Chaos on the Interval
Errata

Erratum in Lemma 5.52 : the beginning of the proof was not well written. Here is the corrected version. We include the whole proof for the sake of clarity.

LEMMA 5.52. *Let f be an interval map that is not chaotic in the sense of Li-Yorke. Let $\varepsilon > 0$. Then there exist finitely many points $y_1, \ldots, y_r$ in $\omega(f)$ and an open set U containing $\omega(f)$ such that, for every point x satisfying*

$$\exists N_0, N_1 \in \mathbb{Z}^+, N_0 \leq N_1,\ \forall n \in [\![N_0, N_1]\!],\ f^n(x) \in U,$$

then there exists $i \in [\![1, r]\!]$ such that

$$\forall n \in [\![N_0, N_1]\!],\ |f^n(x) - f^n(y_i)| \leq \varepsilon.$$

PROOF. According to Proposition 5.49, for every $x \in \omega(f)$ there exists a connected neighborhood $W(x)$ of x such that

$$\forall z \in W(x) \cap \omega(f),\ \forall n \geq 0,\ |f^n(x) - f^n(z)| \leq \frac{\varepsilon}{2}. \tag{5.54}$$

Since $\omega(f)$ is compact by Corollary 5.46, there exist finitely many distinct points $x_1, \ldots, x_s$ in $\omega(f)$ such that $\omega(f) \subset W(x_1) \cup \cdots \cup W(x_s)$. We would like these sets not to overlap too much, so we replace them by smaller but more numerous sets. We define inductively on $k \in [\![1, s]\!]$ a family of connected open sets $(W_k^j)_{1 \leq j \leq \alpha_k}$ that are subsets of $W(x_k)$, and points $(x_k^j)_{1 \leq j \leq \alpha_k}$ such that $x_k^j \in W_k^j \cap \omega(f)$.

Construction at step $k \in [\![1, s]\!]$. Suppose that $(W_i^j)_{1 \leq j \leq \alpha_i}$ and $(x_i^j)_{1 \leq j \leq \alpha_i}$ have been defined for all $i \leq k-1$ (for $k = 1$, these two families are empty). We consider all the connected components C of

$$W(x_k) \setminus \{x_i^j \mid i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]\}$$

such that

$$\left(C \setminus \left(\bigcup_{i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]} W_i^j \right) \right) \cap \omega(f) \neq \emptyset.$$

We call them $W_k^1, \ldots, W_k^{\alpha_k}$ (notice that $W(x_k) \setminus \{x_i^j \mid i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]\}$ has finitely many connected components because $W(x_k)$ is connected and the set $\{x_i^j \mid i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]\}$ is finite). For every $j \in [\![1, \alpha_k]\!]$, we choose a point x_k^j in

$$\left(W_k^j \setminus \left(\bigcup_{i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]} W_i^j \right) \right) \cap \omega(f).$$

This ends the construction at step k. Note that

$$\bigcup_{i \in [\![1, k]\!], j \in [\![1, \alpha_i]\!]} W_i^j \cap \omega(f) = \bigcup_{i=1}^{k} W(x_i) \cap \omega(f).$$

To simplify the notation, we call $V_1, \ldots, V_r$ and $y_1, \ldots, y_r$ the family of sets $(W_i^j)_{i \in [\![1, s]\!], j \in [\![1, \alpha_i]\!]}$ and the associated points $(x_i^j)_{i \in [\![1, s]\!], j \in [\![1, \alpha_i]\!]}$, and we order them in order to have $y_1 < y_2 < \cdots < y_r$. Then V_i is a connected open set containing $y_i \in \omega(f)$, V_i is included in $W(x_j)$ for some $j \in [\![, s]\!]$, and $\omega(f) \subset V_1 \cup \cdots \cup V_r$. Moreover, the construction above ensures that:

$$\forall i, j \in [\![1, r]\!], i \neq j, V_i \cap V_j \subset \langle y_i, y_j \rangle$$

because V_i (resp. V_j) is an interval and does not contain y_j (resp. y_i). This implies that $V_i \cap V_j = \emptyset$ if $|i - j| \geq 2$ (that is, only intervals corresponding to consecutive

points may intersect). We modify once more these sets by an inductive construction for $i = 1, \dots, r-1$:
• if $V_i \cap V_{i+1}$ is not included in $\omega(f)$, we choose a point $x \in (V_i \cap V_{i+1}) \setminus \omega(f)$ and we replace V_i and V_{i+1} by $V_i \cap (-\infty, x)$ and $V_{i+1} \cap (x, +\infty)$ respectively; we still call these sets V_i and V_{i+1};
• if $V_i \cap V_{i+1} \subset \omega(f)$, we do not change the sets at step i.

At the end of this construction, we get intervals $V_1, \dots, V_r$ that are open set and satisfy:

$$\omega(f) \subset V_1 \cup \cdots V_r,$$
$$\forall i \in [\![1, r]\!],\ y_i \in V_i \cap \omega(f),$$
$$\forall i, j \in [\![1, r]\!],\ i \neq j,\ V_i \cap V_j \subset \omega(f).$$

This last condition implies:

$$\forall x \in \bigcup_{i=1}^{r} V_i,\ x \notin \omega(f) \Longrightarrow \exists! i \in [\![1, r]\!],\ x \in V_i. \tag{5.55}$$

Moreover, since $V_i \subset W(x_j)$ for some $j \in [\![1, s]\!]$, the triangular inequality and (5.54) imply:

$$\forall i \in [\![1, r]\!],\ \forall y, z \in V_i \cap \omega(f),\ \forall n \geq 0,\ |f^n(y) - f^n(z)| \leq \varepsilon. \tag{5.56}$$

Let $i \in [\![1, r]\!]$ and $z \in V_i \cap \omega(f)$. According to Lemma 5.50, there exists a positive integer $p_i(z)$ such that

$$\forall n \geq 0,\ f^{np_i(z)}(z) \in V_i. \tag{5.57}$$

We can assume that

$$p_i(z) \text{ is a multiple of } p_i(y_i). \tag{5.58}$$

Since f is continuous, there exists an open neighborhood $U_i(z)$ of z such that:

$$U_i(z) \subset V_i$$
$$f^{p_i(z)}(U_i(z)) \subset V_i \tag{5.59}$$
$$\forall n \in [\![0, p_i(z)]\!],\ \mathrm{diam}\,(f^n(U_i(z))) \leq \varepsilon. \tag{5.60}$$

We set

$$U_i := \bigcup_{z \in \omega(f) \cap V_i} U_i(z) \quad \text{and} \quad U := \bigcup_{i=1}^{r} U_i.$$

The sets U_i are open and satisfy:

$$\forall i \in [\![1, r]\!],\ U_i \cap \omega(f) = V_i \cap \omega(f). \tag{5.61}$$

Indeed, the inclusion $U_i \cap \omega(f) \subset V_i \cap \omega(f)$ is trivial because $U_i \subset V_i$. Conversely, if $z \in V_i \cap \omega(f)$, then $z \in U_i(z) \subset U_i$, so $V_i \cap \omega(f) \subset U_i \cap \omega(f)$.

By definition, the set U is open and contains $\omega(f)$. Let $x_0 \in U$ and $N \geq 0$ be such that

$$\forall n \in [\![0, N]\!],\ f^n(x_0) \in U. \tag{5.62}$$

We are going to show by induction the following:

Fact 1. *There exist integers $k \geq 0$ and $i_0 \in [\![1, r]\!]$ and finite sequences of points $(z_n)_{0 \leq n \leq k}$ and $(x_n)_{0 \leq n \leq k}$ such that, for all $n \in [\![0, k]\!]$,*

$$z_n \in \omega(f) \cap V_{i_0}, \quad x_n \in U_{i_0}(z_n), \quad x_{n+1} = f^{p_{i_0}(z_n)}(x_n).$$

If we set $q_0 := 0$ *and* $q_n := p_{i_0}(z_0) + \cdots + p_{i_0}(z_{n-1})$ *for all* $n \in [\![1, k+1]\!]$*, the integer* k *is such that* $q_k \leq N < q_{k+1}$.

• According to the definition of U, there exists $i_0 \in [\![1, r]\!]$ and $z_0 \in \omega(f) \cap V_{i_0}$ such that $x_0 \in U_{i_0}(z_0)$. If $q_1 := p_{i_0}(z_0) > N$, then the construction is over with $k := 0$.

• Suppose that the points $(z_n)_{0 \leq n \leq j}$ and $(x_n)_{0 \leq n \leq j}$ are already defined up to some integer j with $q_{j+1} \leq N$. We set $x_{j+1} := f^{p_{i_0}(z_j)}(x_j)$. Thus $x_{j+1} = f^{q_{j+1}}(x_0)$. Then $x_{j+1} \in U$ by (5.62) because $q_{j+1} \leq N$. Since $x_j \in U_{i_0}(z_j)$, we have $x_{j+1} \in V_{i_0}$ by (5.59). If $x_{j+1} \in \omega(f)$, then $x_{j+1} \in U_{i_0}$ by (5.61), and we set $z_{j+1} := x_{j+1}$; trivially $x_{j+1} \in U_{i_0}(z_{j+1})$. If $x_{j+1} \notin \omega(f)$, the fact that $x_{j+1} \in U$ implies that there exists $i \in [\![1, r]\!]$ such that $x_{j+1} \in U_i \subset V_i$. Necessarily, $i = i_0$ because of (5.55). Thus there exists $z_{j+1} \in \omega(f) \cap U_{i_0}$ such that $x_{j+1} \in U_{i_0}(z_{j+1})$. If $q_{j+2} := q_{j+1} + p_{i_0}(z_{j+1}) > N$, then the construction is over with $k := j+1$.

Since all the integers $p_{i_0}(z)$ are positive, the sequence (q_n) is increasing, and thus the construction finishes. This ends the proof of Fact 1.

Let x_0 satisfy (5.62) and $n \in [\![0, N]\!]$. We keep the notation of Fact 1. Let $j \in [\![0, k+1]\!]$ be such that $q_j \leq n < q_{j+1}$. We have

$$|f^n(x_0) - f^n(y_{i_0})| \leq |f^{n-q_j}(f^{q_j}(x_0)) - f^{n-q_j}(z_j)| + |f^{n-q_j}(z_j) - f^{n-q_j}(f^{q_j}(y_{i_0}))|. \tag{5.63}$$

Since $n - q_j < q_{j+1} - q_j = p_{i_0}(z_j)$, the fact that the points $x_j = f^{q_j}(x_0)$ and z_j belong to $U_{i_0}(z_j)$, combined with (5.60), implies that

$$|f^{n-q_j}(f^{q_j}(x_0)) - f^{n-q_j}(z_j)| \leq \varepsilon.$$

By (5.57)+(5.58) and the f-invariance of $\omega(f)$, the point $f^{q_j}(y_{i_0})$ is in $V_{i_0} \cap \omega(f)$. Moreover, z_j belongs to $V_{i_0} \cap \omega(f)$. Therefore, (5.56) implies that

$$|f^{n-q_j}(z_j) - f^{n-q_j}(f^{q_j}(y_{i_0}))| \leq \varepsilon.$$

Inserting these inequalities in (5.63), we get

$$\text{if } x_0 \text{ satisfies (5.62)}, \exists i_0 \in [\![1, r]\!], \forall n \in [\![0, N]\!],\ |f^n(x_0) - f^n(y_{i_0})| \leq 2\varepsilon. \tag{5.64}$$

Now, let x be a point and let $N_0 \leq N_1$ be integers such that

$$\forall n \in [\![N_0, N_1]\!],\ f^n(x) \in U.$$

We apply (5.64) to $x_0 := f^{N_0}(x)$ and $N := N_1 - N_0$:

$$\exists i_0 \in [\![1, r]\!],\ \forall n \in [\![0, N]\!],\ |f^n(x_0) - f^n(y_{i_0})| \leq 2\varepsilon. \tag{5.65}$$

Since $f(\omega(f)) = \omega(f)$, there exists $y \in \omega(f)$ such that $f^{N_0}(y) = y_{i_0}$, and this point satisfies: $\forall n \in [\![0, N_1]\!], f^n(y) \in \omega(f) \subset U$. Thus we can apply (5.64) to $x_0 := y$ and $N := N_1$:

$$\exists i \in [\![1, r]\!],\ \forall n \in [\![0, N_1]\!],\ |f^n(y) - f^n(y_i)| \leq 2\varepsilon. \tag{5.66}$$

Combining (5.65) and (5.66), we get, for all $n \in [\![N_0, N_1]\!]$:

$$\begin{aligned} |f^n(x) - f^n(y_i)| &\leq |f^n(x) - f^{n-N_0}(y_{i_0})| + |f^{n-N_0}(y_{i_0}) - f^n(y_i)| \\ &= |f^{n-N_0}(x_0) - f^{n-N_0}(y_{i_0})| + |f^n(y) - f^n(y_i)| \\ &\leq 2\varepsilon + 2\varepsilon = 4\varepsilon, \end{aligned}$$

which gives the expected result. □