Contemporary Mathematics
Volume **218**, 1998
B 0-8218-0988-1-03003-X

Solution of Coercive and Semicoercive Contact Problems by FETI Domain Decomposition

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1. Introduction

A new Neumann-Neumann type domain decomposition algorithm for the solution of contact problems of elasticity and similar problems is described. The discretized variational inequality that models the equilibrium of a system of elastic bodies in contact is first turned by duality to a strictly convex quadratic programming problem with either box constraints or box and equality constraints. This step may be considered a variant of the FETI domain decomposition method where the subdomains are identified with the bodies of the system. The resulting quadratic programming problem is then solved by algorithms proposed recently by the authors. Important new features of these algorithms are efficient adaptive precision control on the solution of the auxiliary problems and effective application of projections, so that the identification of a priori unknown contact interfaces is very fast.

We start our exposition by reviewing a variational inequality in displacements that describes the conditions of equilibrium of a system of elastic bodies in contact without friction. The inequality enhances the natural decomposition of the spatial domain of the problem into subdomains that correspond to the bodies of the system, and we also indicate how to refine this decomposition. After discretization, we get a possibly indefinite quadratic programming problem with a block diagonal matrix.

A brief inspection of the discrete problem shows that its structure is not suitable for computations. The main drawbacks are the presence of general constraints that prevent effective application of projections, and a semidefinite or ill conditioned matrix of the quadratic form that may cause extremely expensive solutions of the auxiliary problems.

A key observation is that both difficulties may be essentially reduced by the application of duality theory. The matrix of the dual quadratic form turns out to be regular, moreover its spectrum is much more favorably distributed for application of the conjugate gradient based methods than the spectrum of the matrix of the

1991 *Mathematics Subject Classification.* Primary 65N55; Secondary 65N30, 35J85, 90C20, 90C90.

This research has been supported by CNPq, FAPESP 95/6574-9 and by grants GAČR 201/97/0421, 105/95/1273.

quadratic form arising from the discretization. These conclusions follow from the close relation of the procedure to the FETI method proposed by Farhat and Roux [**20, 18**] for the solution of linear elliptic problems. Furthermore, the inequality constraints of the dual problem are just non-negativity constraints, so that our recent results on application of projections and adaptive precision control may be used for the solution of these problems.

The structure of the constraints of the dual problem depends on the coercivity of the contact problem under consideration. If the contact problem is *coercive*, i.e. if prescribed equality constraints on the displacement of each body prevent its rigid body motion, then the dual problem has only simple non-negativity constraints. We describe an efficient algorithm for the solution of these problems that uses the conjugate gradient method with projections and inexact solution of the auxiliary subproblems that has been proposed independently by Friedlander and Martínez [**3, 21, 22, 23, 24, 25, 26**] and Dostál [**11**]. The algorithm has been proved to converge to the solution and conditions that guarantee the finite termination property have been established. The algorithm may be implemented with projections so that it can drop or add many couples of nodes on the contact interface whenever the active set is changed. Thus the contact interface may be identified very fast even with a poor initial guess.

Next we consider the solution of *semicoercive* problems, i.e. problems with 'floating' bodies. Application of duality reduces these problems to the solution of quadratic programming problems with simple bounds and equality constraints. In this case, the feasible set is too complex to enable effective evaluations of projections, but we use a variant of the augmented Lagrangian algorithm proposed for the solution of more general non-linear problems by Conn, Gould and Toint [**5, 6**]. The algorithm generates in the outer loop the Lagrange multipliers for equality constraints while auxiliary problems with simple inequality constraints are solved in the inner loop. The precision of the solution of the auxiliary problems is controlled by the norm of the violation of the equality constraints, and an estimate for the error has been obtained that does not have any term that accounts for the precision of the solution of the auxiliary problems with simple bounds. Results on global convergence and boundedness of the penalty parameter are also reported. Moreover, we show that the penalty term in the augmented Lagrangians affects the convergence of the conjugate gradient solution of the auxiliary problems only very mildly. The paper is completed by numerical experiments.

To simplify our exposition, we have restricted our attention to the frictionless contact problems. However, the algorithm may be extended to the solution of contact problems with Coulomb friction [**16**].

2. Conditions of equilibrium of elastic bodies

Consider a system of s homogeneous isotropic elastic bodies, each of which occupies in a reference configuration a domain Ω^p in $I\!R^d, d = 2, 3$ with sufficiently smooth boundary Γ^p as in Figure 1. We assume that the bodies do not interpenetrate each other so that the intersection of any two different domains is empty. Suppose that each Γ^p consists of three disjoint parts Γ^p_U, Γ^p_F and $\Gamma^p_C, \Gamma^p = \Gamma^p_U \cup \Gamma^p_F \cup \Gamma^p_C$, and that the displacements $\mathbf{U}^p : \Gamma^p_U \to I\!R^d$ and forces $\mathbf{F}^p : \Gamma^p_F \to I\!R^d$ are given. The part Γ^p_C denotes the part of Γ^p that may get into contact with some other body. In particular, we shall denote by Γ^{pq}_C the part of Γ^p that can be, in the solution, in

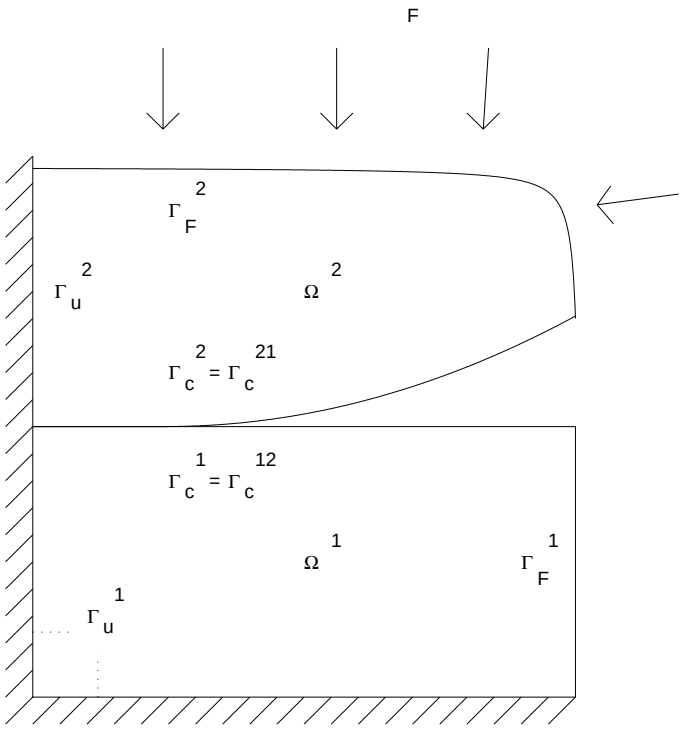

FIGURE 1. Contact problem

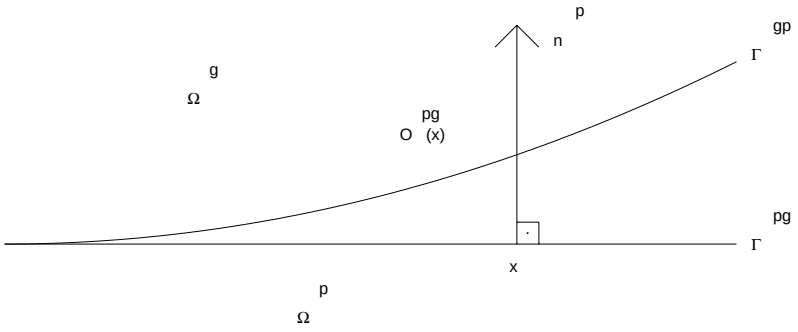

FIGURE 2. Linearized non-interpenetration

contact with the body Ω^q. Finally, let $c^p_{ijk\ell} : \Omega^p \to I\!R^d$ and $\mathbf{g}^p : \Omega^p \to I\!R^d$ denote the entries of the elasticity tensor and a vector of body forces, respectively.

For any sufficiently smooth displacements $\mathbf{u} : \Omega^1 \times \ldots \times \Omega^s \to I\!R^d$, the total potential energy is defined by

$$J(\mathbf{u}) = \sum_{p=1}^{s} \left\{ \frac{1}{2} \int_{\Omega^p} a(\mathbf{u}^p, \mathbf{u}^p) d\Omega - \int_{\Omega^p} (\mathbf{g}^p)^T \mathbf{u}^p d\Omega - \int_{\Gamma^p_F} (\mathbf{F}^p)^T \mathbf{u}^p d\Gamma \right\} \tag{1}$$

where

$$a^p(\mathbf{u}^p,\mathbf{v}^p) = \frac{1}{2}\int_{\Omega^p} c_{ijk\ell}e^p_{ij}(\mathbf{u}^p)e^p_{k\ell}(\mathbf{v}^p)d\Gamma \tag{2}$$

$$e^p_{k\ell}(\mathbf{u}^p) = \frac{1}{2}\left(\frac{\partial u^p_k}{\partial x^p_\ell}+\frac{\partial u^p_\ell}{\partial x^p_k}\right). \tag{3}$$

We suppose that the elasticity tensor satisfies natural physical restrictions so that

$$a^p(\mathbf{u}^p,\mathbf{v}^p) = a(\mathbf{v}^p,\mathbf{u}^p) \ \text{ and } \ a(\mathbf{u}^p,\mathbf{u}^p)\geq 0. \tag{4}$$

To describe the linearized non-interpenetration conditions, let us define for each $p<q$ a one-to-one continuous mapping $\mathbf{O}^{pq}:\Gamma^{pq}_C\to\Gamma^{qp}_C$ that assigns to each $\mathbf{x}\in\Gamma^{pq}_C$ some point of Γ^{qp}_C that is near to $\mathbf{x}$ as in Figure 2. The linearized non-interpenetration condition at $\mathbf{x}\in\Gamma^{pq}_C$ then reads

$$(\mathbf{u}^p(\mathbf{x})-\mathbf{u}^q(\mathbf{O}^{pq}(\mathbf{x})))\mathbf{n}^p\leq(\mathbf{O}^{pq}(\mathbf{x})-\mathbf{x})\mathbf{n}^p,\mathbf{x}\in\Gamma^{pq}_C,\ p<q. \tag{5}$$

Similar conditions may be written for description of non-interpenetration with rigid support.

Now let us introduce the Sobolev space

$$\mathcal{V}=H^1(\Omega^1)^d\times\ldots\times H^1(\Omega^s)^d, \tag{6}$$

and let $\mathbf{K}=\mathbf{K}_{eq}\bigcap\mathbf{K}_{ineq}$ denote the set of all kinematically admissible displacements, where

$$\mathbf{K}_{eq}=\{\mathbf{v}\in\mathcal{V}:\mathbf{v}^p=\mathbf{U} \ \text{ on } \ \Gamma^p_U\} \tag{7}$$

and

$$\mathbf{K}_{ineq}=\{\ \begin{array}{l}\mathbf{v}\in\mathcal{V}:(\mathbf{v}^p(\mathbf{x})-\mathbf{v}^q(\mathbf{O}^{pq}(\mathbf{x})))\mathbf{n}^p\leq(\mathbf{O}^{pq}(\mathbf{x})-\mathbf{x})\mathbf{n}^p,\\ \mathbf{x}\in\Gamma^{pq}_C,\ \ p<q\ \}.\end{array} \tag{8}$$

The displacement $\mathbf{u}\in\mathbf{K}$ of the system of bodies in equilibrium satisfies

$$J(\mathbf{u})\leq J(\mathbf{v}) \ \text{ for any } \ \mathbf{v}\in\mathbf{K}. \tag{9}$$

Conditions that guarantee the existence and uniqueness may be found e.g. in [**4, 28**].

More general boundary conditions that those described by prescribed forces or displacements may be considered, e.g. prescribed normal displacements and zero forces in the tangential plane. Moreover, we can also decompose each body into subdomains as in Figure 3 to obtain optional secondary decomposition. The Sobolev space $\mathcal{V}$ would then be defined on the product of all subdomains and the definition of the set $\mathbf{K}$ would enhance also the interface equality constraints that guarantee continuity of the displacements across new auxiliary interfaces $\Gamma^{p,ij}_A$ in each block Ω^p.

3. Discretized contact problem on interface

If there is no secondary decomposition, then the finite element discretization of $\Omega=\Omega^1\cup\ldots\cup\Omega^s$ with suitable numbering of the nodes results in the quadratic programming (QP) problem

$$\frac{1}{2}u^TKu-f^Tu\to\min \ \text{ subject to } \ Bu\leq c \tag{10}$$

with a symmetric positive definite or positive semidefinite block-diagonal matrix $K=\mathrm{diag}(K_1,\ldots,K_s)$ of order n, an $m\times n$ full rank matrix B, $f\in\mathbb{R}^n$, and

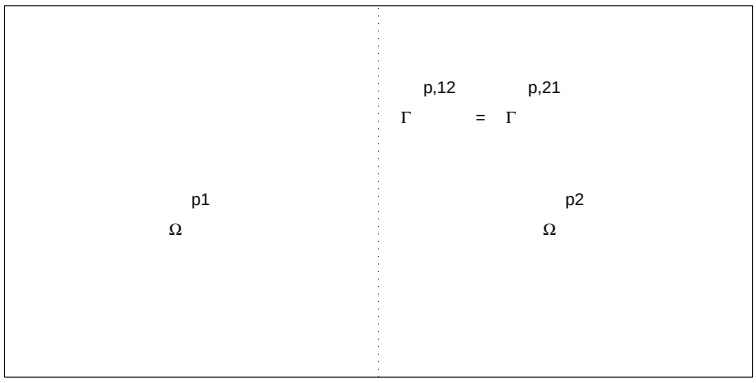

FIGURE 3. Secondary decomposition of Ω^p

$c \in I\!R^m$. The matrix B and the vector c describe the linearized incremental non-interpenetration conditions. The rows b_i of B are formed by zeros and appropriately placed coordinates of outer unit normals, so that the change of normal distance due to the displacement u is given by $u^T b_i$, and the entry c_i of c describes the normal distance of the $i-$th couple of corresponding nodes on the contact interface in the reference configuration. The vector f describes the nodal forces arising from the volume forces and/or some other imposed tractions. Typically n is large and m is much smaller than n. The diagonal blocks K_p that correspond to subdomains Ω^p are positive definite or semidefinite sparse matrices. Moreover, we shall assume that the nodes of the discretization are numbered in such a way that the matrices K_i are banded matrices that can be effectively decomposed, possibly after some regularization, by means of the Cholesky factorization.

If there is a secondary decomposition, then the continuity of the displacements across the auxiliary interface requires $u^T h_i = 0$, where h_i are vectors of order n with zero entries except 1 and -1 in appropriate positions. If H is a matrix formed by the rows h_i, then the discretization of problem (10) with the secondary decomposition results in the QP problem

$$\frac{1}{2}u^T K u - f^T u \to \min \quad \text{subject to} \quad Bu \le c \quad \text{and} \quad Hu = 0. \tag{11}$$

With a suitable enumeration of the nodes, each K_i turns out to be block diagonal with banded diagonal blocks.

Even though (11) is a standard convex quadratic programming problem, its formulation is not suitable for numerical solution. The reasons are that K might be singular and the feasible set is in general so complex that projections cannot be computed to obtain fast identification of the active set at the solution.

The complications mentioned above may be essentially reduced by applying the duality theory of convex programming (e.g. Dostál [**10, 9**]). If there is no secondary decomposition, we get

$$\theta(\lambda) \to \min \text{ subject to } \lambda \ge 0 \text{ and } R^T(f - B^T\lambda) = 0 \tag{12}$$

where

$$\theta(\lambda) = \frac{1}{2}\lambda^T B K^+ B^T \lambda - \lambda^T (B K^+ f - c), \tag{13}$$

R denotes a matrix whose columns span the null space of K, and K^+ denotes a generalized inverse of K that satisfies $KK^+K = K$. Let us recall that

$$K^+ = \mathrm{diag}(K_1^+, \ldots, K_p^+)$$

and that $K_p^+ = K_p^{-1}$ whenever K_p is non-singular. If K_p is singular then it is easy to check that there is a permutation matrix P_p and a non-singular matrix F_p such that

$$P_p^T K_p P_p = \begin{pmatrix} F_p & S_p \\ S_p^T & S_p^T F_p^{-1} \end{pmatrix} \tag{14}$$

and

$$K_p^+ = P_p \begin{pmatrix} F_p^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_p^T. \tag{15}$$

Once the solution λ of (12) is known, the vector u that solves (10) can be evaluated. In particular, if K is positive definite then

$$u = K^{-1}(f - B^T\lambda). \tag{16}$$

If K is singular and R is a full rank matrix then

$$u = R\alpha + K^+(f - B^T\lambda), \tag{17}$$

with

$$\alpha = (R^T\tilde{B}^T\tilde{B}R)^{-1}R^T\tilde{B}^T(\tilde{c} - \tilde{B}A^+(f - B^T\lambda)) \tag{18}$$

and $(\tilde{B}, \tilde{c})$ formed by the rows of (B, c) that correspond to the nonzero entries of λ.

If there is a secondary decomposition, then there are additional Lagrange multipliers for equalities. Thus, the only new feature when compared with the problem without the secondary decomposition is the presence of free Lagrange multipliers in the dual formulation in this case.

The matrix BK^+B^T is invertible when no rigid body displacement can be written as a linear combination of the columns of B^T. Moreover, the matrix BK^+B^T is closely related to the matrix resulting from the application of the FETI method of Farhat and Roux [**20**], so that its spectrum is relatively favorably distributed for the application of the conjugate gradient method (Farhat, Mandel and Roux [**17, 19**]).

4. Solution of coercive problems

An important point in the development of an efficient algorithm for the solution of (12) is the observation that QP problems with simple bounds are much simpler than more general QP problems. Here we shall briefly review our results on the solution of QP problems with simple bounds.

To simplify our notations, let us denote

$$\begin{array}{lcllcl} A & = & BK^+B^T & b & = & BK^+f - c \\ d & = & R^Tc & D & = & R^TB^T \end{array}$$

and let us first assume that K is non-singular, so that problem (12)-(13) reads

$$\theta(x) \to \min \quad \text{subject to} \quad x \geq 0 \tag{19}$$

where

$$\theta(x) = \frac{1}{2}x^T Ax - b^T x. \tag{20}$$

Let us denote by $\mathcal{A}(x)$ and $\mathcal{F}(x)$ the active and free sets of indices of x, respectively, i.e.

$$\mathcal{A}(x) = \{i : x_i = 0\} \quad \text{and} \quad \mathcal{F}(x) = \{i : x_i \neq 0\}. \tag{21}$$

The unbalanced contact gradient g^C and the inner gradient g^I of $\theta(x)$ are defined by

$$g_i^I = g_i \text{ for } i \in \mathcal{F}(x) \text{ and } g_i^I = 0 \text{ for } i \in \mathcal{A}(x) \tag{22}$$

$$g_i^C = 0 \text{ for } i \in \mathcal{F}(x) \text{ and } g_i^C = g_i^- \text{ for } i \in \mathcal{A}(x) \tag{23}$$

where $g = g(x) = \nabla\theta(x)$, $g_i = g_i(x)$ and $g_i^- = \min\{0, g_i\}$. Hence the Kuhn-Tucker conditions for the solution of (19) are satisfied when the projected gradient $g^P = g^I + g^C$ vanishes.

An efficient algorithm for the solution of convex QP problems with simple bounds has been proposed independently by Friedlander and Martínez [**21**] and Dostál [**11**]. The algorithm may be considered a modification of the Polyak algorithm that controls the precision of the solution of auxiliary problems by the norm of g^C in each inner iterate y^i.

If for $\Gamma > 0$ the inequality

$$||g^C(y^i)|| \leq \Gamma||g^I(y^i)||$$

holds then we call y^i proportional. The algorithm explores the face

$$W_I = \{y : y_i = 0 \text{ for } i \in I\}$$

with a given active set I as long as the iterates are proportional. If y^i is not proportional, we generate y^{i+1} by means of the descent direction $d^i = -g^C(y^i)$ in a step that we call proportioning, and then we continue exploring the new face defined by $I = \mathcal{A}(y^{i+1})$. The class of algorithms driven by proportioning may be defined as follows.

ALGORITHM 1. *(General Proportioning Scheme - GPS)*
Let $y^0 \geq 0$ and $\Gamma > 0$ be given. For $i > 0$, choose y^{i+1} by the following rules:
(i) If y^i is not proportional, define y^{i+1} by proportioning.
(ii) If y^i is proportional, choose $y^{i+1} \geq 0$ so that

$$\theta(y^{i+1}) \leq \theta(y^i)$$

and y^{i+1} satisfies at least one of the conditions: $\mathcal{A}(y^i) \subset \mathcal{A}(y^{i+1}), y^{i+1}$ is not proportional, or y^{i+1} minimizes $\theta(\xi)$ subject to $\xi \in W_I, I = \mathcal{A}(y^i)$.

The set relation $\subset$ is used in the strict sense so that it is satisfied if the set on the left is a proper subset of the set on the right. Basic theoretical results have been proved in [**3, 11, 21, 22**].

THEOREM 2. *Let x^k denote an infinite sequence generated by Algorithm GPS with given x^0 and $\Gamma > 0$. Let $\theta(x)$ be a strictly convex quadratic function. Then the following statements are true:*

(i) x^k *converges to the solution* $\bar{x}$ *of (19).*
(ii) If the problem (19) is not degenerate, then there is k *such that* $\bar{x} = x^k$.
(iii) If $\Gamma \geq \kappa(A)^{1/2}$, *where* $\kappa(A)$ *is the spectral condition number of* A, *then there is* k *such that* $\bar{x} = x^k$.

Step (ii) of Algorithm GPS may be implemented by means of the conjugate gradient method. The most simple implementation of this step starts from $y^0 = x^k$ and generates the conjugate gradient iterations $y^1, y^2, \dots$ for $\min\{\theta(y) : y \in \mathcal{W}_I,\ I = \mathcal{A}(y^0)\}$ until y^i is found that is not feasible or not proportional or minimizes $\theta(x)$ subject to $y \geq 0$. If y^i is feasible, then we put $x^{k+1} = y^i$, otherwise $y^i = y^{i-1} - \alpha^i p^i$ is not feasible and we can find $\tilde{\alpha}^i$ so that $x^{k+1} = y^i - \tilde{\alpha}^i p^i$ is feasible and $\mathcal{A}(x^k) \not\subseteq \mathcal{A}(x^{k+1})$. We shall call the resulting algorithm *feasible proportioning* [**11**].

An obvious drawback of feasible proportioning is that the algorithm is usually unable to add more than one index to the active set in one iteration. A simple but efficient alternative is to replace the feasibility condition by $\theta(Py^{i+1}) \leq \theta(Py^i)$, where Py denotes the projection on the set $\Omega = \{y : y \geq 0\}$. If the conjugate gradient iterations are interrupted when condition $\theta(Py^{i+1}) > \theta(Py^i)$ is satisfied, then a new iteration is defined by $x^{k+1} = Py^i$. Resulting modification of the feasible proportioning algorithm is called *monotone proportioning* [**11**]. More details on implementation of the algorithm may be found in [**15**].

5. Solution of semicoercive problems

Now we shall assume that the matrix K is only positive semidefinite, so that problem (12)-(13) with the notations of the previous section reads

$$\theta(x) \to \min \quad \text{subject to} \quad x \geq 0 \text{ and } Dx = d. \tag{24}$$

The algorithm that we propose here may be considered a variant of the algorithm proposed by Conn, Gould and Toint(1991) for identification of stationary points of more general problems.

ALGORITHM 3. *(Simple bound and equality constraints)*
Step 0. { *Initialization of parameters* } *Set* $0 < \alpha < 1$ [$\alpha = .1$] *for equality precision update,* $1 < \beta$ [$\beta = 100$] *for penalty update,* $\rho_0 > 0$ [$\rho_0 = 100$] *for initial penalty parameter,* $\eta_0 > 0$ [$\eta_0 = .001$] *for initial equality precision,* $M > 0$ [$M = \rho_0/100$] *for balancing ratio,* μ^0 [$\mu^0 = 0$] *and* $k = 0$.

Step 1. *Find* x^k *so that*
$||g^P(x^k, \mu^k, \rho_k)|| \leq M||Dx^k - d||$.

Step 2. *If* $||g^P(x^k, \mu^k, \rho_k)||$ and $||Dx^k - d||$ *are sufficiently small then* x^k *is the solution.*

Step 3. $\mu^{k+1} = \mu^k + \rho_k(Dx^k - d)$.

Step 4. *If* $||Dx^k - d|| \leq \eta_k$ *then* $\rho_{k+1} = \rho_k$, $\eta_{k+1} = \alpha\eta_k$

Step 4b. *else* $\rho_{k+1} = \beta\rho_k, \ \ \eta_{k+1} = \eta_k$
end if.

Step 5. *Increase k and return to Step 1.*

In this algorithm, we now denote by g the gradient of the augmented Lagrangian

$$L(x, \mu, \rho) = \theta(x) + \mu^T Dx + \frac{1}{2}\rho||Dx - d||^2$$

so that

$$g(x, \mu, \rho) = Ax - b + D^T(\mu + \rho D(x - d))$$

An implementation of Step 1 is carried out by minimization of the augmented Lagrangian L subject to $x \geq 0$ by means of the algorithm of the previous section. The unique solution $\overline{x} = \overline{x}(\mu, \rho)$ of this auxiliary problem satisfies the Kuhn-Tucker conditions

$$g^P(\overline{x}, \mu, \rho) = 0. \tag{25}$$

Typical values of the parameters are given in brackets.

The essential feature of this algorithm is that it deals completely separately with each type of constraint and that it accepts inexact solutions of the auxiliary box constrained problems in Step 1. For parallel implementation, A should be kept as the product BK^+B since A is just used in the matrix-vector products. The action of K^+ may be evaluated by means of a triangular decomposition.

The algorithm has been proved ([**13**]) to converge for any set of parameters that satisfy the prescribed relations. Moreover, it has been proved that the asymptotic rate of convergence is the same as for the algorithm with exact solution of auxiliary quadratic programming problems (i.e. $M = 0$) and the penalty parameter is uniformly bounded.

The use of the augmented Lagrangian method turned out to be very efficient in spite of the fact that it obviously reintroduces ill conditioning into the auxiliary problems. The explanation is given by the following theorem and by analysis of the conjugate gradient method by Axelsson and Lindskog [**1, 2**], who showed that the rate of convergence is much faster than it could be expected from the conditioning of the problem provided there is a gap in the spectrum.

THEOREM 4. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $D \in \mathbb{R}^{m\times n}$ a full rank matrix, $m < n$ and $\rho > 0$. For any matrix M of order n, let $\delta_1(M) \leq \ldots \leq \delta_n(M)$ denote the eigenvalues of M. Then*

$$\delta_{n-m}(A + \rho D^T D) \ \leq \ \delta_n(A) \tag{26}$$

$$\delta_{n-m+1}(A + \rho D^T D) \ \geq \ \rho\delta_{n-m+1}(D^T D) \ > \ 0. \tag{27}$$

6. Numerical experiments

In this section, we illustrate the practical behavior of our algorithm. First, a model problem used to validate the algorithm is presented. Next, two problems arising in mechanical and mining engineering, respectively, are commented. All the experiments were run in a PC-486 type computer, DOS operating system, Microsoft `Fortran 77` and double precision. The auxiliary problems were solved by `QUACAN` routine developed in the Institute of Mathematics, Statistics and Scientific Computation of UNICAMP.

PROBLEM 1. This is a model problem resulting from the finite difference discretization of the following continuous problem:

$$\begin{aligned} \text{Minimize} \quad & q(u_1,u_2)=\sum_{i=1}^{2}\left(\int_{\Omega_i}|\nabla u_i|^2 d\Omega-\int_{\Omega_i}Pu_i d\Omega\right) \\ \text{subject to} \quad & u_1(0,y)\equiv 0 \text{ and } u_1(1,y)\le u_2(1,y) \text{ for } y\in[0,1], \end{aligned}$$

where $\Omega_1=(0,1)\times(0,1)$, $\Omega_2=(1,2)\times(0,1)$, $P(x,y)=-1$ for $(x,y)\in(0,1)\times[0.75,1)$, $P(x,y)=0$ for $(x,y)\in(0,1)\times(0,0.75)$, $P(x,y)=-1$ for $(x,y)\in(1,2)\times(0,0.25)$ and $P(x,y)=0$ for $(x,y)\in(1,2)\times(0.25,1)$. The discretization scheme consists in a regular grid of 21×21 nodes for each unitary interval. We took the identically zero initial approximation. This problem is such that the matrix of the quadratic function is singular due to the lack of Dirichlet data on the boundary of Ω_2. In order to reduce the residual to 10^{-5}, three simple bounded (SB) problems had to be solved. The total number of iteration used by `QUACAN` was 23, taking 34 matrix-vector products. More details on this problem may be found in [**10**].

PROBLEM 2. The objective of this problem is to identify the contact interface and evaluate the contact stresses of a system of elastic bodies in contact. Some rigid motion is admitted for these bodies. This type of problems is treated in [**14**]. The model problem considered to test our algorithm consists of two identical cylinders that lie one above the other on a rigid support. A vertical traction is applied at the top 1/12 of the circumference of the upper cylinder. Assuming the plane stress, the problem was reduced to 2D and discretized by the boundary element method so that the dimension of the discretized problem was 288 with 14 couples of nodes on the contact interfaces. This problem was first considered admitting vertical rigid motion of the upper cylinder only. A second formulation admitted rigid body motion of both cylinders. The Lagrange multipliers of the solution are the contact nodal forces. To solve the problem with relative precision equal to 10^{-4}, three (SB) problems were solved with $\rho=10^6$, $M=10^4$ and $\Gamma=0.1$. The total number of `QUACAN` iterations was 42.

PROBLEM 3. Finally, we consider a problem of equilibrium of a system of elastic blocks. This problem arises in mining engineering. An example of the solution of such problems under the assumption of plane strain may be found in [**7**]. The difficulties related to the analysis of equilibrium of block structures comprise identification of unknown contact interface, necessity to deal with floating blocks that do not have enough boundary conditions and often large matrices that arise from the finite element discretization of 3D problems. To test the performance of our algorithm we solved a 3D problem proposed by Hittinger in [**27**]. The 2D version of this problem was solved in [**27**] and [**8**]. A description of the problem and the variants solved with our algorithm are in [**12**]. Main characteristics of the two block variant were 4663 nodal variables and 221 dual variables (unknown contact nodal forces), while the three block variant comprised 6419 nodal variables and 382 dual variables. The bandwidth was 165 in both variants. The solution to relative precision 10^{-4} was obtained with three outer iterations, that is, just three (SB) problems were necessary. The number of inner `QUACAN` conjugate gradient iterations for two and three block problems was 105 and 276, respectively.

7. Comments and conclusions

We have described a new algorithm for the solution of coercive and semicoercive contact problems of elasticity without friction based on variational formulation reduced to the boundary. The method directly obtains the tractions on the contact interface. The stress and strain distribution may then be obtained by the solution of standard linear problems for each body separately.

The algorithm combines a variant of the domain decomposition method of the Neumann-Neumann type based on the duality theory of quadratic programming with the new algorithms for the solution of the quadratic programming problems with simple bounds. For the solution of semicoercive problems, these methods are exploited in the augmented Lagrangian algorithm. A new feature of these algorithms is the adaptive control of precision of the solution of auxiliary problems with effective usage of the projections and penalty technique.

The implementation of the algorithm deals separately with each body, so that the algorithm is suitable for parallel implementation. First numerical experiments indicate that the algorithms presented are efficient. We believe that the performance of the algorithms may be considerably improved by the 'coarse grid' preconditioner in combination with the standard regular preconditioners as presented at this conference by F.-X. Roux et al. [**29**].

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