Thus we see that the generalized Blaschke product $B_Z^{z_0}(z)$ is *not* identically zero, has norm at most one in the multiplier space M_H, vanishes on Z, and is positive at z_0. In fact, using the left hand inequality in (C.3), this argument can be reversed and yields the following characterization of nontrivial generalized Blaschke products.

Proposition 5.10 *Suppose H is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. Fix a sequence $Z = \{z_j\}_{j=1}^{\infty}$ and $z_0 \notin Z$. Then $B_Z^{z_0}(z)$ is not identically zero* if and only if

$$B_Z^{z_0}(z_0)^2 \equiv \prod_{n=1}^{\infty} d(z_0, z_n)^2 > 0,$$

if and only if *$\mu_Z = \sum_{j=1}^{\infty} \left\| k_{z_j} \right\|^{-2} \delta_{z_j}$ is a finite measure.*

In particular, Proposition 5.10 gives a sufficient condition for Z to be a zero set for H (i.e. there is $f \in H$ with $f(z) = 0$ if and only if $z \in Z$): $\|\mu_Z\| < \infty$. This recovers some results of Shapiro and Shields when H is the Dirichlet space $\mathcal{D}(\mathbb{D})$ or a generalized Dirichlet space D_φ (see [44] where condition (12) there on φ shows that D_φ has a complete Nevanlinna-Pick kernel). This raises an interesting open question.

Problem 5.11 Find a geometric characterization of the zero sets for H. When $H = H^2(\mathbb{D})$ is the classical Hardy space on the disk, $\|\mu_Z\| < \infty$ is necessary as well as sufficient. However, in general $\|\mu_Z\| < \infty$ is *not* necessary for Z to be a zero set for H, in particular when H is the Dirichlet space $\mathcal{D}(\mathbb{D})$.

consider

We can also ~~characterize~~ separation and Carleson embedding for sequences Z using $\int |\tilde{k}_{z_m}(z)|^2 d\mu_z(z) \leq \|\mu_z\|_{\text{Carleson}}$ for all $m \geq 1$.

Proposition 5.12 *Suppose H is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. Then a sequence $Z = \{z_j\}_{j=1}^{\infty}$ is separated $\frac{|\langle k_{z_n}, k_{z_m}\rangle|}{\|k_{z_n}\|\|k_{z_m}\|} \leq 1 - \varepsilon$, and $\mu_Z = \sum_{j=1}^{\infty} \left\| k_{z_j} \right\|^{-2} \delta_{z_j}$ is a Carleson measure for H* ~~if and~~ only if

$$\inf_{m \geq 1} B_{Z \setminus \{z_m\}}^{z_m}(z_m)^2 \equiv \inf_{m \geq 1} \prod_{n \neq 1} d(z_m, z_n)^2 > 0.$$

At this point we collect what can be said about interpolating sequences for a Hilbert space H with the NP property.

Theorem 5.13 *Suppose H is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. Suppose $Z = \{z_j\}_{j=1}^{\infty}$ is a sequence and let $\mu_Z = \sum_{j=1}^{\infty} \left\| k_{z_j} \right\|^{-2} \delta_{z_j}$ be its associated positive measure. Denote by R the restriction map $Rf = \{f(z_j)\}_{j=1}^{\infty}$, and let $B_{Z \setminus \{z_m\}}^{z_m}$ be the associated generalized Blaschke products. Then the relations*

$$1 \Longleftrightarrow (2+3) \text{ and if } \cancel{z} \text{ is separated } 3 \cancel{\Longleftrightarrow} 4$$

Z →

hold among the following four conditions:

1. *R maps M_H onto $\ell^\infty(Z)$,*
2. *R maps H onto $\ell^2(\mu_Z)$,*
3. *R maps H into $\ell^2(\mu_Z)$,*
4. *$\inf_{m \geq 1} B_{Z \setminus \{z_m\}}^{z_m}(z_m)^2 \equiv \delta_Z > 0$.*

Proof: Conditions **2+3** are equivalent to condition **1** by Theorem 5.6, and implies condition **3** ~~is equivalent to~~ condition **4** by Proposition 5.12.

Remark 5.14 Let $\widetilde{k_{z_n}} = \frac{k_{z_n}}{\|k_{z_n}\|}$ be the normalized reproducing kernel with pole at z_n, and let

$$B^{(n)}(z) = B^{z_n}_{Z\setminus\{z_n\}}(z)$$

be the generalized Blaschke product associated to the set $Z_n = Z \setminus \{z_n\}$ with pole at z_n. In order to prove that condition **4** implies condition **2**, it suffices to show that the Grammian $[\langle g_m, g_n\rangle]_{m,n=1}^{\infty}$ is bounded on ℓ^2 where

$$g_n = \frac{1}{B^{(n)}(z_n)} \mathcal{M}_{B^{(n)}} \widetilde{k_{z_n}} \in H, \qquad n \geq 1.$$

Indeed,

$$\begin{aligned}
\left\langle \widetilde{k_{z_m}}, g_n \right\rangle &= B^{(n)}(z_n)^{-1} \left\langle \widetilde{k_{z_m}}, M_{B^{(n)}} \widetilde{k_{z_n}} \right\rangle \\
&= B^{(n)}(z_n)^{-1} \left\langle M^*_{B^{(n)}} \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle \\
&= \frac{\overline{B^{(n)}(z_m)}}{B^{(n)}(z_n)} \left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle = \delta_n^m.
\end{aligned}$$

If the Gram matrix $[\langle g_m, g_n\rangle]_{m,n=1}^{\infty}$ were bounded on ℓ^2, we would then have for $\xi = \{\xi_j\}_{j=1}^{\infty} \in \ell^2$,

$$\begin{aligned}
\|\xi\|_{\ell^2} &= \sup_{\|\eta\|_{\ell^2}=1} \left| \sum_{n=1}^{\infty} \xi_n \overline{\eta_n} \right| = \sup_{\|\eta\|_{\ell^2}=1} \left| \sum_{m,n=1}^{\infty} \delta_n^m \xi_m \overline{\eta_n} \right| \\
&= \sup_{\|\eta\|_{\ell^2}=1} \left| \sum_{m,n=1}^{\infty} \left\langle \widetilde{k_{z_m}}, g_n \right\rangle \xi_m \overline{\eta_n} \right| \\
&= \sup_{\|\eta\|_{\ell^2}=1} \left| \left\langle \sum_{m=1}^{\infty} \xi_m \widetilde{k_{z_m}}, \sum_{n=1}^{\infty} \eta_n g_n \right\rangle \right| \\
&\leq C^* \left\| \sum_{m=1}^{\infty} \xi_m \widetilde{k_{z_m}} \right\|_H,
\end{aligned}$$

where $C^* \equiv \sup_{\|\eta\|_{\ell^2}=1} \|\sum_{n=1}^{\infty} \eta_n g_n\|_H$ coincides with the norm of the Grammian $[\langle g_m, g_n\rangle]_{m,n=1}^{\infty}$ on ℓ^2. The inequality $\|\xi\|_{\ell^2} \leq C^* \left\|\sum_{m=1}^{\infty} \xi_m \widetilde{k_{z_m}}\right\|_H$ implies condition **2** by standard functional analysis - Corollary A.39 (p. 164) in Appendix A.

Note that using $\delta_Z \leq B^{(n)}(z_n) \leq 1$ and Proposition 5.12, we at least have that the functions g_n are uniformly bounded in H:

$$\|g_n\|_H = B^{(n)}(z_n)^{-1} \left\| M_{B^{(n)}} \widetilde{k_{z_n}} \right\|_H \leq B^{(n)}(z_n)^{-1} \left\| B^{(n)} \right\|_{M_H} \left\| \widetilde{k_{z_n}} \right\|_H \leq \frac{1}{\delta_Z},$$

since $\left\|B^{(n)}\right\|_{M_H} \leq 1$.