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Non-conforming Domain Decomposition Method for Plate and Shell Problems

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1. Introduction

The mortar element method is an optimal domain decomposition method for the approximation of partial differential equations on non-matching grids. There already exists applications of the mortar method to Navier-Stokes, elasticity, and Maxwell problems. The aim of this paper is to provide an extension of the mortar method to plates and shells problems. We first recall the Discrete Kirchhoff Triangles element method (D.K.T.) to approximate the plate and shell equations. The aim of this paper is then to explain what has to be changed in the definition of the D.K.T. method when the triangulation is nonconforming. Numerical results will illustrate the optimality of the mortar element method extended to shell problems and the efficiency of the FETI solution algorithm.

2. Recalling the D.K.T. method

We recall that the Koiter equations are deduced from the Naghdi equations (whose unknowns are the displacement of the mean surface $\vec{u} = (u_1, u_2, w)$ and the rotations $\underline{\beta} = (\underline{\beta}_1, \underline{\beta}_2)$ in the plane tangential to Ω) by imposing the Kirchhoff-Love relations, [**1**], given in (1), between the rotations $\underline{\beta}$ and the components of the displacement

$$\underline{\beta}_\alpha + w_{,\alpha} + b_\alpha^\lambda u_\lambda = 0, \quad \text{where} \qquad (1)$$
$$b_\alpha^\lambda = a^{\lambda\mu} b_{\alpha\mu}$$

and where we denote by $a_{\lambda\mu}$ the first fundamental form, by $b_{\alpha\mu}$ the second fundamental form and by $c_{\alpha\beta}$ the third fundamental form of the mean plane. The covariant derivatives are represented by a vertical bar and the usual derivatives by a comma. We use Greek letters for the indices in $\{1, 2\}$ and Latin letters for the indices in $\{1, 2, 3\}$.

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This paper is based on the PhD thesis [**5**].

2.1. Formulation of the problem. We consider a shell which is
- clamped along $\Gamma_0 \subset \Gamma = \partial\Omega$
- loaded by a body force $\vec{p}$
- loaded by a surface force applied to the part $\Gamma_1 = \Gamma - \Gamma_0 \times] - \frac{e}{2}; \frac{e}{2}[$ of its lateral surface, where e is the thickness.

We shall consider the following problem.

Find $(\vec{u}, \underline{\beta}) \in \vec{Z}$ *such that*

$$a[(\vec{u}, \underline{\beta}); (\vec{v}, \underline{\delta})] = l(\vec{v}, \underline{\delta}) \quad \forall (\vec{v}, \underline{\delta}) \in \vec{Z} \tag{2}$$

with

$$\begin{aligned}
a[(\vec{u}, \underline{\beta}); (\vec{v}, \underline{\delta})] &= \int_\Omega e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12}\chi_{\alpha\beta}(\vec{u}, \underline{\beta})\chi_{\lambda\mu}(\vec{v}, \underline{\delta})]\sqrt{a} ds^1 ds^2 \\
l(\vec{v}, \underline{\delta}) &= \int_\Omega \vec{p}\vec{v}\sqrt{a} ds^1 ds^2 + \int_{\Gamma_1} (\vec{N}\vec{v} - M^\alpha \underline{\delta}_\alpha) d\gamma \\
E^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)}(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}) \\
\gamma_{\alpha\beta}(\vec{v}) &= \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta} w \\
\chi_{\alpha\beta}(\vec{v}, \underline{\delta}) &= \frac{1}{2}(\underline{\delta}_{\alpha|\beta} + \underline{\delta}_{\beta|\alpha}) - \frac{1}{2}(b^\lambda_\alpha v_{\lambda|\beta} + b^\lambda_\beta v_{\lambda|\alpha}) + c_{\alpha\beta} w \\
\sqrt{a} &= \sqrt{det(a_{\alpha\beta})}
\end{aligned}$$

and $\vec{N}$ the resulting force on Γ_1, $M = \epsilon_{\alpha\beta} M^\beta \vec{a}^\alpha$ the resulting moment on Γ_1 and $\vec{Z}$ the space of the displacements/rotations which satisfy the Kirchhoff constraints and the boundary conditions.

The Discrete Kirchhoff Triangle method consists in defining a space of approximation $\vec{Z}_h$ given by

$$\begin{aligned}
\vec{Z}_h = \{ & (\vec{v}_h, \underline{\delta}_h);\ v_{h\alpha} \in V^k_{h1}, \underline{\delta}_{h\alpha} \in V^k_{h1}, v_{h\alpha_{|\Gamma_0}} = 0, \quad \alpha = 1, 2; \\
& w_h \in V^k_{h2}; w_{h_{|\Gamma_0}} = 0 \quad \forall T \in T_h; \underline{\delta}_h \quad \text{clamped} \\
& (\vec{v}_h, \underline{\delta}_h)_T \quad \text{satisfy the discrete Kirchhoff constraints given in } [\mathbf{1}]\}
\end{aligned} \tag{3}$$

such that V^k_{h1} is the space of P_2-Lagrange elements and V^k_{h2} the space of P_3'-Hermite elements, h standing for a discretization parameter.

Remark 1. The Kirchhoff relations are not satisfied at all the points in Ω and therefore the discrete space $\vec{Z}_h$ is not included in $\vec{Z}$. This means that we have a non-conforming approximation of the Koiter equations.

3. The mortar element method for the D.K.T. approximation

Our purpose is to explain what has to be changed in the definition of the D.K.T. method when the triangulation is nonconforming. We recall that in order to match interface fields, we associate to the nonoverlapping decomposition of the domain Ω the skeleton of the decomposition and we choose the mortar and nonmortar sides [**3**].

For the shell equation, many functions have to be matched. First, we have the tangential displacements $v_{h\alpha}$ and then the transversal displacement w_h. We have to match also the rotations $\underline{\beta}_h$ associated with the displacement. For the first two components of the displacement, the matching is easy since these functions are independent and are involved in a second order equation. Their natural space is $H^1(\Omega)$ and the standard mortar method for piecewise parabolic elements is used. We recall that it involves the space of traces W_{h1} of functions of V_{h1}^k on the nonmortar sides and the subspace $\tilde{W}_{h1}$ of W_{h1} of functions that are linear on the first and last (1D) element of the triangulation of this nonmortar side.

Let us state the matching across one particular non-mortar γ^* and denote by $^+$ the mortar (master) side and by $^-$ the nonmortar (slave) side of the decomposition. Then, for any function $v_{h\alpha}$, $\alpha = 1, 2$, we impose

$$\forall \psi_h \in \tilde{W}_{h1}, \quad \int_{\gamma^*} (v_{h\alpha}^- - v_{h\alpha}^+)\psi_h \, d\tau = 0. \tag{4}$$

The space V_{h1} of approximation for the global tangential components of the displacements is thus given by

$$\begin{aligned} V_{h1} = \{ & v_h \in L^2(\Omega),\ v_{h|\Omega^k} \in V_{h1}^k \ \text{ and} \\ & \text{satisfies (4) along any non-mortar } \gamma^* \}. \end{aligned} \tag{5}$$

The originality in the matching presented in this paper lies in the treatment of the out of plane displacement and the associated rotations. We recall that the D.K.T. condition is a relation between the displacements and the rotations, see formula (1). The nonmortar side values are recovered from the mortar side in the following two steps.

$$\begin{array}{lllll}
w^+ & & & & \\
\downarrow \text{D.K.T.} & & & & \\
\underline{\beta}^+ & \xrightarrow{\text{mortar}} & \underline{\beta}^- & & (1) \\
 & & \downarrow & & \\
 & & \nabla w^- & & \\
 & & w^+ & \xrightarrow{\text{mortar}} & w^- \quad (2)
\end{array}$$

We start from w^+ given on the master side of the mortar, and then obtain $\underline{\beta}^+$ by using the D.K.T. condition.

Step (1)

We match $\underline{\beta}^-$ and $\underline{\beta}^+$ by the mortar relations. These are different for the normal and the tangential components (normal and tangential with respect to the interface). First, we match the tangential rotation $\underline{\beta}^{\mathbf{t}-}$ by defining a relation between two piecewise second order polynomials. The relation is naturally the same as for the displacements $v_{h\alpha}$, $\alpha = 1, 2$. We then impose

$$\forall \psi_h \in \tilde{W}_{h1}, \quad \int_{\gamma^*} (\underline{\beta}^{\mathbf{t}-} - \underline{\beta}^{\mathbf{t}+})\psi_h \, d\tau = 0. \tag{6}$$

Let us turn now to the normal rotations. We note first that from the D.K.T. conditions, the normal rotations are piecewise linear on the mortar side, [**2**] . Since we want to preserve, as much as possible, the Kirchhoff conditions, we shall glue the normal rotations as piecewise linear finite element functions. To do this, we define W_{h0} as being the set of continuous piecewise linear functions on γ^* (provided with the nonmortar triangulation) and $\tilde{W}_{h0}$ as the subset of those functions of W_{h0} that are constant on the first and last segment of the (nonmortar) triangulation. We then impose the following relation between the (piecewise linear) normal rotations.

$$\forall \psi_h \in \tilde{W}_{h0}, \quad \int_{\gamma^*} (\underline{\beta}^{\mathbf{n}-} - \underline{\beta}^{\mathbf{n}+})\psi_h \, d\tau = 0. \tag{7}$$

Step (2)
Now that the rotations are completely glued together and are uniquely defined over the interface from the corner values of $\underline{\beta}^-$ and all nodal values of $\underline{\beta}^+$ (themselves derived from v_α^+, w^+), we specify the relations that define w^-. The first set of constraints is to satisfy "inverse D.K.T. conditions" i.e. to match the values of w^- with the rotations $\underline{\beta}^-$. We impose that

• the tangential derivatives of w coincide with $\underline{\beta}^{\mathbf{t}-} + v_\alpha^-$ at any Lagrange node (vertex and middle point). This allows us to define a piecewise P_2 function on the nonmortar elements of γ^* and
• the normal derivative of w coincides with $\underline{\beta}^{\mathbf{n}-}$ at each vertex of the triangulation of γ^*.

Since $\underline{\beta}^{\mathbf{n}-}$ is piecewise linear, the D.K.T. condition is automatically satisfied at the middle node of each element . Futhermore, since $\underline{\beta}^{\mathbf{t}-}$ is piecewise quadratic, it coincides with the tangential derivative of w^- not only at the nodal points but also on the whole interface γ^*. Since the tangential derivative of w^- is determined, it suffices to impose the value of w^- at one of the endpoints of γ^* to determine the value of w^- entirely. We impose, with $\gamma^* = [p_1, p_2]$,

$$\forall \psi \in \tilde{W}_{h0}, \quad \int_{\gamma^*} (w^- - w^+)\psi_h \, d\tau = 0 \tag{8}$$

$$w^-(p_1) = w^+(p_1) \tag{9}$$

$$w^-(p_2) = w^+(p_2) \tag{10}$$

which can be seen as the relations that determine the nodal values of w^- that are lacking. We insist on the fact that this construction leads to finite element functions that satisfy the D.K.T. conditions on each interface. This allows us to define the global space V_{h2} of transversal displacements as follows.

$$V_{h2} = \{w_h \in L^2(\Omega) \; w_{h|\Omega^k} \in V_{h2}^k \text{ and satisfy (6), (7), (9), (8) and (10)}\} \tag{11}$$

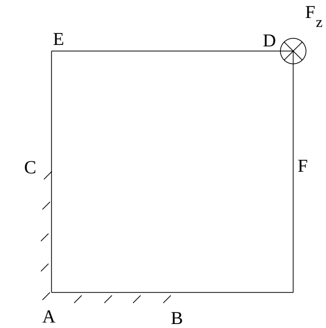

FIGURE 1. Plate configuration

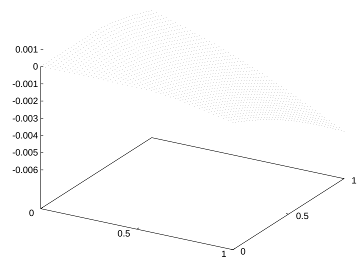

FIGURE 2. Deformation of the plate

4. Numerical results

Description of the problem.
We consider the plate, given in Figure 1, with the following properties.

Thickness	:	$e = 0.05\ m$
Length	:	$L = 1\ m$
Width	:	$w = 1\ m$
Properties	:	$E = 1.0E7\ Pa$ and $\nu = 0.25$
Boundary conditions	:	AB and AC clamped
Loading force	:	on $D : F_z = -1.0\ N$

1. Matching results.
The deformation of the plate loaded at the point D is shown in Figure 2.
The plate is now decomposed as in Figure 3 with non matching grids on the interfaces. The first results, given in Figures 4 and 5, show the good matching of the transversal displacement on the section CF and of the normal derivative of the transversal displacement.
2. Scalability results.
The discretization leads to an algebraic saddle-point problem that can be solved by the FETI method introduced in [**4**] and [**6**]. The FETI method presented here

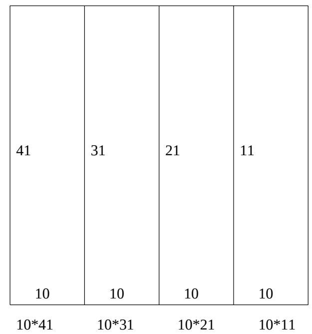

FIGURE 3. Example of decomposition and mesh

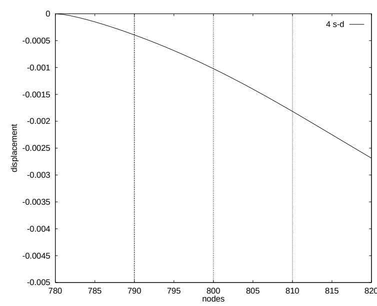

FIGURE 4. Transversal displacement on CF

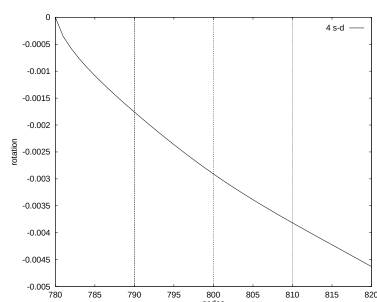

FIGURE 5. Normal derivative of the transversal displacement on CF

results in a scalable substructuring algorithm for solving this saddle point problem iteratively.
For this plate problem approximated by the D.K.T. finite element method, we observe that for a fixed local mesh, the number of iterations is independent of the number of subdomains. Thus, the parallel implementation exhibits good scalability when the right preconditioner is used [**5**], cf. Table 1.

TABLE 1. Scalability results

Number of sub domains	Iterations	Residual
4 (2×2)	49	7.10E-004
8 (4×2)	76	8.202E-004
16 (4×4)	77	7.835E-004
32 (8×4)	96	7.166E-004

Without preconditioner.

Number of sub domains	Iterations	Residual
4 (2×2)	14	5.256E-004
8 (4×2)	16	7.566E-004
16 (4×4)	16	8.966E-004
32 (8×4)	16	9.662E-004
64 (8×8)	16	8.662E-004

With preconditioner.

5. Conclusion

Analysis of the application of the D.K.T. method extended to nonconforming domain decomposition to shell problem illustrate the optimality of the mortar element method and the efficiency of the FETI solution algorithm.

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