b. Construct a segment joining A and B. Then create an equilateral triangle with AB as one of the sides by using the method of Proposition I.1 as given in #2. Label this triangle as △ABD. Hide the extra objects used to construct the triangle.

c. Create a ray starting at D and going through A. Create another ray starting at D and going through B. Select the **Point on Object** tool within the Point toolbox. Click on ray DA to construct new point on ray DA. Name it E. Next, use then click on ray DB to construct new point on ray DB. Name it F. Note that when you select then move these points using the Move tool, their motion is restricted to their respective ray precisely because we created them using the Point on Object tool.

d. Construct a circle with center B and radius BC. Use the **Intersect** tool to construct any points of intersection between this circle and ray DF. Give the name G to the point of intersection such that B is between D and this point G. (This may require renaming the point.)

e. Construct a circle with center D and radius DG. Construct the point of intersection of this circle and ray DE. Call this intersection point L. Use the **Zoom Out** tool or your mouse scroll bar to resize the window if necessary. Likewise, use to shift the visible graphics window as needed.

f. Euclid claims that AL has length equal to BC. Before we measure these lengths, let's change the global settings for rounding. Click for the main menu, then click Settings to access Global Settings. Here you should change the **Rounding** option to Rounding: 3 Decimal Places 3 decimal places. Click Save Settings then X to close this menu.

   Now, to check the claim that AL and BC have the same length, we use the **Distance or Length** tool located in the Measurement toolbox. By selecting A and then L, a length will appear. You can then select B followed by C to show the other length. The two segments should have the same length. Drag A around the diagram. The segment lengths should not change. Drag B around the diagram to verify that the lengths of these line segments change, but they remain equal to each other. Use the Text tool to create a text box that reads "Book I, Proposition 2."

Save this construction with the file name **Lab1p3YourName.ggb**.

How to save an image of the graphics view

Click for the main menu, then click Export Image. To save as a png file, select Download. To paste the image into another application, select Copy to Clipboard. Notice that the image produced is precisely the visible graphics window. To minimize the white space around the diagram, before saving be sure to resize the window in combination with **Move Graphics View** until the diagram is bordered snugly by the boundaries of the window.

4. With Postulate 3, Euclid theoretically constructs a circle with a full sweep of the compass after planting one leg of the compass at the center and the other at a point on the circle. Once he lifts the compass, the legs collapse. So, Postulate 3 provides a *collapsible compass*. We have been using GeoGebra's Circle with Center through Point tool, the virtual equivalent of the collapsible compass since it performs the digital sweep of the compass instantaneously once you provide the center and a point on the circle.

   Proposition I.2 gives us the ability to construct a circle in a different way. We can now construct a circle of any given radius BC at any given center point A, where the radius BC is a segment independent from the center point. This gives us a theoretical rigid compass with legs that can be locked into place. With a rigid compass, we can lock the legs at points B and C, then transfer the desired radius BC to the center point before sweeping out the circle. This new Euclidean tool has an equivalent virtual rigid compass tool that we have added to the Circle toolbox in the starter file for **Lab 2**.

   a. Open **Lab2Starter.ggb** using GeoGebra's file menu. Construct a point A. Now construct a line segment BC in the upper left corner of the screen.

   b. To construct a circle of radius BC at center A, select the **Compass** tool in the Circle toolbox. First select segment BC and then point A. Construct a point on the newly created circle and label it L. Construct the segment AL. Again, measure the length of BC and the length of AL to verify that they are the same. Here you can select the Distance or Length tool and simply select each of the two segments. (Notice that, if the length is 2.345, for example, then only this number is displayed when you choose the segment. If you choose the endpoints of the segment, then the more informative AL=2.345 is displayed.) Drag point B or C around to verify that the lengths of segments AL and BC remain the same.

   Save this construction with the file name **Lab1p4YourName.ggb**.

The rigid compass tool

Since Proposition I.2 establishes the existence of a theoretical rigid compass, from now on we assume that our toolbox contains this tool. Thus, we may use the **Compass** tool in our GeoGebra constructions.

LAB 2

Six Constructions of Neutral Geometry

To accompany sections 3.2 and 3.3 of *Geometry: The Line and the Circle*

Book I of Euclid's *Elements* contains fourteen constructions. Of the ten constructions that are part of Neutral geometry, we saw the first two in the previous lab and we explore six more here: *bisecting an angle* (I.9), *bisecting a line segment* (I.10), *constructing a perpendicular to a segment through a point on the segment* (I.11), *constructing a perpendicular to a segment through a point not on the segment* (I.12), *copying an angle* (I.23), and *constructing a line parallel to a given line through a point not on the line* (I.31). As a starting point, you may wish to consult the textbook to see how Euclid performed each construction. For some of these construction proofs, Euclid's reliance on previous propositions makes his chosen steps inefficient for producing a GeoGebra construction algorithm. You should aim to provide more succinct instructions for these GeoGebra constructions.

For each of the six constructions in this lab you will provide a construction using GeoGebra tools and then, in a text box, prove that your construction has produced the desired result. The proof must justify each step of your construction using Euclid's postulates, common notions and propositions, with one small change. You may **NOT** justify any step of your proof with any of the construction propositions in this lab. Therefore, nowhere in any of your proofs should you write "By I.x" where x=9,10,11,12. More specifically, please note that for the first five constructions (#1 through #5 below), your justification in the text box may use only definitions, common notions, postulates and Propositions I.1 through I.8!

Lab starter files and toolboxes

The tools provided in each Lab$x$Starter file vary. For example, the rigid compass tool is not available in the Circle toolbox of Lab1Starter.ggb but appears in all subsequent starter files. Therefore, beginning a new lab with the proper file is imperative. To do this, **always use GeoGebra's file menu to open the appropriate starter file**. The usual shortcut of double-clicking on the file to start the GeoGebra app should be avoided as it defaults to the toolboxes used in the previous session. Opening the Lab$x$Starter file **after** starting the app ensures that the proper toolboxes are available. To recap, always open a lab starter file using GeoGebra's file menu as follows:

Select ☰ ⇒ File ⇒ Open ⇒ Lab$x$Starter.ggb.

1. Bisecting an Angle (Proposition I.9)

   a. Start GeoGebra. Open **Lab2Starter.ggb** using GeoGebra's file menu. Draw an angle with points B, A and C, where A is the angle vertex by creating rays AB and AC. Bisect angle ∠BAC. In other words, use GeoGebra's point, straightedge and circle construction tools to construct a point F inside ∠BAC such that $\angle BAF \cong \angle CAF$.

   b. Select the **Angle** tool in the Measurement toolbox then select, in order, points C, A, then F to measure ∠CAF. (If your angle measure is greater than 180° then reverse the order of the points to F, A, then C.) Next, measure angle ∠BAF to verify that these angles are equal, and hence, ray AF bisects ∠BAC. Drag points A, B and C to verify that your sketch is dynamic and that these angles remain equal.

   c. In a text box, give the steps of your construction then prove why your construction works using only definitions, common notions, postulates and Propositions I.1 – I. 8. For this proof and all other proofs in this lab, use the text tool to type your proof as part of the GeoGebra document.

Text boxes for proofs

The reliability of the input window for the text box in early versions of GeoGebra Classic 6 exhibited some inconsistencies when entering multi-line text. Though this was fixed in later versions, if you *do* experience any difficulty when typing your proof, then here's a workaround solution: Use a separate program to write and save your proof. When you are finished editing, cut-and-paste the proof into the GeoGebra text input box.

Save this GeoGebra document with the construction of the bisected angle and the proof as **Lab2p1YourName.ggb**.

2. Bisecting a Line Segment (Proposition I.10)

   a. Open **Lab2Starter.ggb**. Construct segment AB. Construct the midpoint, M, of AB. In other words, use GeoGebra's point, straightedge and circle construction tools to construct a point M on AB such that $AM \cong BM$.
   b. Measure the lengths of AM and MB to verify that they are equal. Drag B around to verify that your sketch is dynamic and that M remains as the midpoint.
   c. In a text box, give the steps of your construction then prove why your construction works using only definitions, common notions, postulates and Propositions I.1 – I. 8.

Save this document with the construction of the bisected segment and the proof as **Lab2p2YourName.ggb**.

3. Constructing a Perpendicular Line from a Point on a Segment (Prop. I.11)

   a. Open **Lab2Starter.ggb** using GeoGebra's file menu. Construct a line AB. Construct a point on the line and name it C. Use GeoGebra's point, straightedge and circle construction tools to construct a line through C that is perpendicular to AB.
   b. Construct a point on the perpendicular line and name it D. Measure $\angle$ACD to verify that the lines are perpendicular. Move points around the graphics window to verify that the sketch is dynamic.
   c. In a text box, give the steps of your construction then prove why your construction works using only definitions, common notions, postulates and Propositions I.1 – I. 8.

Save this file with construction and proof as **Lab2p3YourName.ggb**.

4. CONSTRUCTING A PERPENDICULAR LINE FROM A POINT NOT ON A SEGMENT (Prop. I.12)

   a. Open **Lab2Starter.ggb** using GeoGebra's file menu. Construct a line AB. Construct a point not on the line and label it C. Use GeoGebra's point, straightedge and circle construction tools to construct a line through C that is perpendicular to AB. Construct the point of intersection of the two lines and name it D.

   b. Measure angle ∠ADC to verify that the lines are perpendicular. Move points around the graphics window to verify that the sketch responds dynamically.

   c. In a text box, give the steps of your construction then prove why your construction works using only definitions, common notions, postulates and Propositions I.1 – I. 8.

   Save this construction and proof as **Lab2p4YourName.ggb**.

5. COPYING AN ANGLE (Proposition I.23)

   a. Open **Lab2Starter.ggb** using GeoGebra's file menu. Construct a line AB. In the upper-left corner of the window construct an angle ∠DCE which will be our given angle. Use GeoGebra's point, straightedge and circle construction tools to construct a copy of ∠DCE along line AB. In other words, construct a point F so that $\angle FAB \cong \angle DCE$.

   b. Measure the two angles to verify that they are equal. Drag the points of $\angle DCE$ around to ensure sure that your sketch responds dynamically and that the angles remain equal.

   c. In a text box, give the steps of your construction then prove why your construction works using only definitions, common notions, postulates and Propositions I.1 – I. 8.

   Save this GeoGebra construction of the copied angle and the proof as **Lab2p5YourName.ggb**.

6. Constructing a Line Parallel to a Given Line (Proposition I.31)

   a. Open **Lab2Starter.ggb** using GeoGebra's file menu. Construct a line BC and a point A not on the line. Use GeoGebra's point, straightedge and circle construction tools to construct a line EF through A that is parallel to BC.

   b. Check the construction by measuring appropriate angles. Move point A around to verify that your sketch responds dynamically and that the lines remain parallel.

   c. In a text box, give the steps of your construction then prove why your construction works using only definitions, common notions, postulates, Propositions I.1 – I. 8, and Proposition I.27.

   Save this file with the construction of the parallel line and the proof as **Lab2p6YourName.ggb**.

LAB 3

New Built-in Tools and Creating Custom Tools

To accompany sections 3.1 – 3.3 and 7.5 of *Geometry: The Line and the Circle*

In Lab 2 we bisected segments and angles, constructed perpendiculars to a point on or off a given line, constructed a parallel to a given line through a given point, and copied angles. We performed all of these constructions using the three tools in our Line toolbox and the two tools in our Circle toolbox. GeoGebra has more powerful tools to perform all but one of these constructions, and since we have provided justifications for all of these, we have added the equivalent power tools to our toolbox to simplify future constructions.

Construction Proposition	Equivalent GeoGebra tool
I.9	**Angle Bisector**
I.10	**Midpoint or Center**
I.11	**Perpendicular Line**
I.12	**Perpendicular Line**
I.31	**Parallel Line**

There is no built-in GeoGebra tool to copy an angle. So, we must repeat the construction steps of Proposition I.23 as needed.

1. Getting to know our new tools

   a. Bisecting a Segment, Constructing Perpendiculars and Parallels

i. Start GeoGebra. Open **Lab3Starter.ggb** using GeoGebra's file menu. Construct segment AB. Select the **Midpoint or Center** tool then select segment AB to construct point C, the midpoint of AB. Next, select then select A then C to construct D, the midpoint of AC. Select A and D to construct E, the midpoint of AD.

ii. Measure the length of AB and AE. Since we bisected three times, AE will be one-eighth the size of the original segment AB. Create a text box in which you verify this fact.

Dynamic text in a text box

We can incorporate elements in a text box that respond dynamically to changes made to the diagram. Let's demonstrate this by creating a text box in which we have GeoGebra calculate one-eighth the length of segment AB.

Start a new text box. In the input box type the following: *One-eighth of length(AB)* =. Now, just below this input box, click Advanced followed by the GeoGebra symbol . In this list you will see GeoGebra's internal names for every object in your diagram. Click on distanceAB to add this quantity after the equal sign in the text input box above. The input box should appear as follows: One-eighth of length(AB) = distanceAB. Clicking on the small shaded box containing distanceAB allows for editing. Change this to distanceAB/8 then press Enter. The input box should now appear as follows: One-eighth of length(AB) = distanceAB/8. Click OK to view the new text box. Drag point B to see the calculation respond dynamically. Note that this calculation matches the length of segment AE.

iii. Construct a perpendicular line to AB at C: Select the **Perpendicular Line** line tool (in the new Special Line toolbox) then select point C followed by segment AB. Construct a point on this perpendicular line and name it F. Measure $\angle BCF$ to verify that the lines are perpendicular.

iv. Construct a parallel line to AB through F: Select the **Parallel Line** line tool then point F followed by segment AB. Construct a point on this parallel line and name it G. Measure the appropriate angles to verify that AB and FG are parallel.

v. GeoGebra has a **Perpendicular Bisector** tool that combines the Midpoint or Center tool with the Perpendicular Line tool. It can be found in the Special Line toolbox. Use this tool to construct the perpendicular bisector of segment BC.

Save this file as **Lab3p1YourName.ggb**.

b. Bisecting an angle

Open **Lab3Starter.ggb**. Construct an angle $\angle BAC$. Select the **Angle Bisector** tool then select points, in order, B, A then C. In the text, the angle bisector is a ray, but GeoGebra's tool produces the line that contains the ray. This is unusual but will not present any difficulty. If needed, we can produce the ray as follows: construct segment BC then find the intersection of BC and the angle bisector line. With this intersection point named D, then AD is the desired ray. Make ray AD a different color and increase the thickness of its line width so it is prominently displayed.

Save this file as **Lab3p2YourName.ggb**.

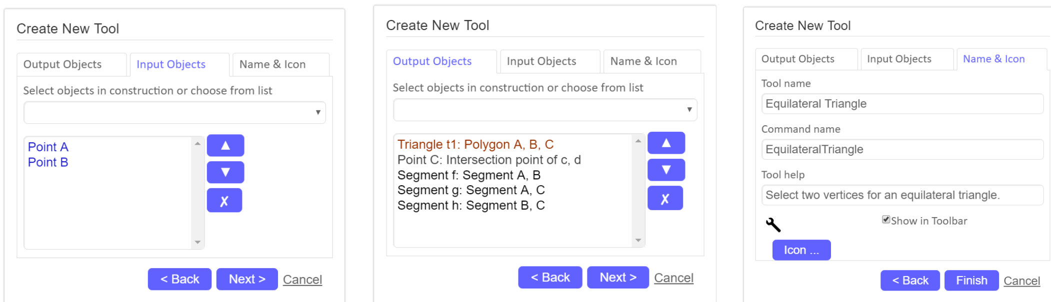
2. CREATING A CUSTOM EQUILATERAL TRIANGLE CONSTRUCTION TOOL (PROP I.1)

When GeoGebra does not offer a tool for a frequently needed construction, then we can create a custom tool. Let's create an equilateral triangle tool.

a. Open **Lab3Starter.ggb** using GeoGebra's file menu. Construct segment AB. Construct a circle of radius AB centered at A. Construct a circle of radius AB centered at B. Construct the intersection points, C and D, of these circles. Join AC and BC. Use the Polygon tool to reconstruct $\triangle ABC$ in order to shade the triangle your favorite color.

b. We can use the current sketch to create a custom GeoGebra tool that will construct an equilateral triangle on a given a line segment. In the main menu ☰, select **Tools**, then **Create New Tool** which opens a Create New Tool window. A tool is essentially a function; it has inputs and outputs. Here, the output will be an equilateral triangle $\triangle ABC$ and the input will be points A and B. In the Create New Tool window, select the **Input Objects** box, click the drop-down box arrow ▼ then select **Point A** to have it appear in the list below. Click the drop-down box arrow ▼ again and select **Point B** to add it to the list below.

The Input list should resemble the first screen image shown below. If yours is significantly different, then start over.



c. Select the **Output Objects** tab. Click the drop-down box arrow and select GeoGebra's Object name for the triangle. It should be similar to **Triangle t1: Polygon A, B, C**. Click the drop-down box arrow again and choose **Point C**. Repeat this process to choose GeoGebra's Object names for segments AB, AC and BC. The Output list should resemble the middle screen image shown above.

d. Select the **Name & Icon** tab to name our new tool.

- Enter *Equilateral Triangle* in the **Tool Name** box. (The **Command Name** will be automatically generated as EquilateralTriangle with no spaces.)
- In the **Tool Help** input box, enter: *Select two vertices for an equilateral triangle.* (These instructions will be displayed whenever you select the tool.)
- Leave the check mark for **Show in Toolbar**. These entries should resemble the last screen image shown above.
- Select **Finish** and the new tool appears rightmost in the tool menu with a wrench icon .

e. Delete every object in the sketch. (Yes! Delete every point, circle and segment!)

f. Verify that the sketch appears empty.

g. Save the sketch as **Lab3EqTri.ggb**.

3. Using the New Custom Equilateral Triangle Tool

Open the file **Lab3EqTri.ggb**. Save this file as **Lab3p3YourName.ggb**. Place two points A and B in the graphics window. Use your new Equilateral Triangle tool to construct an equilateral triangle. [Select the tool then click on A followed by B.] Drag the points around to verify that the sketch responds dynamically. Use the tool to create two more equilateral triangles.

Resave this file as **Lab3p3YourName.ggb**.

4. Creating a Custom Square Construction Tool (Prop I.46)

   a. Open **Lab3Starter.ggb** using GeoGebra's file menu. Construct segment AB. Use GeoGebra's point, line and circle tools to construct square ABCD. When finished constructing the square, use the Polygon tool to color square ABCD.

   b. Follow the instructions in #2b through #2e to create a custom square construction tool. Be sure to include the vertices, sides and interior of the square in the list of Output objects. Delete everything from the sketch before saving this as **Lab3SquareTool.ggb**.

5. Using the New Custom Square Construction Tool

   Open **Lab3SquareTool.ggb** using GeoGebra's file menu. Save this file as **Lab3p5YourName.ggb**. Place two points A and B in the graphics window. Use your new Square tool to construct a square of side AB. [Select the tool then click on A followed by B.] Drag the points around to verify that the sketch responds dynamically.

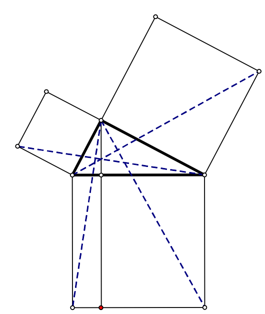
   Resave this sketch as **Lab3p5YourName.ggb**.

LAB 4

Euclid's Proof of the Pythagorean Theorem

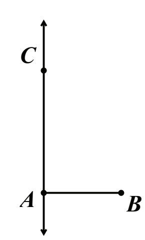
To accompany sections 7.4 – 7.6 of *Geometry: The Line and the Circle*

Book I of Euclid's *Elements* ends with two of the most well-known results in geometry, the Pythagorean Theorem and its converse. The figure shown at right is the diagram Euclid produced in his proof of the Pythagorean Theorem. It has come to be known as the windmill diagram. In order to produce a proof of the Pythagorean Theorem, we construct the diagram then explore the relationships between the figures found within it.

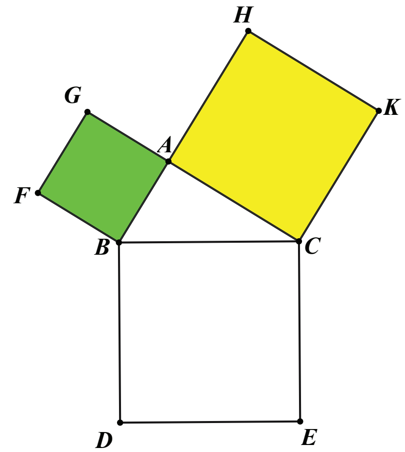


Start GeoGebra. Open **Lab3SquareTool.ggb** using GeoGebra's file menu. Save this file as **Lab4p1YourName.ggb**. First we construct the given right triangle, $\triangle ABC$, of Proposition I.47 with right angle at vertex A. Then we construct squares on the sides.

1. Construct segment AB. Construct a perpendicular line to AB at A. Construct a point C on the perpendicular line. Construct segments AC and BC, then hide the perpendicular line. Triangle $\triangle ABC$ is a right triangle, and we'd like to show that the squares on legs AB and AC, taken together, are equal to the square on hypotenuse BC. To do this, we'll need some squares.



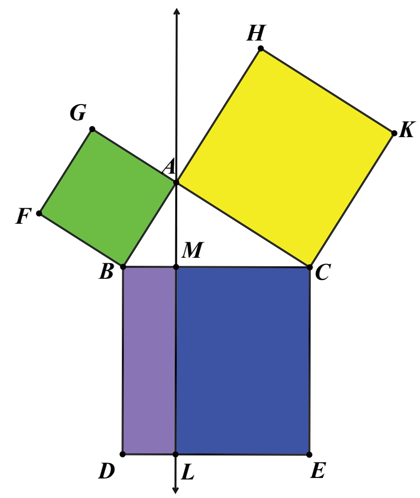
2. Use your **Square** tool to construct a square on each side of the right triangle. [To construct the square on AB, for example, select then click on A followed by B. If any squares overlap $\triangle ABC$ just reverse the order of the points chosen for the tool.] Shade the interiors of the two squares on the legs of right triangle $\triangle ABC$ with different colors, and set the slider bar to 5 for the color of square on the hypotenuse. Reposition the sketch to resemble the windmill diagram shown here by dragging point A as necessary. Then, rename all points to match.



3. Let's establish a few results about BAH and GAC that we'll need later. Why is BAH a straight line? Also, why is BAH parallel to CK? Similarly, why is GAC a straight line? Also, why is GAC parallel to FB? Use the propositions of Book I to give your answers to these questions in a text box. Please start the text box with the descriptor *"Answer to #3:"*.

4. Construct a line through A perpendicular to BC. Construct points M and L where the line through A meets BC and DE, respectively. Quadrilaterals $MCEL$ and $BMLD$ are rectangles. (Why?) Use the Polygon tool to construct and color rectangles $\square MCEL$ and $\square BMLD$ with different colors.

5. Use the **Area** tool in the Measurement toolbox to measure the area of squares $\square BFGA$ and $\square AHKC$, and rectangles $\square MCEL$ and $\square BMLD$. [See the note at the end of this lab for a workaround when a label obscures part of your diagram.] What relationship is there between the areas of these quadrilaterals? Drag the vertices of $\triangle ABC$ to verify that your sketch is dynamic and that this relationship always holds. Give your answer in a text box. (As with the previous question, be sure to start your text box with the descriptor *"Answer to #5"*. Please provide the appropriate descriptor at the start of each text box answer for the questions that follow.)



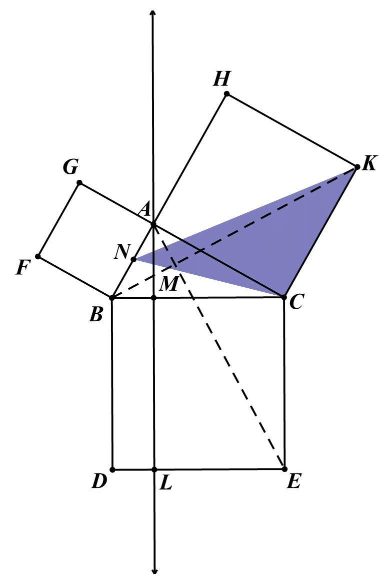
The relationship between the areas of these quadrilaterals is at the heart of Euclid's proof. If the square on leg AC has area equal to that of rectangle $\square MCEL$, and if the square on AB has area equal to that of $\square BMLD$, then this decomposition demonstrates that the squares on the legs of the right triangle are equal to the square on the hypotenuse. While the measurements taken in Step #5 allow us to check this relationship for the given diagram, this specific verification is not a general mathematical proof. After saving the current version of the file, we will add some more elements to the diagram and fill in the necessary details as we follow along with Euclid's proof of the equivalence of these areas.

Resave the current sketch as **Lab4p1YourName.ggb**.

6. Select and set the slider bar to 5 for the color of the four quadrilateral interiors and hide the measurements. Construct BK and AE. Select to change their line styles to dashed.

7. How are triangles $\triangle BCK$ and $\triangle ECA$ related? Drag the vertices of $\triangle ABC$ to verify that your sketch is dynamic and the relationship between the triangles remains valid. In a text box, use Book I propositions to prove that $\triangle BCK \cong \triangle ECA$. Consequently, what is the relationship between the area of these triangles? Give your answers in a text box.

Next, we need to explore the relationship between the area of these congruent triangles and the area of both the square $\square AHKC$ and the rectangle $\square MCEL$. We'll add a dynamic element to the sketch to help us visualize these area relationships.



8. Construct point N on segment AB. Color the triangle interior $\triangle CNK$. Measure its area. [Display note: To display a triangle's interior without its sides, set the slider bar of each side to 0.]

9. Move N along AB. As you do this, does the area of the triangle change? Explain why this is so? [HINT: If you consider CK as the base of $\triangle CNK$, what is its height? Does that change?] What Book I proposition establishes that the area will stay the same? Next, when point N coincides with point A, what relationship between the area of shaded triangle $\triangle CNK$ and square $\square AHKC$ becomes clear? What Book I proposition establishes this relationship? Give your answers in a text box.

10. Move N so that it coincides with B. Use your answer to question #9 to determine the relationship between the area of $\triangle BCK$ and the area of square $\square AHKC$. Give your explanation in a text box.

11. Employ a similar argument to the method used to answer questions #9 and #10 to establish a relationship between the area of $\triangle ECA$ and the area of rectangle $\square MCEL$. Give your answer in a text box.

12. Use the area relationships established in #7, #10 and #11 to explain why the area of square $\square AHKC$ is equal to the area of rectangle $\square MCEL$. Give your explanation in a text box.

Save the current sketch as **Lab4p2YourName.ggb**.

After showing that the square on leg AC of right triangle $\triangle ABC$ has area equal to one of the rectangles on the hypotenuse, we are at the halfway point in the proof of the Pythagorean Theorem. To complete the proof, we need to establish the equality of the area of the square on leg AB with the other rectangle on the hypotenuse, $\square BMLD$. So, we must begin anew at Step #6 by opening **Lab4p1YourName.ggb** using GeoGebra's file menu. We can now proceed along a similar path, adding an asterisk to each step to indicate that we are in the second half of the proof.

6*. Select and set the slider bar to 5 for the color of the four quadrilateral interiors and hide the measurements. Construct AD and FC. Select to change their line styles to dashed.

7*. How are triangles $\triangle FBC$ and $\triangle ABD$ related? Drag the vertices of $\triangle ABC$ to verify that your sketch is dynamic and the relationship between the triangles remains valid. In a text box, use Book I propositions to prove that $\triangle FBC \cong \triangle ABD$. Consequently, what is the relationship between the area of these triangles? Give your answers in a text box.

8*. Construct a point P on segment AC. Color the triangle interior $\triangle PBF$. Measure its area.

Follow the rest of the construction steps to produce a similar argument showing that the area of $\square ABFG$ is equal to the area of $\square BMLD$.

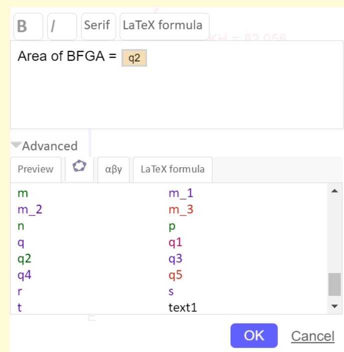
Save this work as **Lab4p3YourName.ggb**.

What to do when a label obscures part of a diagram

GeoGebra restricts the placement of a label to a limited neighborhood of its associated object. This becomes inconvenient when the label obscures part of the diagram. As a workaround, we can utilize a dynamic text box to display the area of $\square BFGA$. This way, we can place it wherever we like and it will still respond dynamically to changes in the diagram.

Recall that in Lab 3 we took a peek behind the scenes to see that GeoGebra keeps its own list of names for each object added to the diagram. Here's a direct way to determine GeoGebra's internal name for an object, let's say $\square BFGA$. Make sure the interior of $\square BFGA$ has been colored. Now, right-click in an open area of this square to reveal a new window; at the top is **Quadrilateral q2: Polygon A,G,F,B**. (Yours may be slightly different.) This means that GeoGebra's internal name for this square is **q2**. Try right-clicking on some other objects to view other internal names.

Let's follow along with steps similar to those in Lab 3. Start a new text box. In the Input box type the following: *Area of BFGA* =. Next, click Advanced then the GeoGebra symbol . Scroll down until you find the internal name for your quadrilateral, then click on it to add it after the equal sign in the Input box. Click OK to view the dynamic text box then move it anywhere you like. Drag point A to verify that the text box responds dynamically.



This project takes its inspiration from Dan Bennett's project for Geometer's Sketchpad, "Euclid's Proof of the Pythagorean Theorem," in the 1998 *Instructor's Resource Guide* for Thomas Q. Sibley's *The Geometric Viewpoint*.

LAB 5

Constructions from Book III of *The Elements*

To accompany sections 10.2 – 10.4 of *Geometry: The Line and the Circle*

1. CENTER OF A CIRCLE

   The first proposition in Book III is the construction of the center of a given circle.

   Start GeoGebra. Open **Lab5Starter.ggb** using GeoGebra's file menu. Construct a circle and hide its center. Pick two points on the circle and label them A and B. Construct segment AB. Using the perpendicular bisector tool, construct the perpendicular bisector of AB. Name the two intersection points of the bisector and the circle as C and D. Use the midpoint tool to construct the midpoint of CD. Name the midpoint E.

   Claim: E is the center of the circle.

   To verify this, pick a point on the circle and name it F. Next, measure the length of EF. Move F around the circle to verify that this distance does not change.

   Save this sketch as **Lab5p1YourName.ggb**.

2. PROPOSITION III.3

   Proposition III.3 states that a diameter (or radius) will bisect a chord that is not a diameter if and only if it is perpendicular to that chord. Let's explore both of these implications.

   a. Open **Lab5Starter.ggb** using GeoGebra's file menu. Construct a circle and name its center A. Pick two points on the circle, naming them B and C. Construct segment BC. Construct the midpoint of BC and name it D. Construct

AD then measure angle $\angle ADB$ to verify that it is a right angle. Also, move points to check that the sketch is dynamic.

Save this sketch as **Lab5p2aYourName.ggb**.

b. Open **Lab5Starter.ggb** using GeoGebra's file menu. Construct a circle and name its center A. Pick two points on the circle and label them B and C. Construct segment BC. Construct a perpendicular from A to BC. Name the intersection of the perpendicular line with BC as D. Measure the lengths of BD and CD and to verify that they are equal. Also, move points to check that the sketch is dynamic.

Save this sketch as **Lab5p2bYourName.ggb**.

3. Tangents to a Circle

Proposition III.17 is the construction of a tangent line to a circle from a point outside of the circle.

Open **Lab5Starter.ggb** using GeoGebra's file menu. Construct a point A. Construct a circle with center B such that A lies outside of the circle. Construct a point on the circle and name it C. We wish to construct a tangent to the circle with center B of radius BC that passes through A. Join AB. Let D be the intersection of AB and the circle. Construct a circle with center B and radius BA. Construct a perpendicular to AB through D. Let E be one of the points of intersection between this line and the circle with center B and radius BA. Hide the other intersection point. Join BE. Let F be the intersection between BE and the original circle with center B and radius BC. Construct line AF and change its color and line width so that it is prominently displayed.

Claim: Line AF is tangent to the circle with center B and radius BC.

To verify this, measure angle $\angle BFA$ to see that it is a right angle. Also, move point B to check that the sketch is dynamic. In a text box, state which proposition from Book III tells us that if $\angle BFA$ is a right angle, then AF is tangent to the circle.

Save this sketch as **Lab5p3YourName.ggb**.

4. Propositions III.20, III.21 and III.22

Proposition III.20 states that in a given circle, the angle at the center is twice its corresponding inscribed angle. The corollary to this proposition, **III.21**, tells us that inscribed angles subtended by the same arc (be it major or minor) must be congruent. A *cyclic quadrilateral* is a 4-gon whose vertices lie on a circle. **Proposition III.22** states that the opposite angles of a cyclic quadrilateral sum to two right angles. Here, we construct a sketch to illustrate these three propositions.

a. Open **Lab5Starter.ggb** using GeoGebra's file menu. Construct a circle. Label its center A. Pick four points on the circle and name them B, C, D and E as you travel counterclockwise around the circle. Labelled this way, D and E should lie on the same side of the circle as determined by chord BC. Join AB and AC using one color, DB and DC using another color, and EB and EC using a third.

   Claim: $\angle BAC = 2\angle BDC$ (III.20) and $\angle BDC = \angle BEC$ (III.21).

   Verify this by measuring angles $\angle BAC$, $\angle BDC$ and $\angle BEC$. (Match the color of the segments for each angle measurement.) Move the points to check that the sketch responds dynamically. Create a dynamic text box that displays these measurements.

> **Performing arithmetic on dynamic text within a text box**
>
> To perform arithmetic on GeoGebra objects within a text box, first follow the instructions given in earlier labs to locate and insert GeoGebra's internal object name in a text box. For example, if GeoGebra's object name for angle $\angle CDB$ is β, insert the following into a text box input window: 2*angle(CDB) = 2* $\boxed{\beta}$ = $\boxed{2*\beta}$. Remember: The trick to producing $\boxed{2 * \beta}$ begins with the input of the single object $\boxed{\beta}$. Clicking on this box $\boxed{\beta}$ allows for editing and thus, entering a formula. In this case, we change the entry within the shaded box to $2 * \beta$.

b. Join BC and DE using a different color. Measure angles $\angle BCD$ and $\angle DEB$. (Use appropriate colors for the angles and vary the transparency of any overlapping angle markers for easy identification.) Verify III.22 by checking the sum of these angles. Create a dynamic text box that displays the sum.

Save this sketch as **Lab5p4YourName.ggb**.

5. PROPOSITION III.35

   Proposition III.35 states that if AC and BD are two intersecting chords of a circle whose intersection point E lies inside the circle, then $AE \cdot EC = BE \cdot ED$. Let's construct a sketch to verify this property.

   Open **Lab5Starter.ggb** using GeoGebra's file menu. Construct a circle and hide the center. Pick four points on the circle and name them A, B, C and D as you travel counterclockwise around the circle. Join AC and BD and let their intersection be E. Hide segments AC and BD and create segments AE, BE, CE and DE. Pick a different color for each of them. Let's verify that $AE \cdot EC = BE \cdot ED$ algebraically and geometrically.

   a. Algebraic verification:

      Measure AE, EC, BE and ED. Since each measurement has a different color, the associated GeoGebra object is easily identified in the Object list. In a text box do the following:

      - type $AE * EC =$
      - under Advanced, pick the object with the same color as AE
      - type *
      - pick the object with the same color as EC
      - type $=$
      - pick the object with the same color as AE
      - in the text box input window, click on the rightmost shaded box, in this example $\boxed{h}$, and edit the inside of this box to multiply by GeoGebra's name for segment EC, in this example j. Hit Enter.
      - The input box should resemble: AE*EC= h * j = h*j
      - Click OK.

      Do the same for BE and ED. Notice that as you move the points, the lengths of the segments change, but the products $AE \cdot EC$ and $BE \cdot ED$ remain equal to each other.

   b. Geometric verification:

      Construct a rectangle on AE where the other side has length CE. Do this so that the rectangle lies on the opposite side of AC as B. Measure its area.

Construct a rectangle on BE where the other side has length DE. Do this so that the rectangle lies on the opposite side of BD as A. Measure its area. Hide the lines and circles used to construct each rectangle. What do you notice about these areas? Move point A to verify that your sketch is dynamic.

Save this sketch as **Lab5p5YourName.ggb**.

LAB 6

Construction of the Four "Centers" of a Triangle

To accompany section 11.2 of *Geometry: The Line and the Circle*

1. CIRCUMCENTER

   Given any triangle, the perpendicular bisectors of the sides of the triangle all meet at a point called the **circumcenter**. The name for the point stems from the word *circumscribed*. A circle is called *circumscribed about a polygon* when the circle contains all vertices of the polygon. Such a circle is called a **circumcircle**. While it is not possible to circumscribe a circle about all polygons, it *is* possible for any triangle. In fact, the circumcircle of any triangle is unique and its center is, therefore, equidistant to the vertices. Hence, this point is called the circumcenter.

   a. Start GeoGebra. Open **Lab6Starter.ggb** using GeoGebra's file menu. Construct a triangle $\triangle ABC$. Construct the circumcenter and the circumscribed circle. Rename the point *Circumcenter* and increase its point size to 6. Be sure to drag the vertices of $\triangle ABC$ to verify that the sketch is dynamic and to see how the position of the circumcenter varies as the triangle's shape changes. Does the circumcenter ever leave the interior of the triangle?

   b. In a text box, give a list of the GeoGebra steps used to construct the circumcenter and circumcircle. [Do not give Euclidean justifications, e.g. "By I.9", but instead give the steps you used within GeoGebra, e.g. "Construct Angle Bisector of $\angle PQR$." The goal here is to write an algorithm of GeoGebra construction steps that anyone familiar with GeoGebra could follow step-by-step to reproduce your dynamic diagram.]

   c. Consult Chapter 11 to give the number of the Euclidean proposition for this construction. Display this in a text box at the top of the graphics window.

Save this sketch as **Lab6circumcenterYourName.ggb**.

2. ORTHOCENTER

An **altitude** of a triangle is the perpendicular segment joining a vertex to its opposite side. Given any triangle, the altitudes of the triangle are concurrent (all meet) at a point called the **orthocenter**. When the three vertices and the orthocenter are distinct, they form an *orthocentric set* of four points. The name of this set derives from the fact that each of the four points is the orthocenter of the triangle formed by the other three points.

a. Open **Lab6Starter.ggb** using GeoGebra's file menu. Construct a triangle $\triangle ABC$. Construct the orthocenter of $\triangle ABC$. Rename the point *Orthocenter* and increase its point size. Be sure to drag the vertices of $\triangle ABC$ to verify that the sketch is dynamic and to see how the position of the orthocenter varies as the triangle's shape changes. Does the orthocenter ever leave the interior of the triangle?

b. In a text box, give a list of the GeoGebra steps used to construct the orthocenter.

Save this sketch as **Lab6orthocenterYourName.ggb**.

3. INCENTER

Given any triangle, the angle bisectors meet at a point called the **incenter**. The name for the point stems from the word *inscribed*. A circle is called *inscribed in a polygon* when the circle touches all sides of the polygon tangentially. Such a circle is called an **incircle**. While it is not possible to inscribe a circle in all polygons, it *is* possible for any triangle. In fact, the incircle of any triangle is unique and its center is equidistant to all three sides of the triangle. Hence, this point is called the incenter.

a. Open **Lab6Starter.ggb** using GeoGebra's file menu. Construct a triangle $\triangle ABC$. Construct the incenter of the triangle, then construct the incircle. Rename the point *Incenter* and increase its point size. Be sure to drag the vertices of $\triangle ABC$ to verify that the sketch is dynamic and to see how the position of the incenter varies as the triangle's shape changes. Does the incenter ever leave the interior of the triangle?

b. In a text box, give a list of the GeoGebra steps used to construct the incenter.

c. Consult Chapter 11 to give the number of the Euclidean proposition for this construction. Display this in a text box at the top of the graphics window.

Save this sketch as **Lab6incenterYourName.ggb**.

4. CENTROID

A **median** of a triangle is a segment that joins a vertex to the midpoint of the opposite side. Given any triangle, the medians of the triangle meet at a point called the **centroid**. The centroid is the center of gravity of the triangle.

a. Open **Lab6Starter.ggb** using GeoGebra's file menu. Construct a triangle $\triangle ABC$. Construct the centroid of the triangle. Rename the point *Centroid* and increase its point size. Be sure to drag the vertices of $\triangle ABC$ to verify that the sketch is dynamic and to see how the position of the centroid varies as the triangle's shape changes. Does the centroid ever leave the interior of the triangle?

b. In a text box, give a list of the GeoGebra steps used to construct the centroid.

Save this sketch as **Lab6centroidYourName.ggb**.

5. EULER LINE

In 1763, Leonhard Euler (1707–1783) was the first mathematician to prove that the circumcenter, orthocenter and centroid of any triangle are collinear. Consequently, this line is now called the **Euler line**. Euler also discovered that the distance from the centroid to the orthocenter is twice the distance from the centroid to the circumcenter.

a. Open **Lab6Starter.ggb** using GeoGebra's file menu. Construct a triangle $\triangle ABC$. Construct the circumcenter, orthocenter and centroid of the triangle. (To distinguish between the various lines in the cluttered diagram, it helps to use a new color for all of the construction elements related to a particular point.) Rename the circumcenter as M, the orthocenter as O, and the centroid as T. Hide any objects that were needed for the constructions but are not needed now. Construct the Euler line. Be sure to drag the vertices of $\triangle ABC$ to verify that the sketch is dynamic.

b. Use the Measurement toolbox to verify Euler's related result, that the distance from centroid T to orthocenter O is twice the distance from the centroid to circumcenter M. Use a dynamic text box to show the comparison of these two measurements. See the end of this lab for a refresher on producing a dynamic text box.

Save this sketch as **Lab6EulerLineYourName.ggb**.

Performing arithmetic on dynamic text within a text box

To perform arithmetic on GeoGebra objects within a text box, first follow the instructions given in earlier labs to locate and insert GeoGebra's internal object name in a text box. For example, if GeoGebra's object name for segment TM is r, insert the following into a text box input window: length(TM) + length(TM) = $\boxed{\text{r}}$ + $\boxed{\text{r}}$ = $\boxed{\text{r+r}}$. The trick to producing $\boxed{\text{r+r}}$ begins with the input of the single object $\boxed{\text{r}}$. Clicking on this $\boxed{\text{r}}$ allows for editing and thus, entering a formula. In this case, we enter $r + r$. Alternatively, try $2r$ to see if that works.

LAB 7

Regular Polygons

To accompany sections 11.3 – 11.5 of *Geometry: The Line and the Circle*

A polygon is **regular** if all of its sides and all of its interior angles are equal. A polygon is **inscribed in a circle** when each vertex of the polygon lies on the circumference of the circle. A polygon is **circumscribed about a circle** when each side of the polygon meets the circumference of the circle tangentially. The focus of Book IV of Euclid's *Elements* is the construction of inscribed and circumscribed regular polygons, including triangles, squares, hexagons, and a fifteen-sided regular polygon called a regular pentadecagon. In this lab we inscribe a regular triangle, square, pentagon, hexagon, octagon and pentadecagon in a given circle.

1. Inscribing a Regular Hexagon (Prop. IV.15) and an Equilateral Triangle

   a. Start GeoGebra. Open **Lab7Starter.ggb** using GeoGebra's file menu. Construct a circle with center O. Construct a point on the circle and name it A. Construct a circle with center A and radius AO. Construct the points of intersection for the two circles. Name one of them B and hide the other. Construct a circle with center B and radius AO. Construct the new point of intersection with the circle centered at O, and label it C. Continue this process to make your way around the circle, labelling the new intersections D, E and F. Join AB, BC, CD, DE, EF and FA to form hexagon $ABCDEF$ inscribed in the original circle Hide the five circles at the circumference of the original circle that were used to construct points B through F, leaving only the hexagon inscribed in the original circle.

   b. Measure the interior angles of the hexagon, starting with $\angle ABC$, to verify that they are equal. Measure the sides of the hexagon, starting with AB, to verify that they are equal. Drag vertex A to verify that the sketch responds dynamically and that these measures remain equal.

c. One of the easiest ways to inscribe an equilateral triangle in a given circle is to pick every other vertex of the inscribed regular hexagon (for example, A, C and E) and join them. Do this using a different color.

d. Measure the sides of $\triangle ACE$ to verify that they are equal. Measure the interior angles of $\triangle ACE$ to verify that they are equal.

Save this sketch as **Lab7p1YourName.ggb** with the original circle and the inscribed regular hexagon and equilateral triangle.

2. Inscribing a Square (Prop. IV.6) and Constructing a Regular Octagon

a. Open **Lab7Starter.ggb** using GeoGebra's file menu. Construct a circle with center O. Construct a point on the circle and name it A. Use a ray to extend AO until it intersects the circle again, and name this intersection point C. Construct a line perpendicular to AC at O. Name the intersection points of this line and the circle as B and D. Join AB, BC, CD and DA to form quadrilateral ABCD inscribed in the original circle. Hide the construction elements used to produce points B, C and D.

b. Measure the sides of quadrilateral ABCD to verify that they are all equal. Measure the interior angles of the quadrilateral to verify that they are all right angles.

c. While the measurements taken in #2b allow us to check whether ABCD meets the conditions of a square for the given diagram, this verification is not a general mathematical proof. In a text box, **prove** that your construction is a square. Do not merely copy Euclid's proof! Be sure to consult the definition of a square and write a proof using modern notation.

d. To construct an octagon from the inscribed square, start by bisecting each side of the square. Create rays from the center of the circle to each midpoint. Construct the intersection of these rays with the circle and label them E, F, G and H. Construct segments from each new point to the adjacent vertices of the square in order to form an octagon inscribed in the original circle. Hide the unneeded construction elements.

e. Measure the sides of the octagon to verify that they are equal. Measure the interior angles at vertices E, F, G and H. Would your construction have worked if, instead, you had bisected each of the square's central angles, ∠AOB, ∠BOC, ∠COD and ∠AOD, to construct E, F, G and H? Explain.

Save this sketch as **Lab7p2YourName.ggb** with the original circle, the inscribed square and regular octagon, the proof, and answer to #2e.

3. Inscribing a Regular Pentagon (Proposition IV.11)

   a. Open **Lab7Starter.ggb** using GeoGebra's file menu. Construct a circle with center O. Construct a point on the circle and name it P_1. (Note: To display a subscript in GeoGebra, change the name to P_1.) Use the construction steps given by Euclid in Proposition IV.10 to construct angle $\angle P_1OP_2 = 72°$, where P_2 is another point on the original circle. [Hint: Yes, you must use the earlier constructions of Propositions II.11, IV.10, and, perhaps I.23, along the way.]
   b. Now that we have a 72° central angle in our given circle, use this to inscribe a regular pentagon in the original circle. Please label the other vertices of the pentagon as P_3, P_4 and P_5 as you construct them. Measure the sides of the pentagon, P_1P_2, P_2P_3, P_3P_4, P_4P_5 and P_5P_1, to verify that they have the same length. Also, check that the interior angles of the pentagon have equal measure.

Save this sketch as **Lab7p3YourName.ggb** with the original circle, the 72° angle construction and the inscribed regular pentagon. There are many elements in this sketch and you may hide what you wish.

   c. Open **Lab7Starter.ggb** using GeoGebra's file menu. Construct a circle with center A. Construct a point on the circle and name it B. The German artist Albrecht Dürer gave the following construction:

      Given a circle with center A, let B be a point on the circle. Bisect the segment AB at a point C. Construct a perpendicular to AB through A and let D be its intersection with the circle. Join CD and construct a circle with center C and radius CD. Let E be its intersection with $\overrightarrow{BA}$. Claim: The length of a side of a regular pentagon inscribed in the circle is equal to DE.

      Using Dürer's instructions, inscribe a pentagon in your circle. Which construction do you prefer?

Save this sketch as **Lab7p3DurerYourName.ggb**

4. Inscribing a Regular Fifteen-sided Polygon (Proposition IV.16)

   a. Open **Lab7Starter.ggb** using GeoGebra's file menu. Construct a circle with center O. Construct a point on the circle and name it A. Inscribe both a regular pentagon and an equilateral triangle that share vertex A. Name the pentagon vertices A, P_2, P_3, P_4 and P_5, and the triangle vertices A, T_2, T_3, proceeding clockwise around the circle for both $\pentagon AP_2P_3P_4P_5$ and $\triangle AT_2T_3$. Hide everything except the original circle, the pentagon and the triangle. (If you constructed a regular hexagon to create your equilateral triangle, hide it.)

   b. When the vertices are named properly, segments AP_2 and AT_2 should be on the same half of the original circle as determined by diameter OA. Consider the arc $\widehat{T_2P_2}$ of the given circle. Since arc $\widehat{AT_2}$ is one-third of the circle and arc $\widehat{AP_2}$ is one-fifth of the circle, then arc $\widehat{T_2P_2} = \widehat{AT_2} - \widehat{AP_2}$ must be two-fifteenths of the circle since $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$. Bisect arc $\widehat{T_2P_2}$ at E. (Explain how you did this.) Use T_2E to construct an inscribed regular fifteen-sided polygon, also called a regular 15-gon or regular pentadecagon. Name the vertices of this 15-gon F_1 through F_{15}. Take some measurements to check your construction. Also, drag points to verify that the sketch is dynamic.

Save this sketch as **Lab7p4YourName.ggb** with the original circle, the pentagon, the triangle and the fifteen-sided polygon. There are many elements in this sketch and you may hide what you wish.

LAB 8

The Poincaŕe Half-plane Model for Hyperbolic Geometry

To accompany sections 12.2 and 13.3 of *Geometry: The Line and the Circle*

In the Poincaré Half-plane model of Hyperbolic geometry, points exist on one side of a horizontal boundary line, for convenience, usually the upper-half Cartesian plane. In this model a hyperbolic line is either bowed (semicircle whose center is on the boundary) or vertical. A segment is, therefore, either an arc of a semicircle or a segment of a vertical line. In this lab we restrict our consideration to the bowed lines of the Poincaré Half-plane model and create new tools for constructing lines and line segments. We use these tools to illustrate the Characteristic Axiom and to measure angles in the model.

1. Creating a Tool to Construct Lines in the Poincaré Half-plane Model

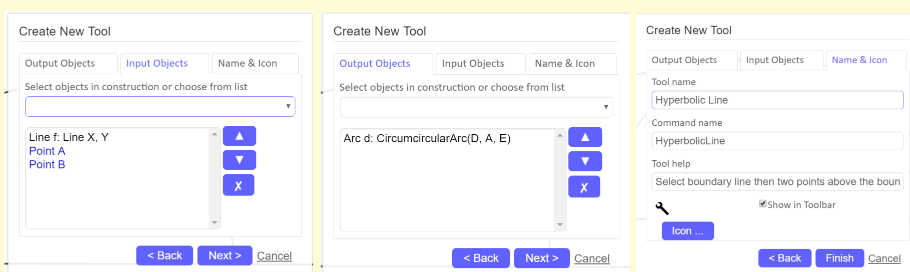
a. Start GeoGebra. Open **Lab8Starter.ggb** using GeoGebra's file menu. Construct a horizontal (Euclidean) line through points named X and Y near the bottom of your window. Next, hide points X and Y. Make the line style dashed. This will be our boundary line. Construct two points, A and B, above the boundary line that are not aligned vertically. Construct the perpendicular bisector of the (Euclidean) segment AB. Construct the intersection of the perpendicular bisector with the boundary line. Label this point C. Construct the circle with center C and radius AC. Construct the intersection points of this circle with the boundary line, and name these points D and E from left to right.

b. Select the new **Circumcircular Arc** tool added to the Circle toolbox, then select, in order, points D, A and E. As the mini tool icon advertises, this tool constructs the arc of the circle from point D through A to E. Hide the perpendicular bisector, the full circle, and points C, D and E. The only visible figures should be A, B, the original dashed boundary line, and the semicircle. Save this file as **Lab8HypLine.ggb**.

c. We can use the current sketch to create a custom GeoGebra tool that will construct a hyperbolic line in the Half-plane model when provided a boundary line and two points above it that are not aligned vertically. We give the steps of the custom tool creation in the box below as a refresher.

Creating the custom hyperbolic line tool

* In the main menu ☰, select **Tools**, then **Create New Tool** which opens a Create New Tool window. A tool is essentially a function; it has inputs and outputs. Here, the inputs will be the boundary line and two points above the boundary line, and the output will be a bowed hyperbolic line through the given points. In the Create New Tool window, select the **Input Objects** box, click the drop-down box arrow ▼, then select GeoGebra's Object name for line XY. It should be **Line f: X,Y**. When selected it will appear in the list below. Next, click the drop-down box arrow again and select **Point A**. Lastly, click the drop-down box arrow again and select **Point B**. After these selections are made in the specified order, the Input list consists of three objects and should resemble the first screen image shown below. If yours is significantly different, then start over.



* Select the **Output Objects** tab. Click the drop-down box arrow ▼, and select GeoGebra's Object name for the circumcircular arc through A and B. It should be similar to **Arc d: CircumcircularArc(D, A, E)**,

and the Output list should resemble the middle screen image shown above.

* Select the **Name & Icon** tab to name our new tool.
  - Enter *Hyperbolic Line* in the **Tool Name** box. (The **Command Name** will be automatically generated as HyperbolicLine with no spaces.)
  - In the **Tool Help** input box, enter *Select boundary line then two points above the boundary.* (These instructions will be displayed whenever you select the tool.)
  - Leave the check mark for **Show in Toolbar**.

  These entries should resemble the last screen image shown above.
* Select **Finish** and the new tool appears in the tool menu with a wrench icon.

d. Delete the semicircle, the line XY, and all points. (Yes, delete everything!)

e. Verify that the sketch appears empty.

Resave this sketch as **Lab8HypLine.ggb**.

2. Constructing Lines in the Poincaré Half-plane Model

a. Open **Lab8HypLine.ggb** using GeoGebra's file menu. Save this file as **Lab8p2YourName.ggb**.

b. Create a horizontal line near the bottom of your window. Change the line style to dashed. Hide the points used to construct the line.

c. Use your new Hyperbolic Line tool to create several hyperbolic lines in the Poincaré Half-plane model. [Begin by clicking on the boundary line, then choose two points above the boundary line.] Drag the points around to verify that the sketch responds dynamically.

Resave this sketch as **Lab8p2YourName.ggb**.

3. Illustrating the Characteristic Axiom

The Characteristic Axiom for Hyperbolic geometry is: *Through a point not on a given straight line, there exist at least two straight lines that are parallel to the given* *line*. Let's use the new tool to illustrate the Characteristic Axiom for two cases in the Poincaré Half-plane model.

a. Open **Lab8HypLine.ggb** using GeoGebra's file menu. Save this file as **Lab8p3YourName.ggb**.

b. Create a horizontal line near the bottom of your window. Change the line style to dashed. Hide the points used to construct the line.

c. Use your new Hyperbolic Line tool to draw a hyperbolic line between two points labelled A and B. Color this line.

d. Construct a new point C above the boundary line but not on hyperbolic line AB. The two cases we consider depend on the location of C: either C lies in the region between line AB and the boundary line, or outside this region. To illustrate the first case of the Characteristic Axiom, move C to the region between line AB and the boundary line. Illustrate the Characteristic Axiom of Hyperbolic geometry by constructing two hyperbolic lines through C that do not intersect (and hence are parallel to) hyperbolic line AB. Color C and the lines through C using the same color, but a different color than line AB.

e. Next, pick a point outside the region between line AB and the boundary line and label it D. Construct two hyperbolic lines through D that do not intersect (and hence are parallel to) line AB. Color D and the lines through D with a new color.

Resave this sketch as **Lab8p3YourName.ggb**.

4. Creating a Tool to Construct Line Segments in the Half-plane Model

a. Open **Lab8Starter.ggb** using GeoGebra's file menu. Construct the horizontal boundary line through points named X and Y near the bottom of your window. Change the line style to dashed. Next, hide points X and Y. Construct two points, A and B, above the boundary line. Construct the perpendicular bisector between A and B. Construct the intersection of the perpendicular bisector with the boundary line. Label this point Z. Construct the circle with center Z and radius AZ. Construct the intersection points of this circle with the boundary line, and name these points C and D from left to right.

b. Construct the semicircle above the boundary line by using the **Circumcircular Arc** tool, and selecting, in order, points C, A and D. Next, hide the full circle.

Construct the intersection of the perpendicular bisector and the semicircle, and name this point E.

c. Select the **Circumcircular Arc** tool then select, in order, points A, E and B. Hide the perpendicular bisector, the semicircle, and points C, D and E. Hide the label for point Z, but leave the point in the sketch. The only visible figures should be A, B, the hyperbolic line segment AB, the original dashed boundary line, and boundary point Z without its label. Drag point A to verify that the sketch is dynamic, and that the other elements of the sketch respond accordingly. Save this file as **Lab8HypSegment.ggb**.

d. Use the instructions in #1c as a guide to create a new Hyperbolic Segment tool. The Inputs should be the boundary line XY, point A and point B. The Outputs should be the arc from A to B [Arc e: CircumcircularArc(A, E, B)] and point Z. Name the tool **Hyperbolic Segment**. For **Tool Help**, enter *Select boundary line then two points above the boundary.*

e. Delete the arc, the line XY, and all points. (Yes, everything!)

f. Verify that the sketch appears empty.

Resave this sketch as **Lab8HypSegment.ggb**.

5. HYPERBOLIC TRIANGLES

A triangle is a three-sided polygon where each side is a line segment. Let's use our new hyperbolic line segment tool to create a hyperbolic triangle in the Poincaré Half-plane model.

a. Open **Lab8HypSegment.ggb** using GeoGebra's file menu. Save this as **Lab8p5YourName.ggb**.

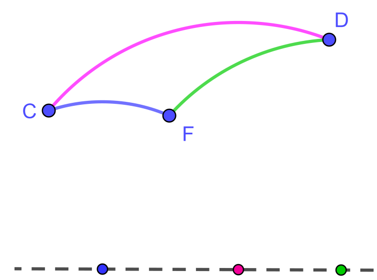
b. Create a horizontal line near the bottom of your window and change the line style to dashed. Hide the points used to construct the line.

c. Use your new **Hyperbolic Segment** tool to create a hyperbolic segment CD in the Poincaré Half-plane model. [Begin by clicking on the boundary line, then choose two points above the boundary line.] Drag the points to verify that the sketch responds dynamically. Color the arc CD and its associated point on the boundary line using the same color.

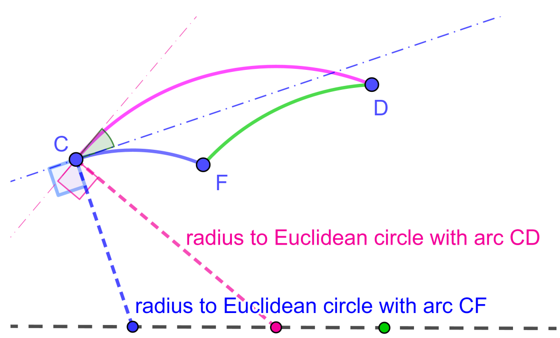
d. Create a new hyperbolic segment CF, where F is a new point. Color the arc CF and its associated point on the boundary line using the same color, but a different color than that of CD and its associated point.

e. Create the hyperbolic segment FD. Color FD and its associated point on the boundary line using the same color, but a different color than those already in use.

f. This creates a hyperbolic triangle, $\triangle CDF$. Notice that each side of the triangle is paired with one of the points on the boundary line, and we can identify the associated boundary point by its color. (Remember, every point on an arc is equidistant to the boundary point associated with the arc using the Euclidean distance metric.) Move the vertices of $\triangle CDF$ to resize the triangle and verify that the sketch is dynamic.



g. In the Half-plane model, the measure of $\angle DCF$, the interior angle of $\triangle CDF$ at vertex C, is the measure of the angle formed by the Euclidean tangent lines to the hyperbolic lines CD and CF. To measure this angle, we must first construct the two tangent lines at C. Since hyperbolic lines CD and CF are arcs of Euclidean circles, each tangent line is tangent to a Euclidean circle that goes through C and has its center at a point on the dashed boundary line. Recall from Proposition III.16 and III.18 that a line is tangent to a circle if and only if it is perpendicular to the radius at the point of tangency. Therefore, construct the two radii at C, then construct the perpendiculars to the radii. By III.16, these perpendiculars *are* the tangent lines at C, and the angle between them is the desired interior angle of $\triangle CDF$ at C. Use this to measure $\angle DCF$.



To display a dynamic LaTex-formatted angle measurement, for example, $\angle DCF = 31.12°$

- Change the caption for the angle to: `$\angle DCF = %v$`.
- Change the option for Show Label to: Caption.

Here, %v is the code used to display the value of the GeoGebra object.

h. Repeat this process at vertices D and F to determine the other interior angles of hyperbolic triangle $\triangle CDF$, $\angle CDF$ and $\angle DFC$.

i. After you have found all three interior angles of $\triangle CDF$, create a dynamic text box to display the sum of the interior angles. Drag the vertices of the triangle to verify that the calculation is dynamic and that, in accordance with I.32$_H$, the sum is always less than two right angles.

j. Before saving, tidy up the sketch by hiding, coloring, or minimizing objects so that the clutter of the tangent lines does not overwhelm the sketch.

Resave this sketch as **Lab8p5YourName.ggb**.

LAB 9

Isometries of the Plane: Reflections

To accompany section 15.2 of *Geometry: The Line and the Circle*

An **isometry** is a mapping that preserves distance. In this lab we explore one type of isometry, a reflection about a line, with the help of a GeoGebra tool in the new Transformation toolbox: **Reflect about Line**. The other tools in this toolbox will be utilized in the next lab when we explore two other types of isometries.

1. REFLECTIONS ABOUT A LINE

   A reflection acting on any geometric figure is determined by a line.

   a. Start GeoGebra. Open **Lab9Starter.ggb** using GeoGebra's file menu. Construct $\triangle ABC$ and color its interior.

   b. Construct a new line and name it k.

   c. Select the **Reflect about Line** tool, then choose the interior of $\triangle ABC$ followed by line k. This produces $\triangle A'B'C'$, the image of $\triangle ABC$ under the reflection about line k.

   d. Move line k to verify that your sketch is dynamic, and note the effect on the image of $\triangle ABC$ in order to answer the following questions on general properties of reflections.

   e. In general, what are the fixed points of a reflection about a line ℓ? Give your answer to this question, and all questions that follow, in a text box. Please start each new text box with the appropriate heading, e.g. *"Answer to #1e:"*, labelling the others accordingly.

   f. In general, how is a point related to its image under a reflection about a line ℓ? Give your answer in a text box.

g. In general, describe the inverse of a reflection about a line ℓ. Give your answer in a text box.

h. Construct a new line. Name it m and give it a different color than k.

i. We compose two reflections by performing one followed by the other. Reflect $\triangle ABC$ across k, then reflect the resulting image across line m. You may have to zoom out to see all figures of your sketch. Leave the intermediate image and the final image in your sketch. Give both of these triangles the same color, but a different color than $\triangle ABC$. Give the final image the caption, "k followed by m."

j. Perform the composition of reflections again, but reverse the order of the lines. Leave the intermediate image and the final image in your sketch. Give both of these triangles the same color, but a different color than any other color in your sketch. Give the final image the caption, "m followed by k."

k. Does order of composition matter when we compose two different reflections? Give your answer in a text box.

Save this sketch as **Lab9p1YourName.ggb**.

2. Proof of Theorem 15.15

While there are other types of isometries, the reflection is the queen of isometries since all others can be reproduced through a careful combination of reflections. Theorem 15.15 establishes this with the statement:

Every isometry of Neutral geometry is the composition of three or fewer reflections.

In order to explore the proof of this theorem, we must state Corollary 15.10: *If two isometries agree at three noncollinear points, then they agree everywhere.* Thus, we start with two congruent triangles in the plane and then work to see why one can be mapped to the other with the composition of three or fewer reflections.

a. Open **Lab9Starter.ggb** using GeoGebra's file menu. Construct $\triangle ABC$ where the vertices are oriented alphabetically in a clockwise direction. In this case, we say that $\triangle ABC$ has clockwise orientation. Next, color its interior.

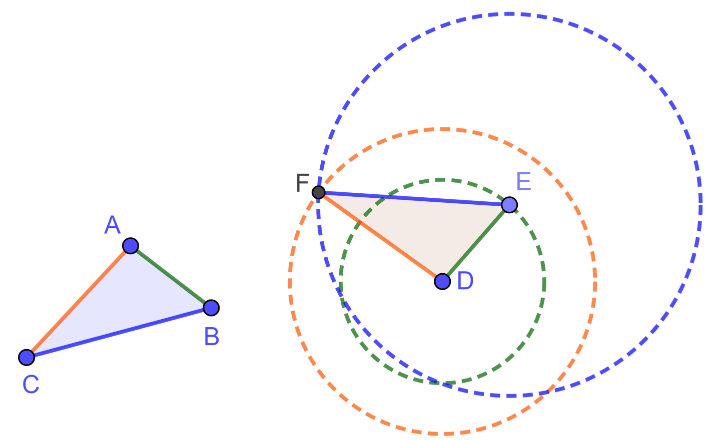
b. Construct a point D in the plane where D does not lie on or inside $\triangle ABC$.

c. Using D as a center, construct two circles, one with radius AB and the other with radius AC.

d. Pick an arbitrary point on the circle with center D and radius AB, and label it E.

e. Using E as its center, construct a circle with radius BC.

f. Label one of the two intersections of the circle with center D and radius AC, and the circle with center E and radius BC, as F.



g. Create triangle $\triangle DEF$. Hide the three circles used to create triangle $\triangle DEF$.

h. Note the congruency of $\triangle ABC$ and $\triangle DEF$. Drag the vertices of $\triangle ABC$ to verify that the sketch is dynamic, and that $\triangle DEF$ remains congruent. In a text box, prove that $\triangle ABC \cong \triangle DEF$.

i. What orientation does $\triangle DEF$ have, clockwise or counterclockwise? Move the vertices of your original triangle so that $\triangle ABC$ and $\triangle DEF$ have the same orientation.

j. Move the vertices of $\triangle ABC$ so that $\triangle ABC$ and $\triangle DEF$ have opposite orientations.

Save a copy of this sketch as **Lab9p2YourName.ggb**.

3. Finding the lines of reflection in the proof of Theorem 15.15

a. CASE 1: THREE REFLECTIONS

We start with the case where $\triangle ABC$ and $\triangle DEF$ have opposite orientations.

i. Continue with your current sketch where $\triangle ABC$ and $\triangle DEF$ have opposite orientations. Join AD. Construct the perpendicular bisector for AD and label this line as k.

ii. Reflect $\triangle ABC$ about line k. To do this, select the **Reflect about Line** tool, then choose the interior of $\triangle ABC$ followed by line k. As demonstrated in Figure 9.1, this produces $\triangle A'B'C'$, the image of $\triangle ABC$ under the reflection about line k. In particular, this transformation maps A to A', B to B', and C to C'. Notice that A' coincides with point D. In a text box, explain why this is the case.

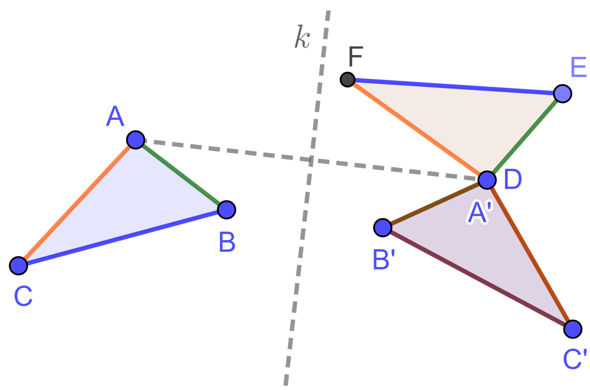


Figure 9.1: Line k is the first line of reflection.

iii. Join $B'E$ and construct its perpendicular bisector, naming this line ℓ.

iv. Construct $\triangle DB'C'$ then reflect it about line ℓ. As demonstrated in Figure 9.2, this produces $\triangle D'B''C''$, where D maps to D', B' maps to B'', and C' maps to C''. Notice that B'' coincides with E, and D' coincides with D. Thus, D is a fixed point of this reflection. In a text box, explain why $B'' = E$ and $D' = D$.

v. What line of reflection will transform $\triangle DEC''$ into a triangle that coincides with $\triangle DEF$? Construct this line then verify that the reflection of $\triangle DEC''$ about this line produces the desired result. In a text box, explain why this is the case.

Save a copy of this sketch as **Lab9p3Case1YourName.ggb**.

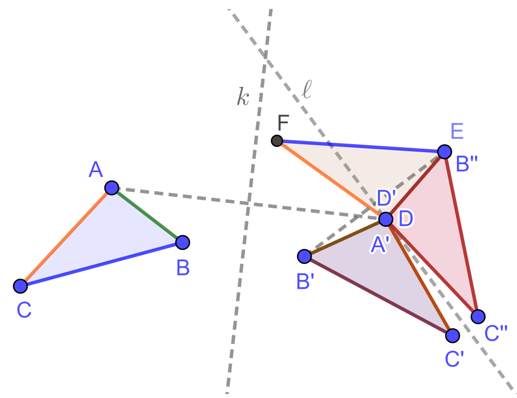


Figure 9.2: Line ℓ is the second line of reflection.

b. Case 2: Two reflections

By Corollary 15.10, when two isometries agree at three noncollinear points then they agree everywhere. Thus, the images of any three noncollinear points determine an isometry. In Case 1, the noncollinear points A, B and C were mapped to D, E and F, respectively, through the composition of three reflections. Therefore, we know that every isometry can be written as the composition of at most three reflections. As we will see in this case, when $\triangle ABC$ and $\triangle DEF$ have the same orientation, exactly two reflections are required to map one triangle to the other.

i. Continue with your current sketch. Modify the sketch so that $\triangle ABC$ and $\triangle DEF$ have the same orientation. What is the image of $\triangle DB'C'$ after it is reflected about line ℓ? In a text box, use the orientation of triangles $\triangle ABC$ and $\triangle DEF$ to explain why exactly two reflections are needed in this case.

Save a copy of this sketch as **Lab9p3Case2YourName.ggb**.

ii. Keeping the orientation of $\triangle ABC$ and $\triangle DEF$ the same, drag the vertices of $\triangle ABC$ so that the two lines of reflection intersect, as demonstrated in Figure 9.3. Also, hide labels D', B'', C'', and the text box. In the next lab, we will see that a single isometry maps $\triangle ABC$ to $\triangle DEF$ in this case. Since we'll reexamine this sketch after exploring two more isometries in the next lab, save a copy of this sketch with the intersecting lines of reflection as **Lab9Case2IntersectYourName.ggb**.

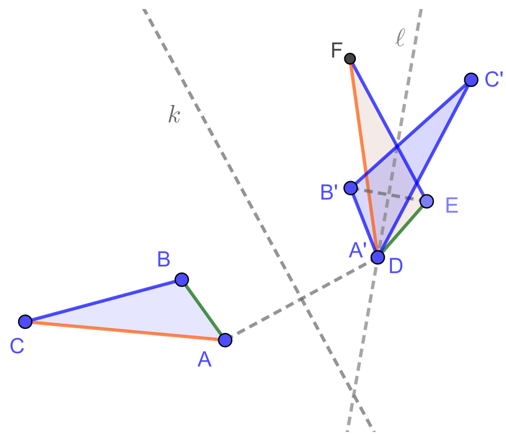


Figure 9.3: Reflection lines k and ℓ intersect

iii. Continue with your current sketch. Keeping the orientation of $\triangle ABC$ and $\triangle DEF$ the same, this time drag the vertices of $\triangle ABC$ so that the two lines of reflection are parallel, as demonstrated in Figure 9.4. In the next lab, we will see that a single isometry maps $\triangle ABC$ to $\triangle DEF$ in this case as well. Since we'll reexamine this sketch after exploring two more isometries in the next lab, save a copy of this sketch with the parallel lines of reflection as **Lab9Case2ParallelYourName.ggb**.

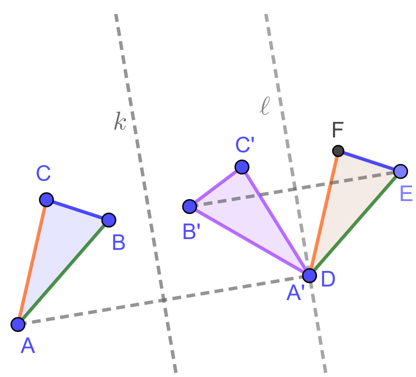


Figure 9.4: Lines of reflection are parallel

c. CASE 3: ONE REFLECTION.

In the last case, only one reflection is required to map one triangle to the other. Using GeoGebra's file menu, open **Lab9p2YourName.ggb**, where $\triangle ABC$ and $\triangle DEF$ have opposite orientation. Save this sketch as **Lab9p3Case3YourName.ggb**.

i. Delete reflection line ℓ. This will delete $\triangle D'B''C''$ simultaneously.

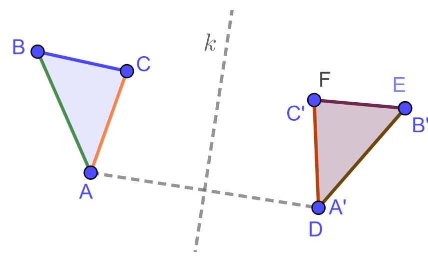


Figure 9.5: One reflection line

ii. Move the vertices of triangle $\triangle ABC$ so that its reflection across k coincides with $\triangle DEF$, as demonstrated in Figure 9.5. How are the orientations of triangles $\triangle ABC$ and $\triangle DEF$ related? Answer this question in a text box, and be as specific as possible.

Resave this sketch as **Lab9p3Case3YourName.ggb**.

LAB 10

Isometries of the Plane: Translations and Rotations

To accompany sections 15.2 and 15.3 of *Geometry: The Line and the Circle*

An **isometry** is a mapping that preserves distance. In this lab we explore two other types of isometries, translations and rotations, with the help of a few new GeoGebra tools in the Transformation toolbox: **Translate by Vector** and **Rotate Around Point**.

1. TRANSLATIONS

   A translation of any geometric figure is determined by a vector, say $\vec{v}$.

   a. Start GeoGebra. Open **Lab10Starter.ggb** using GeoGebra's file menu. Construct $\triangle ABC$ and color its interior.

   b. Construct segment MN.

   c. Select the **Translate by Vector** tool, then choose the interior of $\triangle ABC$ followed by point M then N. This produces $\triangle A'B'C'$, the image of $\triangle ABC$ under the translation by the direction and length of vector $\overrightarrow{MN}$. [Since vector notation is nearly identical to the notation for the ray $\overrightarrow{MN}$, we will be careful to clearly state our intended meaning.]

   d. Move the endpoints of vector $\overrightarrow{MN}$ to verify that your sketch is dynamic, and note the effect on the image of $\triangle ABC$.

   e. In general, describe the fixed points of a translation by a vector $\overrightarrow{PQ}$. Give your answer to this question, and all questions that follow, in a text box. Please start each new text box with the appropriate heading, e.g. *"Answer to #1e:"*, labelling the others accordingly.

f. In general, how is a point related to its image under a translation by a vector $\overrightarrow{PQ}$? Give your answer in a text box.

g. In general, describe the inverse of a translation by a vector $\overrightarrow{PQ}$. Give your answer in a text box.

h. Construct a new segment LK and give it a different color than MN.

i. We compose two translations by performing one followed by the other. Translate $\triangle ABC$ by vector $\overrightarrow{MN}$, then translate the resulting image by vector $\overrightarrow{LK}$. You may have to zoom out to see all figures of your sketch. Leave the intermediate image and the final image in your sketch. Give both of these triangles the same color, but a different color than $\triangle ABC$. Give the final image the caption, "vector MN followed by vector LK."

j. Perform the composition of translations again, but reverse the order of the vectors. Leave the intermediate image and the final image in your sketch. Give both of these triangles the same color, but a different color than any other color in your sketch. Give the final image the caption, "vector LK followed by vector MN."

k. Does order of translation matter when we compose two different translations? Give your answer in a text box.

Save this sketch as **Lab10p1YourName.ggb**.

2. ROTATIONS

A rotation is determined by the *center* of rotation, O, and the *angle* of rotation, α.

a. Open **Lab10Starter.ggb** using GeoGebra's file menu. Construct $\triangle ABC$ and color its interior.

b. Construct a new point and name it O.

c. Construct an angle $\angle DEF$. Measure the angle, and show the label and measurement.

d. Select the **Rotate Around Point** tool. This tool requires three inputs: the object to be rotated followed by the center of rotation then the angle of rotation. First, note the name that GeoGebra gave $\angle DEF$. We will work under the assumption that it is named α. After selecting tool , choose the interior of $\triangle ABC$ followed by point O. We now enter the angle of rotation in the box of the new window that opens. (To input the Greek letter α for the angle measure, hold the ALT key and type the letter a.) Leave the **counterclockwise** direction option checked for this and all other rotations. This produces $\triangle A'B'C'$, the image of $\triangle ABC$ under the rotation by angle $\alpha = \angle DEF$ about center O.

e. Move point O to verify that your sketch is dynamic, and note the effect on the image of $\triangle ABC$.

f. Move point E to change $\angle DEF$ and verify that your sketch is dynamic. Note the effect on the image of $\triangle ABC$.

g. In general, describe the fixed points of a rotation by angle θ about a center Q. Give your answer to this question, and all questions that follow, in a text box. Please start each new text box with the appropriate heading, e.g. *"Answer to #2g:"*, labelling the others accordingly.

h. In general, how is a point related to its image under a rotation by angle θ about a center Q? Give your answer in a text box.

i. In general, describe the inverse of a rotation by angle θ about a center Q. Give your answer in a text box.

j. Pick two **different** centers and rotate $\triangle ABC$ about one center and then the other, using $\angle DEF$ for both rotations. Give the intermediate image and final image the same color. Give the final image the caption, "about center X first", as appropriate.

k. Perform the composition of rotations again, but reverse the order of the centers. Leave the intermediate image and the final image in your sketch. Give both of these triangles the same color, but a different color than any other color in your sketch. Give this final image the appropriate caption, "about center X first."

l. Does order of rotation matter when we compose two rotations with different centers? Give your answer in a text box.

Save this sketch as **Lab10p2YourName.ggb**.

3. COMPOSITION OF ISOMETRIES

Let's try to determine whether order of composition matters when composing two different types of isometries.

a. Rotation and Translation

   i. Open **Lab10Starter.ggb** using GeoGebra's file menu. Construct $\triangle ABC$ and color its interior.
   ii. Pick any rotation and translation. First, rotate $\triangle ABC$ then translate the resulting image. Leave both the intermediate and final images in your sketch. Give both of these triangles the same color, but a different color than $\triangle ABC$. Give the final image the caption, "rotation followed by translation."
   iii. Next, reverse the order by translating $\triangle ABC$ then rotating the resulting image. Leave both the intermediate and the final images in your sketch. Give both of these triangles the same color, but a different color than any other color in your sketch. Give this final image the caption, "translation followed by rotation."
   iv. Does either order of composition produce the same final image? Give your explanation in a text box.

   Save this sketch as **Lab10p3RoTrYourName.ggb**.

b. Rotation and Reflection

   i. Open **Lab10Starter.ggb** using GeoGebra's file menu. Construct $\triangle ABC$ and color its interior.
   ii. Pick any rotation and reflection. First, rotate $\triangle ABC$ then reflect the resulting image. Leave both the intermediate and final images in your sketch. Give both of these triangles the same color, but a different color than $\triangle ABC$. Give the final image the caption, "rotation followed by reflection."
   iii. Next, reflect $\triangle ABC$ then rotate the resulting image. Leave both the intermediate and the final images in your sketch. Give both of these triangles the same color, but a different color than any other color in your sketch. Give this final image the caption, "reflection followed by rotation."
   iv. Does either order of composition produce the same final image? Give your explanation in a text box.

   Save this sketch as **Lab10p3RoReYourName.ggb**.

c. Reflection and Translation

i. Open **Lab10Starter.ggb** using GeoGebra's file menu. Construct $\triangle ABC$ and color its interior.
ii. Pick any reflection and translation. First, reflect $\triangle ABC$ then translate the resulting image. Leave both the intermediate and final images in your sketch. Give both of these triangles the same color, but a different color than $\triangle ABC$. Give the final image the caption, "reflection followed by translation."
iii. Next, translate $\triangle ABC$ then reflect the resulting image. Leave both the intermediate and the final images in your sketch. Give both of these triangles the same color, but a different color than any other color in your sketch. Give this final image the caption, "translation followed by reflection."
iv. Does either order of composition produce the same final image? Give your explanation in a text box.

Save this sketch as **Lab10p3ReTrYourName.ggb**.

4. Follow-up question from Lab 9

Now that we have explored reflections, rotations and translations, let's answer two follow-up questions related to our exploration of Case 2 of the proof of Theorem 15.15.

a. Open **Lab9Case2IntersectYourName.ggb** using GeoGebra's file menu. In this case, what single isometry maps $\triangle ABC$ to $\triangle DEF$? Answer this question in a text box, and be as specific as possible.

Save this file as **Lab10IntersectYourName.ggb**.

b. Open **Lab9Case2ParallelYourName.ggb** using GeoGebra's file menu. In this case, what single isometry maps $\triangle ABC$ to $\triangle DEF$? Answer this question in a text box, and be as specific as possible.

Save this sketch as **Lab10ParallelYourName.ggb**.

To choose between radian or degree angle measure

In the upper-right corner, select File settings . Next, select  on the right for Algebra settings. From the drop-down menu for Angle Unit on the left, choose radians or degrees

as desired.

LAB 11

Constructible Lengths and Hippasian Numbers

To accompany section 16.2 of *Geometry: The Line and the Circle*

A **constructible length** is the length of any segment that can be constructed with a straightedge and compass given a segment of length 1 called a unit length. Every constructible length is a constructible number. A **Hippasian number** is a number that can be written as a combination of the addition, subtraction, multiplication, division or square root operations on the rational numbers. In this lab we will demonstrate that the Hippasian numbers are constructible lengths. To do this, we show that lengths $a + b$, $a - b$, ab, $\frac{a}{b}$, and $\sqrt{a}$ are constructible from the given lengths a, b and a unit length.

1. Addition and Subtraction

   a. Start GeoGebra. Open **Lab11Starter.ggb** using GeoGebra's file menu. In the upper-left corner of the window, create a segment that we will use as our given unit length. GeoGebra will not allow us to name this segment 1, so we need a caption.

> Caption Refresher
>
> To create and display a caption, right-click on the segment then select Settings. In the Caption box, enter 1. Below, put a check mark in the Show Label box and change the descriptor displayed to Caption as shown here: ☑ Show Label: Caption ▾. When finished, close the settings menu by clicking X.

If we intended to use GeoGebra's measurement tools, then we would need to ensure that GeoGebra's measure of this segment is 1 by shortening or lengthening this segment appropriately. This is unnecessary, however, since we will not be taking any measurements. Draw two other segments of different lengths under the unit length in the upper-left corner. Name the longer segment a, and the shorter b. Save this sketch as **Lab11Givens.ggb** since we will use it as a starting point for the other parts of this lab.

b. Create a line in the middle of the window. Use the given segments a and b to construct a segment of length $a + b$ on this line. Color the new segment, then name it $a + b$ using a caption, and increase its line width so that it is prominently displayed. Drag the endpoints of the given segments a and b to verify that the sketch responds dynamically.

c. Create another line below the first line. Use the given segments a and b to construct a segment of length $a - b$ on this line. Color the new segment, then name it $a - b$ using a caption, and increase its line width so that it is prominently displayed. Drag the endpoints of the given segments a and b to verify that the sketch responds dynamically.

Save this sketch as **Lab11p1YourName.ggb**.

2. Multiplication and Division

a. Open **Lab11Givens.ggb** using GeoGebra's file menu. Use one of the two diagrams given below as a guide to construct a segment of length ab. Color the new segment, then name it ab using a caption, and increase its line width so that it is prominently displayed. In a text box, prove why the construction works.

Save this sketch as **Lab11p2abYourName.ggb**.

b. Open **Lab11Givens.ggb** using GeoGebra's file menu. Use the other diagram as a guide to construct a segment of length $\frac{a}{b}$. Color the new segment, then name it a/b using a caption, and increase its line width so that it is prominently displayed. In a text box, prove why the construction works.

Save this sketch as **Lab11p2aOverbYourName.ggb**.

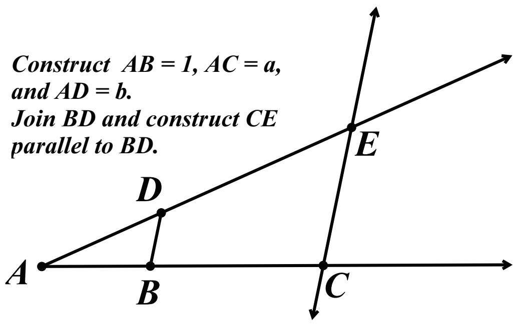


Diagram 1

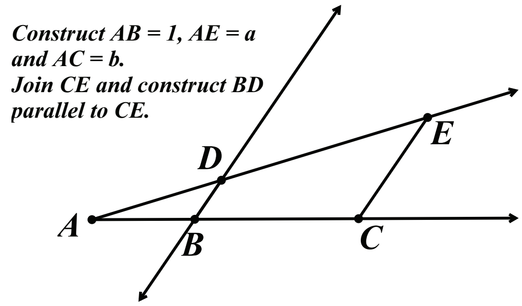


Diagram 2

3. Square Root

Open **Lab11Givens.ggb** using GeoGebra's file menu. Use the diagram given in Figure 11.1 as a guide to construct a segment of length $\sqrt{a}$. (Which segment in the diagram has length $\sqrt{a}$?) Prove why the construction works in a text box.

Save this sketch as **Lab11p3YourName.ggb**.

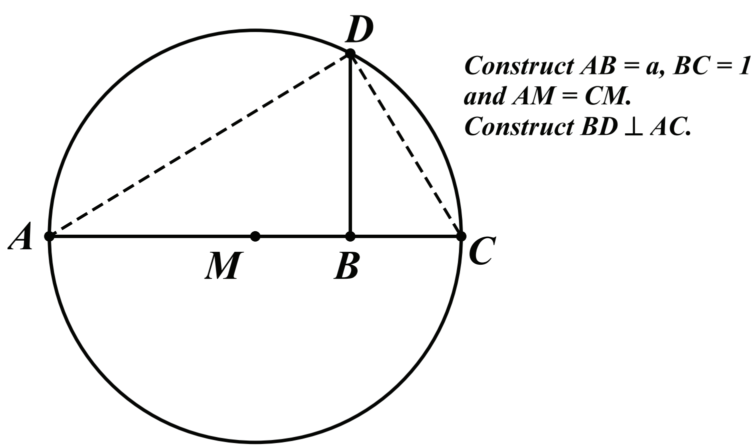


Figure 11.1: Square root construction