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Domain Decomposition Algorithms for Saddle Point Problems

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1. Introduction

In this paper, we introduce some domain decomposition methods for saddle point problems with or without a penalty term, such as the Stokes system and the mixed formulation of linear elasticity. We also consider more general nonsymmetric problems, such as the Oseen system, which are no longer saddle point problems but can be studied in the same abstract framework which we adopt.

Several approaches have been proposed in the past for the iterative solution of saddle point problems. We recall here:
- Uzawa's algorithm and its variants (Arrow, Hurwicz, and Uzawa [**1**], Elman and Golub [**24**], Bramble, Pasciak, and Vassilev [**10**], Maday, Meiron, Patera, and Rønquist [**38**]);
- multigrid methods (Verfürth [**54**], Wittum [**55**], Braess and Blömer [**7**], Brenner [**11**]);
- preconditioned conjugate gradient methods for a positive definite equivalent problem (Bramble and Pasciak [**8**]);
- block–diagonal preconditioners (Rusten and Winther [**50**], Silvester and Wathen [**51**], Klawonn [**31**]);
- block–triangular preconditioners (Elman and Silvester [**25**], Elman [**23**], Klawonn [**32**], Klawonn and Starke [**34**], Pavarino [**43**]).

Some of these approaches allow the use of domain decomposition techniques on particular subproblems, such as the inexact blocks in a block preconditioner. In this paper, we propose some alternative approaches based on the application of domain decomposition techniques to the whole saddle point problem, discretized with either h-version finite elements or spectral elements. We will consider both a) overlapping Schwarz methods and b) iterative substructuring methods. We refer to Smith, Bjørstad, and Gropp [**52**] or Chan and Mathew [**18**] for a general introduction to domain decomposition methods.

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a) Early work by Lions [**37**] and Fortin and Aboulaich [**29**] extended the original overlapping Schwarz method to Stokes problems, but these methods were based on a positive definite problem obtained by working in the subspace of divergence-free functions and they did not have a coarse solver, which is essential for obtaining scalability. Later, the overlapping Schwarz method was also extended to the mixed formulations of scalar-second order elliptic problems (see Mathew [**40, 41**], Ewing and Wang [**27**], Rusten, Vassilevski, and Winther [**49**]) and to indefinite, nonsymmetric, scalar-second order elliptic problems (see Cai and Widlund [**13, 14**]). In Section 6, we present a different overlapping Schwarz method based on the solution of local saddle point problems on overlapping subdomains and the solution of a coarse saddle point problem. The iteration is accelerated by a Krylov space method, such as GMRES or QMR. The resulting method is the analog for saddle point problems of the method proposed and analyzed by Dryja and Widlund [**20, 21**] for symmetric positive definite elliptic problems. As in the positive definite case, our method is parallelizable, scalable, and has a simple coarse problem. This work on overlapping methods is joint with Axel Klawonn of the Westfälische Wilhelms-Universität Münster, Germany.

b) Nonoverlapping domain decomposition preconditioners for Stokes problems have been considered by Bramble and Pasciak [**9**], Quarteroni [**47**] and for spectral element discretizations by Fischer and Rønquist [**28**], Rønquist [**48**], Le Tallec and Patra [**36**], and Casarin [**17**]. In Section 7, we present a class of iterative substructuring methods in which the saddle point Schur complement, obtained after the elimination of the internal velocities and pressures in each subdomain, is solved with a block preconditioner. The velocity block can be constructed using wire basket or Neumann-Neumann techniques. In the Stokes case, this construction is directly based on the original scalar algorithms, while in the elasticity case it requires an extension of the scalar techniques. The iteration is accelerated by a Krylov space method, such as GMRES or PCR. The resulting algorithms are parallelizable and scalable, but the structure of the coarse problem is more complex than in overlapping methods. This work on nonoverlapping methods is joint with Olof B. Widlund of the Courant Institute, New York University, USA.

2. Model saddle point problems

The Stokes system. Let $\Omega \subset R^d, d = 2,3$ be a polyhedral domain and $L_0^2(\Omega)$ be the subset of $L^2(\Omega)$ consisting of functions with zero mean value. Given $\mathbf{f} \in (H^{-1}(\Omega))^d$ and, for simplicity, homogeneous Dirichlet boundary conditions, the Stokes problem consists in finding the velocity $\mathbf{u} \in \mathbf{V} = (H_0^1(\Omega))^d$ and the pressure $p \in U = L_0^2(\Omega)$ of an incompressible fluid with viscosity μ by solving:

$$(1) \qquad \begin{cases} \mu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} dx - \int_\Omega \mathrm{div} \mathbf{v} p dx & = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathbf{V}, \\ - \int_\Omega \mathrm{div} \mathbf{u} q dx & = 0 \qquad \forall q \in U. \end{cases}$$

Linear elasticity in mixed form. The following mixed formulation of the system of linear elasticity describes the displacement $\mathbf{u}$ and the variable $p = -\lambda \mathrm{div} \mathbf{u}$ of an almost incompressible material with Lamé constants λ and μ. The material is fixed along $\Gamma_0 \subset \partial\Omega$, subject to a surface force of density $\mathbf{g}$ along $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ and subject to an external force $\mathbf{f}$:

$$(2)\quad \begin{cases} 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx & - & \int_\Omega \mathrm{div}\mathbf{v}\ p\ dx & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V}, \\ -\int_\Omega \mathrm{div}\mathbf{u}\ q\ dx & - & \frac{1}{\lambda}\int_\Omega pq\ dx & = & 0 & \forall q \in L^2(\Omega). \end{cases}$$

Here $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\Gamma_0} = 0\}$, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ are the components of the linearized strain tensor $\epsilon(\mathbf{u})$, and $<\mathbf{F},\mathbf{v}> = \int_\Omega \sum_{i=1}^3 f_i v_i\ dx + \int_{\Gamma_1} \sum_{i=1}^3 g_i v_i\ ds$. It is well known that this mixed formulation is a good remedy for the locking and ill-conditioning problems that arise in the pure displacement formulation when the material becomes almost incompressible; see Babuška and Suri [**2**]. The incompressibility of the material can be characterized by λ approaching infinity or, equivalently, by the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ approaching $1/2$.

The Oseen system (linearized Navier-Stokes). An example of nonsymmetric problem is given by the Oseen system. Linearizing the Navier-Stokes equations by a fixed-point or Picard iteration, we have to solve in each step the following Oseen problem: given a divergence-free vector field $\mathbf{w}$, find the velocity $\mathbf{u} \in \mathbf{V} = (H_0^1(\Omega))^d$ and the pressure $p \in U = L_0^2(\Omega)$ of an incompressible fluid with viscosity μ satisfying

(3)

$$\begin{cases} \mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} dx + \int_\Omega [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} dx - \int_\Omega \mathrm{div}\mathbf{v} p dx & = & \int_\Omega \mathbf{f}\cdot\mathbf{v} dx & \forall \mathbf{v} \in \mathbf{V}, \\ -\int_\Omega \mathrm{div}\mathbf{u} q dx & = & 0 & \forall q \in U. \end{cases}$$

Here the right-hand side $\mathbf{f}$ is as in the Stokes problem and the convection term is given by the skew-symmetric bilinear form $\int_\Omega [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} dx = \int_\Omega \sum_{i,j=1}^3 w_j \frac{\partial u_i}{\partial x_j} v_i dx.$

3. An abstract framework for saddle point problems and generalizations

In general, given two Hilbert spaces $\mathbf{V}$ and U, the algorithms described in this paper apply to the following generalization of abstract saddle point problems with a penalty term. An analysis and a more complete treatment can be found in Brezzi and Fortin [**12**].

Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(4)\quad \begin{cases} a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) & - & t^2 c(p,q) & = & <G,q> & \forall q \in U \quad t \in [0,1], \end{cases}$$

where $\mathbf{F} \in \mathbf{V}'$ and $G \in U'$. When $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric, (4) is a saddle point problem, but we keep the same terminology also in the more general nonsymmetric case. In order to have a well-posed problem, we assume that the following properties are satisfied. Let $B : \mathbf{V} \longrightarrow U'$ and its transpose $B^T : U \longrightarrow \mathbf{V}'$ be the linear operators defined by

$$(B\mathbf{v}, q)_{U'\times U} = (\mathbf{v}, B^T q)_{V\times V'} = b(\mathbf{v}, q) \quad \forall \mathbf{v} \in \mathbf{V},\ \forall q \in U.$$

i) $a(\cdot,\cdot) : \mathbf{V}\times\mathbf{V} \longrightarrow R$ is a continuous, positive semidefinite bilinear form, invertible on the kernel $KerB$ of B, i.e.

$$\exists \alpha_0 > 0 \;\text{ such that }\; \begin{cases} \inf_{\mathbf{u}\in KerB} \sup_{\mathbf{v}\in KerB} \dfrac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_V \|\mathbf{v}\|_V} \geq \alpha_0, \\ \\ \inf_{\mathbf{v}\in KerB} \sup_{\mathbf{u}\in KerB} \dfrac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_V \|\mathbf{v}\|_V} \geq \alpha_0; \end{cases}$$

ii) $b(\cdot,\cdot) : \mathbf{V}\times U \longrightarrow R$ is a continuous bilinear form satisfying the inf-sup condition

$$\exists \beta_0 > 0 \;\text{ such that }\; \sup_{v\in V} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_V} \geq \beta_0 \|q\|_{U/KerB^T};$$

iii) $c(\cdot,\cdot) : U \times U \longrightarrow R$ is a symmetric, continuous, U-elliptic bilinear form.
More general conditions could be assumed; see Brezzi and Fortin [**12**] and Braess [**6**].

For simplicity, we adopt in the following the Stokes terminology, i.e. we call the variables in $\mathbf{V}$ velocities and the variables in U pressures.

4. Mixed finite element methods: $P_1(h) - P_1(2h)$ and $Q_1(h) - P_0(h)$ stabilized

The continuous problem (4) is discretized by introducing finite element spaces $\mathbf{V}^h \subset \mathbf{V}$ and $U^h \subset U$. For simplicity, we consider uniform meshes, but more general nonuniform meshes may be used. We consider two choices of finite element spaces, in order to illustrate our algorithms for both stable and stabilized discretizations, with continuous and discontinuous pressures respectively.

a) $P_1(h) - P_1(2h)$ (also known as $P2 - iso - P1$). Let τ_{2h} be a triangular finite element mesh of Ω of characteristic mesh size $2h$ and let τ_h be a refinement of τ_{2h}. We introduce finite element spaces consisting of continuous piecewise linear velocities on τ_h and continuous piecewise linear pressures on τ_{2h} :

$$\begin{aligned} \mathbf{V}^h &= \{\mathbf{v} \in (C(\Omega))^d \cap \mathbf{V} : \mathbf{v}|_T \in P_1, T \in \tau_h\}, \\ U^h &= \{q \in C(\Omega) \cap U : q|_T \in P_1, T \in \tau_{2h}\}. \end{aligned}$$

This is a stable mixed finite element method, i.e. it satisfies a uniform inf-sup condition (see Brezzi and Fortin [**12**]).

b) $Q_1(h) - P_0(h)$ stabilized. Here the velocities are continuous piecewise trilinear (bilinear in 2D) functions on a quadrilateral mesh of size h and the pressures are piecewise constant (discontinuous) functions on the same mesh :

$$\begin{aligned} \mathbf{V}^h &= \{\mathbf{v} \in (C(\Omega))^d \cap \mathbf{V} : \mathbf{v}|_T \in Q_1, T \in \tau_h\}, \\ U^h &= \{q \in U : q|_T \in P_0, T \in \tau_h\}. \end{aligned}$$

This couple of finite element spaces does not satisfy the inf-sup condition, but can be stabilized as shown in Kechkar and Silvester [**30**] by relaxing the discrete incompressibility condition. In the Stokes case in two dimensions, this stabilization is achieved by defining a nonoverlapping macroelement partitioning $\mathcal{M}_h$ such that each macroelement $M \in \mathcal{M}_h$ is a connected set of adjoining elements from τ_h. Denoting by Γ_M the set of interelement edges in the interior of M and by $e \in \Gamma_M$ one of these interior edges, the original bilinear form $c(p,q) = 0$ is replaced by

$$c_h(p,q) = \beta \sum_{M\in\mathcal{M}_h} \sum_{e\in\Gamma_M} h_e \int_e [\![p]\!]_e [\![q]\!]_e ds. \tag{5}$$

Here $[\![p]\!]_e$ is the jump operator across $e \in \Gamma_M$, h_e is the length of e, and β is a stabilization parameter; see [**30**] for more details and an analysis.

By discretizing the saddle point problem (4) with these mixed finite elements, we obtain the following discrete saddle point problem:

$$K_h x = \begin{bmatrix} A_h & B_h^T \\ B_h & -t^2 C_h \end{bmatrix} x = f_h \ . \tag{6}$$

The matrix K_h is symmetric and indefinite whenever A_h is symmetric, as in the Stokes and elasticity cases. The penalty parameter t^2 is zero in the Stokes, Oseen and incompressible elasticity cases when discretized with stable elements, such as $P_1(h) - P_1(2h)$; it is nonzero in the case of almost incompressible elasticity or when stabilized elements, such as $Q_1(h) - P_0(h)$ stabilized, are used.

5. Mixed spectral element methods: $Q_n - Q_{n-2}$ and $Q_n - P_{n-1}$

The continuous problem (4) can also be discretized by conforming spectral elements. Let Ω_{ref} be the reference cube $(-1,1)^3$, let $Q_n(\Omega_{\text{ref}})$ be the set of polynomials on Ω_{ref} of degree n in each variable, and let $P_n(\Omega_{\text{ref}})$ be the set of polynomials on Ω_{ref} of total degree n. We assume that the domain Ω can be decomposed into N nonoverlapping finite elements Ω_i, each of which is an affine image of the reference cube. Thus, $\Omega_i = \phi_i(\Omega_{\text{ref}})$, where ϕ_i is an affine mapping.

a) $Q_n - Q_{n-2}$. This method was proposed by Maday, Patera, and Rønquist [**39**] for the Stokes system. $\mathbf{V}$ is discretized, component by component, by continuous, piecewise polynomials of degree n:

$$\mathbf{V}^n = \{\mathbf{v} \in \mathbf{V} : v_k|_{\Omega_i} \circ \phi_i \in Q_n(\Omega_{\text{ref}}),\ i = 1, \cdots, N, \quad k = 1,2,3\}.$$

The pressure space is discretized by piecewise polynomials of degree $n-2$:

$$U^n = \{q \in U : q|_{\Omega_i} \circ \phi_i \in Q_{n-2}(\Omega_{\text{ref}}),\ i = 1, \cdots, N\}.$$

We note that the elements of U^n are discontinuous across the boundaries of the elements Ω_i. These mixed spectral elements are implemented using Gauss-Lobatto-Legendre (GLL) quadrature, which also allows the construction of a very convenient tensor-product basis for $\mathbf{V}^n$. Denote by $\{\xi_i, \xi_j, \xi_k\}_{i,j,k=0}^n$ the set of GLL points of $[-1,1]^3$, and by σ_i the quadrature weight associated with ξ_i . Let $l_i(x)$ be the Lagrange interpolating polynomial of degree n which vanishes at all the GLL nodes except ξ_i, where it equals one. Each element of $Q_n(\Omega_{\text{ref}})$ is expanded in the GLL basis

$$u(x,y,z) = \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n u(\xi_i, \xi_j, \xi_k) l_i(x) l_j(y) l_k(z),$$

and each L^2-inner product of two scalar components u and v is replaced by

$$(u,v)_{n,\Omega} = \sum_{s=1}^N \sum_{i,j,k=0}^n (u \circ \phi_s)(\xi_i, \xi_j, \xi_k)(v \circ \phi_s)(\xi_i, \xi_j, \xi_k)|J_s| \sigma_i \sigma_j \sigma_k, \tag{7}$$

where $|J_s|$ is the determinant of the Jacobian of ϕ_s. Similarly, a very convenient basis for U^n consists of the tensor-product Lagrangian nodal basis functions associated with the internal GLL nodes. Another basis associated with the Gauss-Legendre (GL) nodes has been studied in [**28**] and [**38**]. We refer to Bernardi and Maday [**3, 4**] for more details and the analysis of the resulting discrete problem.

The $Q_n - Q_{n-2}$ method satisfies the nonuniform inf-sup condition

$$\sup_{\mathbf{v} \in \mathbf{V}^n} \frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1}} \geq C n^{-(\frac{d-1}{2})} \|q\|_{L^2} \quad \forall q \in U^n, \tag{8}$$

where $d = 2, 3$ and the constant C is independent of n and q; see Maday, Patera, and Rønquist [**39**] and Stenberg and Suri [**53**]. However, numerical experiments, reported in Maday, Meiron, Patera, and Rønquist, see [**38**] and [**39**], have also shown that for practical values of n, e.g., $n \leq 16$, the inf-sup constant β_n of the $Q_n - Q_{n-2}$ method decays much slower than what would be expected from the theoretical bound.

b) $Q_n - P_{n-1}$. This method uses the same velocity space $\mathbf{V}^n$ described before, together with an alternative pressure space given by piecewise polynomials of total degree $n - 1$:

$$\{q \in U : q|_{\Omega_i} \circ \phi_i \in P_{n-1}(\Omega_{ref}),\ i = 1, \cdots, N\}.$$

This choice has been studied by Stenberg and Suri [**53**] and more recently by Bernardi and Maday [**5**], who proved a uniform inf-sup condition for it. Its practical application is limited by the lack of a standard tensorial basis for P_{n-1}; however, other bases, common in the p-version finite element literature, can be used.

Other interesting choices for U^n have been studied in Canuto [**15**] and Canuto and Van Kemenade [**16**] in connection with stabilization techniques for spectral elements using bubble functions.

Applying GLL quadrature to the abstract problem (4), we obtain again a discrete saddle point problem of the form

$$K_n x = \begin{bmatrix} A_n & B_n^T \\ B_n & -t^2 C_n \end{bmatrix} x = f_n \ . \tag{9}$$

As before, K_n is a symmetric indefinite matrix in the Stokes and elasticity case, while it is a nonsymmetric matrix in the Oseen case.

6. Overlapping Schwarz Methods

We present here the basic idea of the method for the additive variant of the preconditioner and $P_1(h){-}P_1(2h)$ finite elements on uniform meshes (see Section 4). More general multiplicative or hybrid variants, unstructured meshes and spectral element discretizations can be considered as well. See Klawonn and Pavarino [**33**] for a more complete treatment.

Let τ_H be a coarse finite element triangulation of the domain Ω into N subdomains Ω_i of characteristic diameter H. A fine triangulation τ_h is obtained as a refinement of τ_H and H/h will denote the number of nodes on each subdomain side. In order to have an overlapping partition of Ω, each subdomain Ω_i is extended to a larger subdomain Ω_i', consisting of all elements of τ_h within a distance δ from Ω_i.

Our overlapping additive Schwarz preconditioner $\widehat{K}_{OAS}^{-1}$ for K_h is based on the solutions of local saddle point problems on the subdomains Ω_i' and on the solution of a coarse saddle point problem on the coarse mesh τ_H. In matrix form:

$$\widehat{K}_{OAS}^{-1} = R_0^T K_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T K_i^{-1} R_i, \tag{10}$$

where $R_0^T K_0^{-1} R_0$ represents the coarse problem and $R_i^T K_i^{-1} R_i$ represents the $i-$th local problem. This preconditioner is associated with the following decomposition

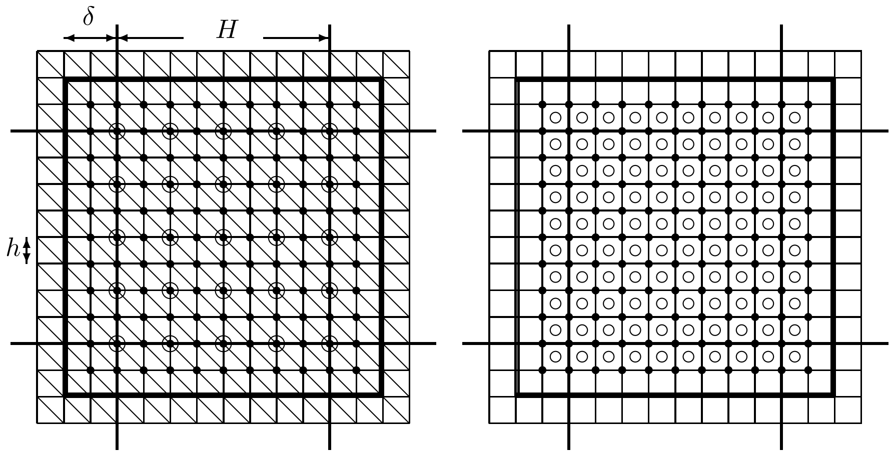

FIGURE 1. Local spaces associated with an interior subdomain Ω_i'. $P_1(h)-P_1(2h)$ (left) and $Q_1(h)-P_0(h)$ stabilized (right): velocity degrees of freedom are denoted by bullets (•), pressure degrees of freedom are denoted by circles (∘). Subdomain size $H/h = 8$, overlap $\delta = 2h$.

of the discrete space $\mathbf{V}^h \times U^h$ into a coarse space $\mathbf{V}_0^h \times U_0^h$ and local spaces $\mathbf{V}_i^h \times U_i^h$, associated with the subdomains Ω_i':

$$\mathbf{V}^h \times U^h = \mathbf{V}_0^h \times U_0^h + \sum_{i=1}^{N} \mathbf{V}_i^h \times U_i^h.$$

a) Coarse problem. For $P_1(h) - P_1(2h)$ elements, the coarse space is defined as

$$\mathbf{V}_0^h = \mathbf{V}^{H/2}, \qquad U_0^h = U^{H/2}.$$

The associated coarse stiffness matrix is $K_0 = K_{H/2}$, obtained using $P_1(H/2) - P_1(H)$ mixed elements and R_0^T represents the standard piecewise bilinear interpolation matrix between coarse and fine degrees of freedom, for both velocities and pressures. We use $H/2$ as the mesh size of the coarse velocities because we choose H as the mesh size of the coarse pressures.

For $Q_1(h) - P_0(h)$ stabilized elements, the coarse space is defined as

$$V_0^h = V^H, \qquad U_0^h = U^H.$$

The associated coarse stiffness matrix is $K_0 = K_H$ and R_0^T is the standard piecewise bilinear interpolation matrix between coarse and fine velocities and the standard injection matrix between coarse and fine pressures.

b) Local problems. For $P_1(h) - P_1(2h)$ finite elements (with continuous pressures), the local spaces consist of velocities and zero mean value pressures satisfying zero Dirichlet boundary conditions on the internal subdomain boundaries $\partial\Omega_i' \setminus \partial\Omega$:

$$\mathbf{V}_i^h = \mathbf{V}^h \cap (H_0^1(\Omega_i'))^d,$$

$$U_i^h = \{q \in U^h \cap L_0^2(\Omega_i') : q = 0 \ \text{ on } \ \partial\Omega_i' \setminus \partial\Omega \ \text{ and outside } \ \Omega_i'\}.$$

Here, the minimal overlap is one pressure element, i.e. $\delta = 2h$.

For $Q_1(h)-P_0(h)$ stabilized elements, the pressures are discontinuous piecewise constant functions and there are no degrees of freedom associated with $\partial\Omega_i' \setminus \partial\Omega$. In this case, we set to zero the pressure degrees of freedom in the elements that touch $\partial\Omega_i' \setminus \partial\Omega$. The associated local pressure spaces are:

$$U_i^h = \{q \in L_0^2(\Omega_i') : q|_T = 0 \ \forall T : \overline{T} \cap (\partial\Omega_i' \setminus \partial\Omega) \neq \emptyset\}.$$

Here, the minimal overlap is $\delta = h$. In matrix terms, the matrices R_i in (10) are restriction matrices returning the degrees of freedom associated with the interior of Ω_i' and $K_i = R_i K_h R_i^T$ are the local stiffness matrices. Each discrete local problem (and its matrix representation K_i) is nonsingular because of the zero mean-value constraint for the local pressure solution. See Figure 1 for a graphic representation of these local spaces in two dimensions.

We remark that $\widehat{K}_{OAS}^{-1}$ is a nonsingular preconditioner, since K_0 and $K_i, i = 1, \cdots, N$, are nonsingular matrices. In the symmetric cases (Stokes and elasticity), $\widehat{K}_{OAS}^{-1}$ is a symmetric indefinite preconditioner. If we need to work with global zero mean-value pressures, as in the Stokes and Oseen problems or in the incompressible limit of the mixed linear elasticity problem, we enforce this constraint in each application of the preconditioner.

7. Iterative substructuring methods for spectral element discretizations

The elimination of the interior unknowns in a saddle point problem is somewhat different than the analogous process in a positive definite problem. In this section, we illustrate this process for the spectral element discretization (see Section 5) of the Stokes problem. We will see that the remaining interface unknowns and constant pressures in each spectral element satisfy a reduced saddle point problem, analogous to the Schur complement in the positive definite case. We refer to Pavarino and Widlund [**45**] for a more complete treatment.
The interface Γ of the decomposition $\{\Omega_i\}$ of Ω is defined by

$$\Gamma = (\cup_{i=1}^N \partial\Omega_i) \setminus \partial\Omega.$$

The discrete space of restrictions to the interface is defined by

$$\mathbf{V}_\Gamma^n = \{\mathbf{v}|_\Gamma, \quad \mathbf{v} \in \mathbf{V}^n\}.$$

Γ is composed of N_F faces F_k (open sets) of the elements and the wire basket W, defined as the union of the edges and vertices of the elements, i.e.

$$\Gamma = \cup_{k=1}^{N_F} F_k \cup W. \tag{11}$$

We first define local subspaces consisting of velocities with support in the interior of individual elements,

$$\mathbf{V}_i^n = \mathbf{V}^n \cap H_0^1(\Omega_i)^3, \qquad i = 1, \cdots, N, \tag{12}$$

and local subspaces consisting of pressures with support and zero mean value in individual elements

$$U_i^n = U^n \cap L_0^2(\Omega_i), \qquad i = 1, \cdots, N. \tag{13}$$

The velocity space $\mathbf{V}^n$ is decomposed as

$$\mathbf{V}^n = \mathbf{V}_1^n + \mathbf{V}_2^n + \cdots + \mathbf{V}_N^n + \mathbf{V}_{\mathcal{S}}^n,$$

where the local spaces $\mathbf{V}_i^n$ have been defined in (12) and

$$\mathbf{V}_{\mathcal{S}}^n = \mathcal{S}^n(\mathbf{V}_\Gamma^n) \tag{14}$$

is the subspace of interface velocities. The discrete Stokes extension $\mathcal{S}^n$ is the operator that maps any $\mathbf{u} \in \mathbf{V}_\Gamma^n$ into the velocity component of the solution of the following Stokes problem on each element:
Find $\mathcal{S}^n\mathbf{u} \in \mathbf{V}^n$ and $p \in (\sum_{i=1}^N U_i^n)$ such that on each Ω_i

$$
\left\{
\begin{array}{llll}
s_n(\mathcal{S}^n\mathbf{u},\mathbf{v}) + b_n(\mathbf{v},p) & = & 0 & \forall \mathbf{v} \in \mathbf{V}_i^n \\
b_n(\mathcal{S}^n\mathbf{u},q) & = & 0 & \forall q \in U_i^n \\
\mathcal{S}^n\mathbf{u} = \mathbf{u} \quad \text{on} \quad \partial\Omega_i. & & &
\end{array}
\right.
\tag{15}
$$

Here the discrete bilinear forms are $s_n(\mathbf{u},\mathbf{v}) = \mu(\nabla\mathbf{u} : \nabla\mathbf{v})_{n,\Omega}$ and $b_n(\mathbf{u},p) = -(\mathrm{div}\mathbf{u},p)_{n,\Omega}$, where the discrete L^2-inner product has been defined in (7). In the elasticity case, an analogous interface space $\mathbf{V}_\mathcal{M}^n$ can be defined using a discrete mixed elasticity extension operator. The pressure space U^n is decomposed as

$$U^n = U_1^n + U_2^n + \cdots + U_N^n + U_0,$$

where the local spaces U_i^n have been defined in (13) and

$$U_0 = \{q \in U^n : q|_{\Omega_i} = \text{ constant}, \ i = 1,\cdots,N\}$$

consists of piecewise constant pressures in each element. The vector of unknowns is now reordered placing first the interior unknowns, element by element, and then the interface velocities and the piecewise constant pressures in each element:

$$(\mathbf{u},p)^T = (\mathbf{u}_1 \ p_1, \mathbf{u}_2 \ p_2, \ \cdots, \ \mathbf{u}_N \ p_N, \ \mathbf{u}_\Gamma \ p_0)^T.$$

After this reordering, our saddle point problem (9) has the following matrix structure:

$$
\begin{bmatrix}
A_{11} & B_{11}^T & \cdots & 0 & 0 & A_{1\Gamma} & 0 \\
B_{11} & 0 & \cdots & 0 & 0 & B_{1\Gamma} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & A_{NN} & B_{NN}^T & A_{N\Gamma} & 0 \\
0 & 0 & \cdots & B_{NN} & 0 & B_{N\Gamma} & 0 \\
A_{\Gamma 1} & B_{1\Gamma}^T & \cdots & A_{\Gamma N} & B_{N\Gamma}^T & A_{\Gamma\Gamma} & B_0^T \\
0 & 0 & \cdots & 0 & 0 & B_0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\ p_1 \\ \vdots \\ \mathbf{u}_N \\ p_N \\ \mathbf{u}_\Gamma \\ p_0
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{b}_1 \\ 0 \\ \vdots \\ \mathbf{b}_N \\ 0 \\ \mathbf{b}_\Gamma \\ 0
\end{bmatrix}.
\tag{16}
$$

The leading block of this matrix is the direct sum of N local saddle point problems for the interior velocities and pressures $(\mathbf{u}_i, p_i)$. In addition there is a reduced saddle point problem for the interface velocities and piecewise constant pressures $(\mathbf{u}_\Gamma, p_0)$. These subsystems are given by

$$
\left\{
\begin{array}{lll}
A_{ii}\mathbf{u}_i + B_{ii}^T p_i & = & \mathbf{b}_i - A_{i\Gamma}\mathbf{u}_\Gamma \\
B_{ii}\mathbf{u}_i & = & -B_{i\Gamma}\mathbf{u}_\Gamma
\end{array}
\right.
\qquad i = 1,2,\cdots,N,
\tag{17}
$$

and

$$
\left\{
\begin{array}{lll}
A_{\Gamma\Gamma}\mathbf{u}_\Gamma + A_{\Gamma 1}\mathbf{u}_1 + \cdots + A_{\Gamma N}\mathbf{u}_N + B_{1\Gamma}^T p_1 + \cdots + B_{N\Gamma}^T p_N + B_0^T p_0 & = & \mathbf{b}_\Gamma \\
B_0\mathbf{u}_\Gamma & = & 0.
\end{array}
\right.
\tag{18}
$$

The local saddle point problems (17) are uniquely solvable because the local pressures are constrained to have zero mean value. The reduced saddle point problem

(18) can be written more clearly by introducing the linear operators R_i^b, R_i^Γ and P_i^b, P_i^Γ representing the solutions of the $i-$th local saddle point problem:

$$\mathbf{u}_i = R_i^b \mathbf{b}_i + R_i^\Gamma \mathbf{u}_\Gamma, \qquad p_i = P_i^b \mathbf{b}_i + P_i^\Gamma \mathbf{u}_\Gamma, \qquad i = 1, 2, \cdots, N.$$

Then (18) can be rewritten as

$$\left\{ \begin{array}{lcccl} S_\Gamma \mathbf{u}_\Gamma & + & B_0^T p_0 & = & \tilde{\mathbf{b}}_\Gamma \\ B_0 \mathbf{u}_\Gamma & & & = & 0, \end{array} \right. \tag{19}$$

where

$$S_\Gamma = A_{\Gamma\Gamma} + \sum_{i=1}^N A_{\Gamma i} R_i^\Gamma + \sum_{i=1}^N B_{i\Gamma}^T P_i^\Gamma, \qquad \tilde{\mathbf{b}}_\Gamma = \mathbf{b}_\Gamma - \sum_{i=1}^N A_{\Gamma i} R_i^b \mathbf{b}_i - \sum_{i=1}^N B_{i\Gamma}^T P_i^b \mathbf{b}_i.$$

As always, the matrices R_i^b, R_i^Γ and P_i^b, P_i^Γ need not be assembled explicitly; their action on given vectors is computed by solving the corresponding local saddle point problem. Analogously, S_Γ need not be assembled, since its action on a given vector can be computed by solving the N local saddle point problems (17) with $\mathbf{b}_i = 0$. The right-hand side $\tilde{\mathbf{b}}_\Gamma$ is formed from an additional set of solutions of the N local saddle point problems (17) with $\mathbf{u}_\Gamma = 0$.

The saddle point Schur complement (19) satisfies a uniform inf-sup condition (see [**45**] for a proof):

LEMMA 1.

$$\sup_{\mathcal{S}^n \mathbf{v} \in \mathbf{V}_{\mathcal{S}}^n} \frac{(\mathrm{div} \mathcal{S}^n \mathbf{v}, q_0)^2}{s_n(\mathcal{S}^n \mathbf{v}, \mathcal{S}^n \mathbf{v})} \geq \beta_\Gamma^2 \|q_0\|_{L^2}^2 \qquad \forall q_0 \in U_0,$$

where β_Γ is independent of $q_0, n,$ and N.

An analogous stability result holds for the incompressible elasticity case.

7.1. Block preconditioners for the saddle point Schur complement. We solve the saddle point Schur complement system (19) by some preconditioned Krylov space method such as PCR, if we use a symmetric positive definite preconditioner and the problem is symmetric, or GMRES if we use a more general preconditioner. Let S be the coefficient matrix of the reduced saddle point problem (19)

$$S = \begin{bmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{bmatrix}. \tag{20}$$

We will consider the following block-diagonal and lower block-triangular preconditioners (an upper block-triangular preconditioner could be considered as well):

$$\widehat{S}_D = \begin{bmatrix} \widehat{S}_\Gamma & 0 \\ 0 & \widehat{C}_0 \end{bmatrix} \qquad \widehat{S}_T = \begin{bmatrix} \widehat{S}_\Gamma & 0 \\ B_0 & -\widehat{C}_0 \end{bmatrix},$$

where $\widehat{S}_\Gamma$ and $\widehat{C}_0$ are good preconditioners for S_Γ and the coarse pressure mass matrix C_0, respectively. We refer to Klawonn [**31, 32**] for an analysis of block preconditioners. We consider two choices for $\widehat{S}_\Gamma$, based on wire basket and Neumann-Neumann techniques, and we take $\widehat{C}_0 = C_0$.

a) A wire basket preconditioner for Stokes problems. We first consider a simple Laplacian-based wire basket preconditioner

$$\widehat{S}_\Gamma = \begin{bmatrix} \widehat{S}_W & 0 & 0 \\ 0 & \widehat{S}_W & 0 \\ 0 & 0 & \widehat{S}_W \end{bmatrix}, \tag{21}$$

where we use on each scalar component the scalar wire basket preconditioner introduced in Pavarino and Widlund [**44**] and extended to GLL quadrature based approximations in [**46**],

$$\widehat{S}_W^{-1} = R_0 \widehat{S}_{WW}^{-1} R_0^T + \sum_{k=1}^{N_F} R_{F_k} S_{F_k F_k}^{-1} R_{F_k}^T.$$

Here R_0 is a matrix representing a change of basis in the wire basket space, $R_{F_k}^T$ are restriction matrices returning the degrees of freedom associated with the face $F_k, k = 1, \cdots, N_F$, and $\widehat{S}_{WW}$ is an approximation of the original wire basket block. This is an additive preconditioner with independent parts associated with each face and the wire basket of the elements, defined in (11). It satisfies the following bound, proven in [**45**].

THEOREM 2. *Let the blocks of the block-diagonal preconditioner $\widehat{S}_D$ be the wire basket preconditioner $\widehat{S}_\Gamma$ defined in (21) and the coarse mass matrix C_0. Then the Stokes saddle point Schur complement S preconditioned by $\widehat{S}_D$ satisfies*

$$cond(\widehat{S}_D^{-1} S) \leq C \frac{(1 + \log n)^2}{\beta_n},$$

where C is independent of n and N.

The mixed elasticity case is more complicated, but an analogous wire basket preconditioner can be constructed and analyzed; see [**45**].

b) A Neumann-Neumann preconditioner for Stokes problems. In the Stokes case, we could also use a Laplacian-based Neumann-Neumann preconditioner on each scalar component; see Dryja and Widlund [**22**], Le Tallec [**35**] for a detailed analysis of this family of preconditioners for $h-$version finite elements and Pavarino [**42**] for an extension to spectral elements. In this case,

$$\widehat{S}_\Gamma = \begin{bmatrix} \widehat{S}_{NN} & 0 & 0 \\ 0 & \widehat{S}_{NN} & 0 \\ 0 & 0 & \widehat{S}_{NN} \end{bmatrix}, \tag{22}$$

where

$$\widehat{S}_{NN}^{-1} = R_H^T K_H^{-1} R_H + \sum_{j=1}^{N} R_{\partial\Omega_j}^T D_j^{-1} \widehat{S}_j^{\dagger} D_j^{-1} R_{\partial\Omega_j}$$

is an additive preconditioner with independent coarse solver K_H^{-1} and local solvers $\widehat{S}_j^{\dagger}$, respectively associated with the coarse triangulation determined by the elements and with the boundary $\partial\Omega_j$ of each element. Here $R_{\partial\Omega_j}$ are restriction matrices returning the degrees of freedom associated with the boundary of Ω_j, D_j are diagonal matrices and † denotes an appropriate pseudo-inverse for the singular Schur complements associated with interior elements; see [**22, 42**] for more details. Also for this preconditioner, a polylogarithmic bound is proven in [**45**].

THEOREM 3. *Let the blocks of the block-diagonal preconditioner $\widehat{S}_D$ be the Neumann-Neumann preconditioner $\widehat{S}_\Gamma$ defined in (22) and the coarse mass matrix C_0. Then the Stokes saddle point Schur complement S preconditioned by $\widehat{S}_D$ satisfies*

$$cond(\widehat{S}_D^{-1} S) \leq C \frac{(1+\log n)^2}{\beta_n},$$

where C is independent of n and N.

Other scalar iterative substructuring preconditioners could also be applied in this fashion to the Stokes system; see Dryja, Smith, and Widlund [**19**].

8. Numerical results

In this section, we report the results of numerical experiments with the overlapping additive Schwarz method described in Section 6 and with some of the iterative substructuring methods described in Section 7. The two sets of results cannot be directly compared because the overlapping method is applied to h-version discretizations in two dimensions, while the iterative substructuring methods are applied to spectral element discretizations in three dimensions. All the computations were performed in MATLAB.

8.1. Overlapping Schwarz methods for h-version discretizations in two dimensions. In the following tables, we report the iteration counts for the iterative solution of our three model saddle point problems (Stokes, mixed elasticity, and Oseen) with the overlapping additive Schwarz method of Section 6, i.e. with the preconditioner $\widehat{K}_{OAS}^{-1}$ defined in (10). In each application of our preconditioner, we solve the local and coarse saddle point problems directly by gaussian elimination. Inexact local and/or coarse solvers could also be considered, as in positive definite problems. We accelerate the iteration with GMRES, with zero initial guess and stopping criterion $\|r_i\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_i is the $i-$th residual. Other Krylov space accelerators, such as BiCGSTAB or QMR, could be used. The computational domain Ω is the unit square, subdivided into $\sqrt{N} \times \sqrt{N}$ square subdomains. More complete results for Stokes problems, including multiplicative and other variants of the preconditioner, can be found in Klawonn and Pavarino [**33**].

a) $P_1(h) - P_1(2h)$ finite elements for the Stokes problem. Table 1 reports the iteration counts (with and without coarse solver) and relative errors in comparison with the direct solution (in the max norm) for the Stokes problem (1) discretized with $P_1(h) - P_1(2h)$ finite elements. Here $u = 0$ on $\partial\Omega$ and f is a uniformly distributed random vector. The overlap δ is kept constant and minimal, i.e. the size $\delta = 2h$ of one pressure element. h is refined and N is increased so that the subdomain size is kept constant at $H/h = 8$ (scaled speedup). The global problem size varies from 531 to 14,163 unknowns. The empty entry in the table (-) could not be run due to memory limitations. The results indicate that the number of iterations required by the algorithm is bounded by a constant independent of h and N. As in the positive definite case, the coarse problem is essential for scalability: without the coarse problem, the number of iterations grows with N. These results are also plotted in Figure 2 (left). The convergence history of GMRES (with and without $\widehat{K}_{OAS}^{-1}$ as preconditioner) is shown in Figure 3 (left), for the case with 16 subdomains and $h = 1/32$.

TABLE 1. Stokes problem with $P_1(h) - P_1(2h)$ finite elements: iteration counts and relative errors for GMRES with the overlapping additive Schwarz preconditioner $\widehat{K}_{OAS}^{-1}$; constant subdomain size $H/h = 8$, minimal overlap $\delta = 2h$.

		with coarse		no coarse	
$\sqrt{N}$	h^{-1}	iter.	$\|x^m - x\|_\infty / \|x\|_\infty$	iter.	$\|x^m - x\|_\infty / \|x\|_\infty$
2	16	17	3.42e-7	21	9.05e-7
3	24	18	1.04e-6	33	1.82e-6
4	32	19	5.72e-7	43	9.53e-6
5	40	19	1.84e-6	53	1.07e-5
6	48	19	1.75e-6	63	1.39e-5
7	56	20	1.10e-6	73	3.17e-5
8	64	20	1.42e-6	86	4.60e-5
9	72	20	1.13e-6	-	-
10	80	20	1.79e-6	-	-

TABLE 2. Lid-driven cavity Stokes flow with $Q_1(h) - P_0(h)$ stab. finite elements: iteration counts and relative errors for GMRES with the overlapping additive Schwarz preconditioner $\widehat{K}_{OAS}^{-1}$; constant subdomain size $H/h = 8$.

			with coarse		no coarse	
overlap	$\sqrt{N}$	h^{-1}	iter.	$\|x^m - x\|_\infty / \|x\|_\infty$	iter.	$\|x^m - x\|_\infty / \|x\|_\infty$
	2	16	18	5.58e-7	14	4.22e-1
$\delta = h$	4	32	27	2.04e-6	27	4.88e-1
	8	64	31	7.35e-7	44	5.14e-1
	2	16	16	1.93e-6	16	1.20e-7
$\delta = 2h$	4	32	21	3.84e-7	37	8.06e-7
	8	64	22	2.51e-7	81	1.96e-6

b) $Q_1(h) - P_0(h)$ stabilized finite elements for the Stokes problem. Table 2 reports the iteration counts and relative errors in comparison with the direct solution for the Stokes problem (1) discretized with $Q_1(h) - P_0(h)$ stabilized finite elements, using the MATLAB software of Elman, Silvester, and Wathen [**26**], which requires $1/h$ to be a power of two. Here the boundary conditions and right-hand side are imposed to obtain a lid-driven cavity Stokes flow. The default value of the stabilization parameter β in (5) is $1/4$. The global problem size varies from 834 to 12,546 unknowns and, as before, we study the scaled speedup of the algorithm with $H/h = 8$. We could run only three cases ($N = 4, 16, 64$), but the iteration counts seem to behave as in the corresponding cases in Table 1 for $P_1(h) - P_1(2h)$ finite elements. Therefore the experiments seem to indicate a constant bound on the number of iterations that is independent of h and N. Again, the coarse space is essential for obtaining scalability. Here we can use a minimal overlap of $\delta = h$ since both velocities and pressures use the same mesh τ_h. We also report the results for $\delta = 2h$ to allow a comparison with the results of Table 1 (where the minimal

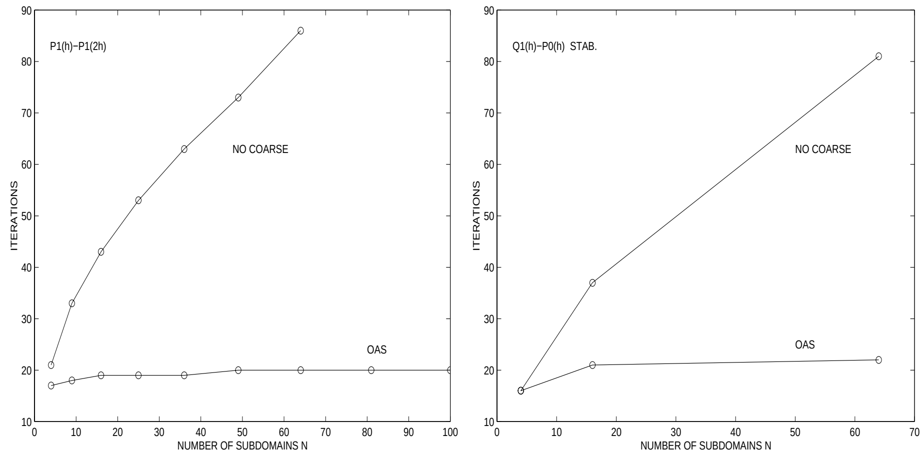

FIGURE 2. Iteration counts for GMRES with overlapping additive Schwarz preconditioner $\widehat{K}_{OAS}^{-1}$ (with and without coarse problem): subdomain size $H/h = 8$, overlap $\delta = 2h$, Stokes problem with $P_1(h) - P_1(2h)$ finite elements (left), lid-driven cavity Stokes flow with $Q_1(h) - P_0(h)$ stab. elements (right).

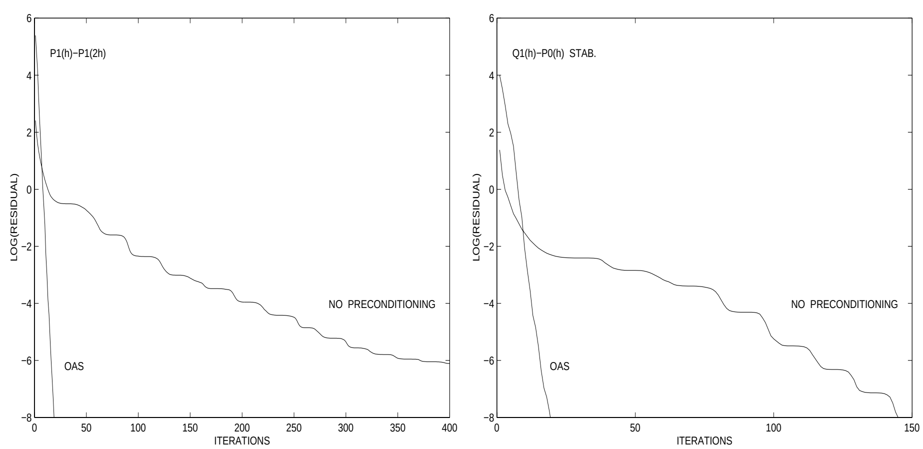

FIGURE 3. Convergence history for GMRES with and without overlapping additive Schwarz preconditioner $\widehat{K}_{OAS}^{-1}$: $N = 16$, subdomain size $H/h = 8$, overlap $\delta = 2h$, Stokes problem with $P_1(h) - P_1(2h)$ finite elements (left), lid-driven cavity Stokes flow with $Q_1(h) - P_0(h)$ stab. elements (right).

overlap is $\delta = 2h$). These results are also plotted in Figure 2 (right). The convergence history of GMRES (with and without $\widehat{K}_{OAS}^{-1}$ as preconditioner) for the case $N = 16, h = 1/32, \delta = 2h$, is plotted in Figure 3 (right).

c) $P_1(h) - P_1(2h)$ finite elements for mixed elasticity. Analogous results were obtained for the mixed formulation of the elasticity system (2), discretized with $P_1(h) - P_1(2h)$ finite elements, with $u = 0$ on $\partial\Omega$. The results of Table 3 indicate

TABLE 3. Mixed linear elasticity with $P_1(h) - P_1(2h)$ finite elements: iteration counts for GMRES with the overlapping additive Schwarz preconditioner $\widehat{K}_{OAS}^{-1}$; subdomain size $H/h = 8$, minimal overlap $\delta = 2h$.

		Poisson ratio ν						
$\sqrt{N}$	$1/h$	0.3	0.4	0.49	0.499	0.4999	0.49999	0.5
2	16	15	15	17	17	17	17	17
3	24	17	17	18	18	18	18	18
4	32	18	18	19	19	19	19	19
5	40	18	18	19	19	19	19	19
6	48	18	18	19	19	19	19	19
7	56	19	19	19	20	20	20	20
8	64	19	19	20	20	20	20	20
9	72	19	19	20	20	20	20	20
10	80	19	19	20	20	20	20	20

TABLE 4. Oseen problem with $Q_1(h) - P_0(h)$ stabilized finite elements and circular vortex $\mathbf{w} = (2y(1-x^2), -2x(1-y^2))$: iteration counts and relative errors for GMRES with the overlapping additive Schwarz preconditioner $\widehat{K}_{OAS}^{-1}$, constant subdomain size $H/h = 8$, overlap $\delta = h$.

			with coarse		no coarse	
	$\sqrt{N}$	h^{-1}	iter.	err.	iter.	err.
$\mu = 1$	2	16	19	1.12e-6	14	7.08e-7
	4	32	25	7.28e-7	31	2.71e-6
	8	64	30	7.86e-7	79	1.15e-6
$\mu = 0.1$	2	16	21	3.81e-7	15	5.47e-7
	4	32	26	6.03e-7	32	7.37e-7
	8	64	27	9.89e-7	99	3.27e-6
$\mu = 0.02$	2	16	29	9.43e-7	22	9.16e-7
	4	32	39	4.84e-7	42	4.81e-7
	8	64	42	1.15e-6	118	1.69e-6
$\mu = 0.01$	2	16	35	9.34e-7	29	9.53e-7
	4	32	51	2.02e-6	53	1.80e-6
	8	64	58	1.62e-6	211	1.45e-5

that the convergence rate of our method is bounded independently of h, N, and the Poisson ratio when approaching the incompressible limit $\nu = 0.5$.

d) $Q_1(h) - P_0(h)$ stabilized finite elements for the Oseen problem. Table 4 reports the iteration counts for GMRES with $\widehat{K}_{OAS}^{-1}$ and $\delta = h$, and the relative errors in comparison with the direct solution, for the Oseen problem (3), using the MATLAB software of Elman, Silvester, and Wathen [**26**]. The divergence-free field $\mathbf{w}$ is a circular vortex, $\mathbf{w} = (2y(1-x^2), -2x(1-y^2))$, and the stabilization parameter is 1/4. We study the scaled speedup of the algorithm with $H/h = 8$, running the three cases $N = 4, 16, 64$ for each given value of the diffusion parameter

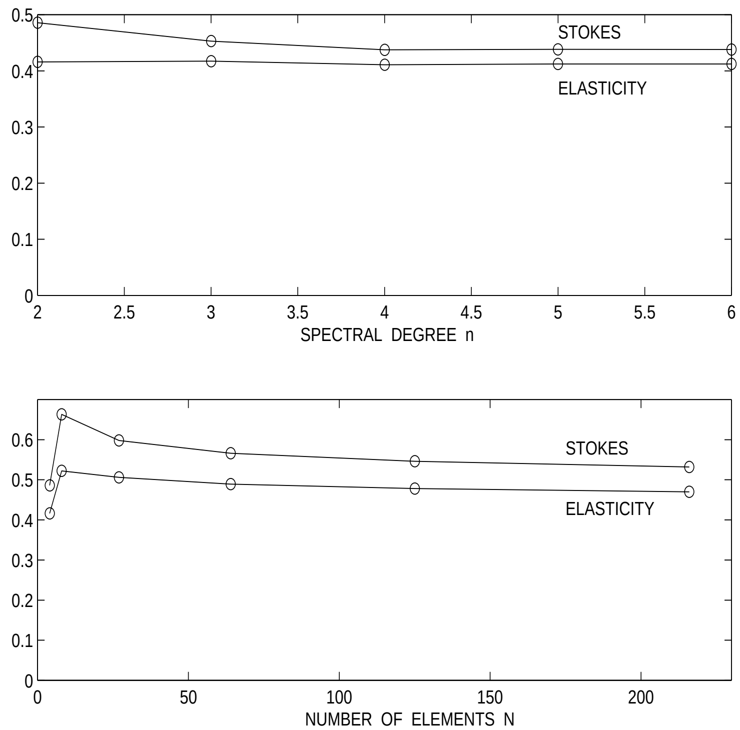

FIGURE 4. Inf-sup constant β_Γ for the Stokes and incompressible mixed elasticity saddle point Schur complement ($Q_n - Q_{n-2}$ spectral elements)

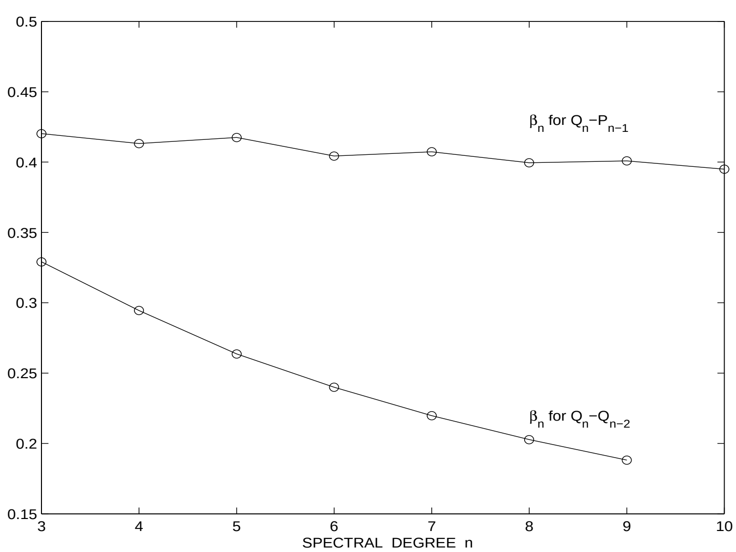

FIGURE 5. Inf-sup constant β_n for the discrete Stokes problem ($Q_n - Q_{n-2}$ and $Q_n - P_{n-1}$ spectral elements)

Table 5. Linear elasticity in mixed form: local condition number $cond(\widehat{S}_\Gamma^{-1} S_\Gamma)$ of the local saddle point Schur complement with wire basket preconditioner (with original wire basket block) on one interior element; $Q_n - Q_{n-2}$ method.

n	Poisson ratio ν						
	0.3	0.4	0.49	0.499	0.4999	0.49999	0.5
2	9.06	9.06	9.06	9.06	9.06	9.06	9.06
3	17.54	20.19	44.92	58.26	60.12	60.31	60.33
4	24.45	29.69	62.30	85.35	88.77	89.13	89.17
5	34.44	38.68	76.69	106.72	111.49	111.99	112.05
6	40.97	46.84	90.97	129.73	136.38	137.09	137.17
7	51.23	55.65	107.19	153.29	161.97	162.90	162.99
8	59.70	64.60	122.13	176.32	187.45	188.66	188.66

Table 6. Generalized Stokes problem: local condition number $cond(\widehat{S}_\Gamma^{-1} S_\Gamma)$ of the local saddle point Schur complement with wire basket preconditioner (with original wire basket block) on one interior element; $Q_n - Q_{n-2}$ method.

n	Poisson ratio ν						
	0.3	0.4	0.49	0.499	0.4999	0.49999	0.5
2	4.89	4.89	4.89	4.89	4.89	4.89	4.89
3	14.13	17.31	36.55	44.79	45.88	45.99	46.00
4	19.18	24.24	54.33	73.08	75.76	76.04	76.07
5	24.18	30.56	66.25	86.85	89.92	90.24	90.28
6	28.71	36.29	87.52	121.36	126.52	127.07	127.13
7	33.44	42.15	95.50	130.82	136.25	136.82	136.89
8	38.36	48.71	114.89	163.55	171.49	172.34	172.43

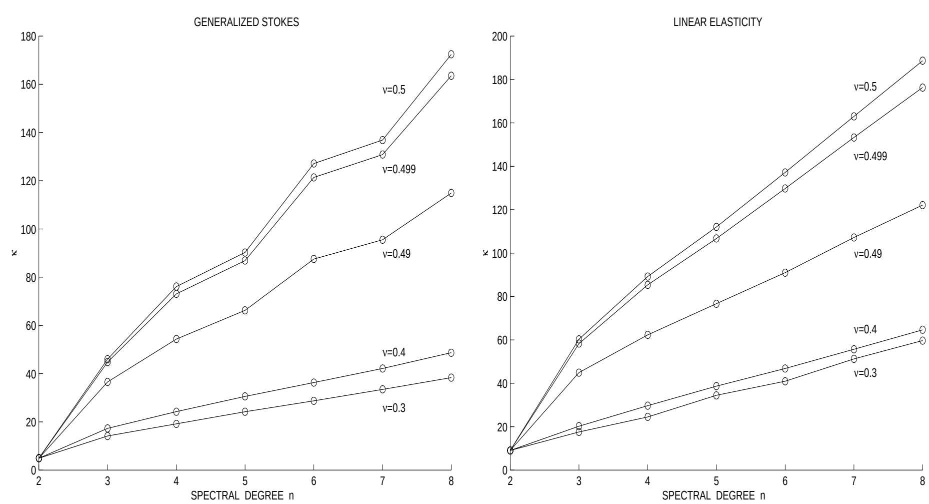

Figure 6. Local condition number $cond(\widehat{S}_\Gamma^{-1} S_\Gamma)$ from Tables 5 and 6; generalized Stokes problem (left), mixed elasticity (right)

μ. The results indicate a bound on the number of iterations that is independent of h and N, but that grows with the inverse of the diffusion parameter μ.

8.2. Iterative substructuring for spectral element discretizations in three dimensions. We first computed the discrete inf-sup constant β_Γ of the saddle point Schur complement (20), for both the mixed elasticity and Stokes system discretized with $Q_n - Q_{n-2}$ spectral elements. β_Γ is computed as the square root of the minimum nonzero eigenvalue of $C_0^{-1}B_0^T S_\Gamma^{-1} B_0$, where S_Γ and B_0 are the blocks in (20) and C_0 is the coarse pressure mass matrix. The upper plot in Figure 4 shows β_Γ as a function of the spectral degree n while keeping fixed a small number of elements, $N = 2 \times 2 \times 1$. The lower plot in Figure 4 shows β_Γ as a function of the number of spectral elements N for a small fixed spectral degree $n = 2$. Both figures indicate that β_Γ is bounded by a constant independent of N and n, in agreement with Lemma 1. We also computed the discrete inf-sup constant β_n of the whole Stokes problem on the reference cube by computing the square root of the minimum nonzero eigenvalue of $C_n^{-1}B_n^T A_n^{-1} B_n$, where A_n, B_n, and C_n are the blocks in (9). The results are plotted in Figure 5. The inf-sup parameter of the $Q_n - P_{n-1}$ method is much better than that of the $Q_n - Q_{n-2}$ method, in agreement with the theoretical results of [**5**] and the experiments in [**43**].

We next report on the local condition numbers of $\widehat{S}_\Gamma^{-1} S_\Gamma$ for one interior element. Here S_Γ is the velocity block in the saddle point Schur complement (20) and $\widehat{S}_\Gamma^{-1}$ is the wire basket preconditioner described in Section 7 for the Stokes case. We report only the results obtained with the original wire basket block of the preconditioner, while we refer to Pavarino and Widlund [**45**] for more complete results. Table 5 presents the results for the mixed elasticity problem, while Table 6 gives the results for the generalized Stokes problem (in which there is a penalty term of the form $-t^2(p,q)_{L^2}$). These results are also plotted in Figure 6. In both cases, the incompressible limit is clearly the hardest, yielding condition numbers three or four times as large as those of the corresponding compressible case. For a given value of ν, the condition number seems to grow linearly with n, which is consistent with our theoretical results in Theorem 2 and 3.

References

1. K. J. Arrow, L. Hurwicz, and H. Uzawa, *Studies in linear and non-linear programming*, Stanford University Press, Stanford, CA, 1958.
2. I. Babuška and M. Suri, *Locking effects in the finite element approximation of elasticity problems*, Numer. Math. **62** (1992), 439–463.
3. C. Bernardi and Y. Maday, *Approximations spectrales de problèmes aux limites elliptiques*, Springer-Verlag France, Paris, 1992.
4. ———, *Spectral Methods*, Handbook of Numerical Analysis, Volume V: Techniques of Scientific Computing (Part 2), North-Holland, 1997, pp. 209–485.
5. ———, *Uniform inf-sup conditions for the spectral element discretization of the Stokes problem*, Tech. Report 97034, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie – Centre National de la Recherche Scientifique, October 1997.
6. D. Braess, *Stability of saddle point problems with penalty*, RAIRO M^2AN **30** (1996), no. 6, 731–742.
7. D. Braess and C. Blömer, *A multigrid method for a parameter dependent problem in solid mechanics*, Numer. Math. **57** (1990), 747–761.
8. J.H. Bramble and J.E. Pasciak, *A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems*, Math. Comp. **50** (1988), 1–17.
9. ———, *A domain decomposition technique for Stokes problems*, Appl. Numer. Math. **6** (1989/90), 251–261.

10. J.H. Bramble, J.E. Pasciak, and A. Vassilev, *Analysis of the inexact Uzawa algorithm for saddle point problems*, SIAM J. Numer. Anal. **34** (1997), no. 3, 1072–1092.
11. S.C. Brenner, *Multigrid methods for parameter dependent problems*, RAIRO M^2AN **30** (1996), no. 3, 265–297.
12. F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, Berlin, 1991.
13. X.-C. Cai and O. B. Widlund, *Domain decomposition algorithms for indefinite elliptic problems*, SIAM J. Sci. Statist. Comput. **13** (1992), no. 1, 243–258.
14. ______, *Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems*, SIAM J. Numer. Anal. **30** (1993), no. 4, 936–952.
15. C. Canuto, *Stabilization of spectral methods by finite element bubble functions*, Comp. Meths. Appl. Mech. Eng. **116** (1994), no. 1–4, 13–26.
16. C. Canuto and V. Van Kemenade, *Bubble-stabilized spectral methods for the incompressible Navier-Stokes equations*, Comp. Meths. Appl. Mech. Eng. **135** (1996), no. 1–2, 35–61.
17. M. A. Casarin, *Schwarz preconditioners for spectral and mortar finite element methods with applications to incompressible fluids*, Ph.D. thesis, Dept. of Mathematics, Courant Institute of Mathematical Sciences, New York University, March 1996.
18. T. F. Chan and T. P. Mathew, Domain decomposition algorithms, Acta Numerica (1994), 61–143, Acta Numerica (1994), Cambridge University Press, 1994, pp. 61–143.
19. M. Dryja, B. F. Smith, and O. B. Widlund, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal. **31** (1994), no. 6, 1662–1694.
20. M. Dryja and O. B. Widlund, *An additive variant of the Schwarz alternating method for the case of many subregions*, Tech. Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.
21. ______, *Domain decomposition algorithms with small overlap*, SIAM J. Sci.Comput. **15** (1994), no. 3, 604–620.
22. ______, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Comm. Pure Appl. Math. **48** (1995), no. 2, 121–155.
23. H. C. Elman, *Preconditioning for the steady-state Navier–Stokes equations with low viscosity*, Tech. Report CS-TR-3712, UMIACS-TR-96-82, University of Maryland, 1996.
24. H. C. Elman and G. Golub, *Inexact and preconditioned Uzawa algorithms for saddle point problems*, SIAM J. Numer. Anal. **31** (1994), no. 6, 1645–1661.
25. H. C. Elman and D. Silvester, *Fast nonsymmetric iterations and preconditioning for Navier-Stokes equations*, SIAM J. Sci. Comp. **17** (1996), 33–46.
26. H. C. Elman, D. J. Silvester, and A. J. Wathen, *Iterative methods for problems in computational fluid dynamics*, Iterative Methods in Scientific Computing (T. Chan R. Chan and G. Golub, eds.), Springer-Verlag, 1997.
27. R. E. Ewing and J. Wang, *Analysis of the Schwarz algorithm for mixed finite element methods*, Math. Model. Anal. Numer. **26** (1992), 739–756.
28. P.F. Fischer and E. Rønquist, *Spectral element methods for large scale parallel Navier-Stokes calculations*, Comp. Meths. Appl. Mech. Eng. **116** (1994), 69–76.
29. M. Fortin and D. Aboulaich, *Schwarz's decomposition method for incompressible flow problems*, First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Philadelphia) (R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds.), SIAM, 1988.
30. N. Kechkar and D. J. Silvester, *Analysis of locally stabilised mixed finite element methods for the Stokes problem*, Math. Comp. **58** (1992), 1–10.
31. A. Klawonn, *An optimal preconditioner for a class of saddle point problems with a penalty term*, Tech. report, Westfälische Wilhelms-Universität Münster, 1994, To appear in SIAM J. Sci. Comp., Vol. 19, #2 (1998).
32. ______, *Block-triangular preconditioners for saddle point problems with a penalty term*, Tech. report, Westfälische Wilhelms-Universität Münster, 1995, To appear in SIAM J. Sci. Comp., Vol. 19, #1 (1998).
33. A. Klawonn and L. F. Pavarino, *Overlapping Schwarz methods for mixed linear elasticity and Stokes problems*, Tech. Report 15-97 N, Westfälische Wilhelms-Universität Münster, 1997, To appear in Comp. Meths. Appl. Mech. Eng.

34. A. Klawonn and G. Starke, *Block Triangular Preconditioners for Nonsymmetric Saddle Point Problems: Field–of–Values Analysis*, Tech. Report 04/97-N, Westfälische Wilhelms-Universität Münster, Germany, Schriftenreihe Angewandte Mathematik und Informatik, March 1997.
35. P. Le Tallec, *Domain decomposition methods in computational mechanics*, Computational Mechanics Advances (J. Tinsley Oden, ed.), vol. 1 (2), North-Holland, 1994, pp. 121–220.
36. P. Le Tallec and A. Patra, *Non-overlapping domain decomposition methods for adaptive hp approximations of the Stokes problem with discontinuous pressure fields*, Comp. Meths. Appl. Mech. Eng. **145** (1997), 361–379.
37. P. L. Lions, *On the Schwarz alternating method. I.*, First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Philadelphia) (R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds.), SIAM, 1988.
38. Y. Maday, D. Meiron, A. Patera, and E. Rønquist, *Analysis of iterative methods for the steady and unsteady Stokes problem: application to spectral element discretizations*, SIAM J. Sci. Comp. **14** (1993), no. 2, 310–337.
39. Y. Maday, A. Patera, and E. Rønquist, *The $P_N \times P_{N-2}$ method for the approximation of the Stokes problem*, Tech. Report 92009, Dept. of Mech. Engr., M.I T., 1992.
40. T. P. Mathew, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part I: Algorithms and Numerical results*, Numer. Math. **65** (1993), no. 4, 445–468.
41. ———, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part II: Theory*, Numer. Math. **65** (1993), no. 4, 469–492.
42. L. F. Pavarino, *Neumann-Neumann algorithms for spectral elements in three dimensions*, RAIRO M^2AN **31** (1997), no. 4, 471–493.
43. ———, *Preconditioned mixed spectral element methods for elasticity and Stokes problems*, SIAM J. Sci. Comp. (1998), To appear.
44. L. F. Pavarino and O. B. Widlund, *A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions*, SIAM J. Numer. Anal. **33** (1996), no. 4, 1303–1335.
45. ———, *Iterative substructuring methods for spectral element discretizations of elliptic systems. II: Mixed methods for linear elasticity and Stokes flow*, Tech. Report 755, Dept. of Computer Science, Courant Institute of Mathematical Sciences, December 1997.
46. ———, *Iterative substructuring methods for spectral elements: Problems in three dimensions based on numerical quadrature*, Comp. Math. Appl. **33** (1997), no. 1/2, 193–209.
47. A. Quarteroni, *Domain decomposition algorithms for the Stokes equations*, Domain Decomposition Methods (Philadelphia) (T. Chan, R. Glowinski, J. Périaux, and O. B. Widlund, eds.), SIAM, 1989.
48. E. Rønquist, *A domain decomposition solver for the steady Navier-Stokes equations*, Proc. of ICOSAHOM '95 (A.V. Ilin and L.R. Scott, eds.), 1996.
49. T. Rusten, P. S. Vassilevski, and R. Winther, *Interior penalty preconditioners for mixed finite element approximations of elliptic problems*, Math. Comp. **65** (1996), no. 214, 447–466.
50. T. Rusten and R. Winther, *A preconditioned iterative method for saddle point problems*, SIAM J. Matr. Anal. Appl. **13** (1992), 887–904.
51. D. Silvester and A. Wathen, *Fast iterative solution of stabilised Stokes systems. Part II: Using general block preconditioners*, SIAM J. Numer. Anal. **31** (1994), no. 5, 1352–1367.
52. B. F. Smith, P. Bjørstad, and W. D. Gropp, *Domain decomposition: Parallel multilevel methods for elliptic partial differential equations*, Cambridge University Press, 1996.
53. M. Suri and R. Stenberg, *Mixed hp finite element methods for problems in elasticity and Stokes flow*, Numer. Math. **72** (1996), no. 3, 367–390.
54. R. Verfürth, *A multilevel algorithm for mixed problems*, SIAM J. Numer. Anal. **21** (1984), 264–271.
55. G. Wittum, *Multigrid methods for the Stokes and Navier-Stokes equations*, Numer. Math. **54** (1989), 543–564.

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