Contemporary Mathematics
Volume **218**, 1998
B 0-8218-0988-1-03032-6

A Robin-Robin Preconditioner for an Advection-Diffusion Problem

Yves Achdou and Frédéric Nataf

1. Introduction

We propose a generalization of the Neumann-Neumann preconditioner for the Schur domain decomposition method applied to a advection diffusion equation. Solving the preconditioner system consists of solving boundary value problems in the subdomains with suitable Robin conditions, instead of Neumann problems. Preliminary tests assess the good behavior of the preconditioner.

The Neumann-Neumann preconditioner is used for the Schur domain decomposition method applied to symmetric operators, [**5**]. The goal of this paper is to propose its generalization to non symmetric operators. We replace the Neumann boundary conditions by suitable Robin boundary conditions which take into account the non symmetry of the operator. The choice of these conditions comes from a Fourier analysis, which is given in Sec. 2. When the operator is symmetric the proposed Robin boundary conditions reduce to Neumann boundary conditions. Also as in the symmetric case, the proposed preconditioner is exact for two subdomains and a uniform velocity. The preconditioner is presented in the case of a domain decomposed into non overlapping strips: the case of a more general domain decomposition will be treated in a forthcoming work as well as the addition of a coarse space solver.

The paper is organized as follows. In Sec. 2, the method is defined at the continuous level. In Sec. 3, the proposed preconditioner is constructed directly at the algebraic level. This may be important, if the grid is coarse and if upwind methods are used because the preconditioner defined at the continuous level is not relevant. In Sec. 4, we propose an extension to the case of nonmatching meshes (mortar method) [**3**] [**1**]. In Sec. 5, numerical results are shown for both conforming and nonconforming domain decompositions (mortar method).

2. The Continuous Case

We consider an advection-diffusion equation

$$\mathcal{L}(u) = cu + \vec{a}.\nabla u - \nu\Delta u = f \quad \text{in } \Omega =]0, L[\times]0, \eta[,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

1991 *Mathematics Subject Classification.* Primary 65N55; Secondary 76D30.

The positive constant c may arise from a time discretization by an Euler implicit scheme of the time dependent equation. The equation is solved by a primal Schur method. We focus on the case when the domain is decomposed into non overlapping vertical strips $\Omega_k =]l_k,\, l_{k+1}[\times]0,\eta[$, $1 \le k \le N$. Let $\Gamma_{k,k+1} = \{l_{k+1}\}\times]0,\eta[$.

REMARK 1. The general case of an arbitrary domain decomposition will be treated in a forthcoming work.

We introduce

$$\begin{aligned} \mathcal{S} &: (H_{00}^{1/2}(]0,\eta[))^{N-1} \times L^2(\Omega) \to (H^{-1/2}(]0,\eta[))^{N-1} \\ ((u_k)_{1\le k\le N-1}, f) &\mapsto (\frac{1}{2}\nu(\frac{\partial v_k}{\partial n_k} + \frac{\partial v_{k+1}}{\partial n_{k+1}})_{\Gamma_{k,k+1}})_{1\le k\le N-1} \end{aligned}$$

where v_k satisfies

$$\mathcal{L}(v_k) = f \text{ in } \Omega_k, \tag{1}$$

$$v_k = u_k \text{ on } \Gamma_{k,k+1} \text{ for } 1 \le k \le N-1, \tag{2}$$

$$v_k = u_{k-1} \text{ on } \Gamma_{k-1,k} \text{ for } 2 \le k \le N, \tag{3}$$

$$v_k = 0 \text{ on } \partial\Omega \cap \partial\Omega_k. \tag{4}$$

It is clear that $U = (u_{\Gamma_{k,k+1}})_{1\le k\le N-1}$ satisfies

$$\mathcal{S}(U,0) = -\mathcal{S}(0,f).$$

At the continuous level, we propose an approximate inverse of $\mathcal{S}(.,0)$ defined by

$$\begin{aligned} \mathcal{T} &: (H^{-1/2}(]0,\eta[))^{N-1} \to (H_{00}^{1/2}(]0,\eta[))^{N-1}, \\ (g_k)_{1\le k\le N-1} &\mapsto (\frac{1}{2}(v_k + v_{k+1})_{\Gamma_{k,k+1}})_{1\le k\le N-1}, \end{aligned}$$

where v_k satisfies

$$\mathcal{L}(v_k) = 0 \text{ in } \Omega_k, \tag{5}$$

$$(\nu\tfrac{\partial}{\partial n_k} - \tfrac{\vec{a}.\vec{n}_k}{2})(v_k) = g_k \text{ on } \Gamma_{k,k+1} \text{ for } 1 \le k \le N-1, \tag{6}$$

$$(\nu\tfrac{\partial}{\partial n_k} - \tfrac{\vec{a}.\vec{n}_k}{2})(v_k) = g_{k-1} \text{ on } \Gamma_{k-1,k} \text{ for } 2 \le k \le N, \tag{7}$$

$$v_k = 0 \text{ on} \partial\Omega \cap \partial\Omega_k. \tag{8}$$

REMARK 2. Our approach is different from that used in [**7**] or [**4**], where the interface conditions $\nu\frac{\partial}{\partial n_k} - \min(\vec{a}.\vec{n}_k, 0)$ are used in the framework of Schwarz algorithms.

The Robin boundary conditions in (6)-(7) are not standard and lead nevertheless to a well-posed problem:

PROPOSITION 3. *Let ω be an open set of $\mathbb{R}^2$, $f \in L^2(\omega)$, $\lambda \in H^{-1/2}(\partial\omega)$, $\vec{a} \in (C^1(\bar{\omega}))^2$, $c \in \mathbb{R}$ s.t. $c - \frac{1}{2}div(\vec{a}) \ge \alpha > 0$ for some $\alpha \in \mathbb{R}$. Then, there exists a unique $u \in H^1(\omega)$ s.t.*

$$\begin{aligned} &\int\int_\omega c\,v\,w + (\vec{a}.\nabla v)w + \nu\nabla v.\nabla w - \int_{\partial\omega} \frac{\vec{a}.\vec{n}}{2} vw \\ &= < \lambda, w >_{H^{-1/2}\times H^{1/2}} + \int\int_\omega fw, \quad \forall w \in H^1(\omega). \end{aligned}$$

Proof. When using the Lax-Milgram theorem, the only thing which is not obvious is the coercivity of the bilinear form

$$(v,w) \mapsto \int\int_\omega c\,v\,w + (\vec{a}.\nabla v)w + \nu \nabla v.\nabla w - \int_{\partial\omega} \frac{\vec{a}.\vec{n}}{2}\, vw$$

Integrating by parts leads to

$$\int\int c\,v^2 + (\vec{a}\nabla v)v + \nu|\nabla v|^2 - \int_{\partial\omega} \frac{\vec{a}.\vec{n}}{2}\, v^2 = \int\int (c - \frac{1}{2}div(\vec{a}))v^2 + \nu|\nabla v|^2$$
$$\geq \min(\alpha,\nu)\, \|v\|^2_{H^1(\omega)}.$$

□

Proposition 4. *In the case where the plane $\mathbb{R}^2$ is decomposed into the left ($\Omega_1 =]-\infty, 0[\times\mathbb{R}$) and right ($\Omega_2 =]0,\infty[\times\mathbb{R}$) half-planes and where the velocity $\vec{a}$ is uniform, we have that*

$$\mathcal{T} \circ \mathcal{S}(.,0) = Id.$$

Proof. A point in $\mathbb{R}^2$ is denoted by (x,y). The vector $\vec{a}$ is denoted $\vec{a} = (a_x, a_y)$. The unit outward normal and tangential vectors to domain Ω_k are denoted by $\vec{n}_k$ and $\vec{\tau}_k$ respectively. The proof is based on the Fourier transform in the y direction and the Fourier variable is denoted by ξ. The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$. Let us compute $\mathcal{S}(u_0, 0)$ for $u_0 \in H^{1/2}(\mathbb{R})$. Let w_k be the solution to (1)-(4), with $f = 0$ and u_0 as a Dirichlet data. The Fourier transform of (1) w.r.t. y yields

$$(c + a_x\partial_x + a_y i\xi - \nu\partial_{xx} + \nu\xi^2)(\hat{w}_k(x,\xi)) = 0$$

where $i^2 = -1$. For a given ξ, this equation is an ordinary differential equation in x whose solutions have the form $\alpha_k(\xi)e^{\lambda_k(\xi)|x|} + \beta_k(\xi)e^{\tilde{\lambda}_k(\xi)|x|}$ where

$$\lambda_k(\xi) = \frac{-\vec{a}.\vec{n}_k - \sqrt{4\nu c + (\vec{a}.\vec{n}_k)^2 + 4i\vec{a}.\vec{\tau}_1\xi\nu + 4\xi^2\nu^2}}{2\nu}$$

and

$$\tilde{\lambda}_k(\xi) = \frac{-\vec{a}.\vec{n}_k + \sqrt{4\nu c + (\vec{a}.\vec{n}_k)^2 + 4i\vec{a}.\vec{\tau}_1\xi\nu + 4\xi^2\nu^2}}{2\nu}$$

The solutions w_k must be bounded at infinity so that $\beta_k = 0$. The Dirichlet boundary conditions at $x = 0$ give $\alpha_k(\xi) = \hat{u}_0(\xi)$. Finally, we have that $w_k = \mathcal{F}^{-1}(\hat{u}_0(\xi)e^{\lambda_k(\xi)|x|})$ satisfy (1)-(3). Hence,

$$\mathcal{S}(u_0,0) = \frac{1}{2}\mathcal{F}^{-1}(\sqrt{4\nu c + (\vec{a}.\vec{n}_k)^2 + 4i\vec{a}.\vec{\tau}_2\xi\nu + 4\xi^2\nu^2}\ \hat{u}_0(\xi)).$$

In the same way, it is possible to compute $\mathcal{T}(g)$ for $g \in H^{-1/2}(\mathbb{R})$. Indeed, let v_1 (resp. v_2) be the solution to (5)-(7) in domain Ω_1 (resp. Ω_2). The function v_k may be sought in the form $v_k = \mathcal{F}^{-1}(\alpha_k(\xi)e^{\lambda_k(\xi)|x|})$. The boundary conditions (6)-(7) give:

$$\hat{g}(\xi) = (-\nu\lambda_k(\xi) - \frac{\vec{a}.\vec{n}_k}{2})\alpha_k(\xi)$$
$$= \frac{\sqrt{4\nu c + (\vec{a}.\vec{n}_k)^2 + 4i\vec{a}.\vec{\tau}_2\xi\nu + 4\xi^2\nu^2}}{2}\alpha_k(\xi)$$

Hence, $\hat{v}_k(0,\xi) = \frac{2}{\sqrt{4\nu c+(\vec{a}.\vec{n}_k)^2+4i\vec{a}.\vec{\tau}_2\xi\nu+4\xi^2\nu^2}}\hat{g}(\xi)$ and $\widehat{\mathcal{T}(g)} = \frac{1}{2}(\hat{v}_1(0,\xi)+\hat{v}_2(0,\xi))$ i.e.

$$\mathcal{T}(g) = 2\mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{\sqrt{4\nu c+(\vec{a}.\vec{n}_k)^2+4i\vec{a}.\vec{\tau}_2\xi\nu+4\xi^2\nu^2}}\right).$$

Hence, it is clear that $\mathcal{T}\circ\mathcal{S}(.,0) = Id$. ☐

REMARK 5. The same kind of computation shows that if $max(\frac{cL}{|\vec{a}.\vec{n}|}, L\sqrt{\frac{c}{\nu}}) \gg 1$, we still have $\mathcal{T}\circ\mathcal{S}(.,0)\simeq Id$. This means the preconditioner $\mathcal{T}$ remains efficient for an arbitrary number of subdomains as long as the advective term is not too strong or the viscosity is small enough. Moreover, in the case of simple flows, we expect that the preconditioned operator is close to a nilpotent operator whose nilpotency is the number of subdomains, [**2**]. In this case, the convergence does not depend on the parameter c and the method works well for large δt.

3. The Discrete Case

We suppose for simplicity that the computational domain is $\mathbb{R}^2$ discretized by a Cartesian grid. Let us denote $A = (A_{ij}^{kl})_{i,j,k,l\in\mathbb{Z}}$ the matrix resulting from a discretization of the advection-diffusion problem. We suppose that the stencil is a 9-point stencil ($A_{ij}^{kl} = 0$ for $|i-k|\geq 2$ or $|j-l|\geq 2$). This is the case, for instance, for a Q1-SUPG method or for a classical finite difference or finite volume scheme. We have to solve $AU = F$ where $U = (u_{ij})_{i,j\in\mathbb{Z}}$ is the vector of the unknowns. The computational domain is decomposed into two half planes ω_1 and ω_2. We introduce a discretized form $\mathcal{S}_h$ of the operator $\mathcal{S}$ (we adopt the summation convention of Einstein over all repeated indices)

$$\mathcal{S}_h : \mathbb{R}^{\mathbb{Z}}\times\mathbb{R}^{\mathbb{Z}\times\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}} \tag{9}$$

$$((u_{0j})_{j\in\mathbb{Z}}, (F_{ij})_{i,j\in\mathbb{Z}}) \mapsto (A_{0j}^{-1l}v_{-1l}^1 + B_j^{1\ l}v_{0l}^1 + A_{0j}^{1l}v_{1l}^2 + B_j^{2\ l}v_{0l}^2 - F_{0j})_{j\in\mathbb{Z}} \tag{10}$$

where (v_{ij}^m) satisfy

$$A_{ij}^{kl}v_{kl}^m = F_{ij} \quad \text{for } i<0 \text{ if } m=1 \quad \text{and } i>0 \text{ if } m=2,\ j\in\mathbb{Z},$$
$$v_{0j}^m = u_{0j}, \qquad \text{for } j\in\mathbb{Z}.$$

The coefficients $B_j^{m\ l}$ are the contributions of the domain ω_m to A_{0j}^{0l}, $A_{0j}^{0l} = B_j^{1\ l} + B_j^{2\ l}$. For example, if $\vec{a}=0$, $B_j^{1\ l} = B_j^{2\ l} = A_{0j}^{0l}/2$. For example, for a 1D case with a uniform grid and an upwind finite difference scheme ($a>0$) and $c=0$,

$$A_0^{-1} = -\frac{\nu}{h^2}-\frac{a}{h},\quad A_0^0 = \frac{2\nu}{h^2}+\frac{a}{h},\quad A_0^1 = -\frac{\nu}{h^2},\quad B^1 = \frac{\nu}{h^2}+\frac{a}{h} \quad\text{and}\quad B^2 = \frac{\nu}{h^2}.$$

$\mathcal{S}_h(u_0,F)$ is the residual of the equation on the interface. It is clear that $U_0 = (u_{0j})_{j\in\mathbb{Z}}$ satisfies

$$\mathcal{S}_h(U_0,0) = -\mathcal{S}_h(0,F). \tag{11}$$

We propose for an approximate inverse of $\mathcal{S}_h(.,0)$, $\mathcal{T}_h$ defined by

$$\mathcal{T}_h : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}} \tag{12}$$

$$(g_j)_{j\in\mathbb{Z}} \mapsto \tfrac{1}{2}(v_{0j}^1 + v_{0j}^2)_{j\in\mathbb{Z}}, \tag{13}$$

where $(v_{ij}^1)_{i\leq 0,\, j\in\mathbb{Z}}$ and $(v_{ij}^2)_{i\geq 0,\, j\in\mathbb{Z}}$ satisfy

$$A_{ij}^{kl}v_{kl}^1 = 0,\ i<0, j\in\mathbb{Z},\quad A_{0j}^{-1l}v_{-1l}^1 + \frac{A_{0j}^{0l}}{2}v_{0l}^1 = g_j \tag{14}$$

and

$$A_{ij}^{kl} v_{kl}^2 = 0,\ i > 0, j \in \mathbb{Z},\quad A_{0j}^{1l} v_{1l}^2 + \frac{A_{0j}^{0l}}{2} v_{0l}^2 = g_j. \tag{15}$$

REMARK 6. For a Neumann-Neumann preconditioner, in place of (14) and (15), we would have $A_{0j}^{-1l} v_{-1l}^1 + B_j^{1\ l} v_{0l}^1 = g_j$ and $A_{0j}^{1l} v_{1l}^2 + B_j^{2\ l} v_{0l}^2 = g_j$.

REMARK 7. For a constant coefficient operator $\mathcal{L}$ and a uniform grid, a discrete Fourier analysis can be performed similarly to that of the previous section. It can then be proved that

$$\mathcal{T}_h \circ \mathcal{S}_h(.,0) = Id_h$$

REMARK 8. The last equations of (14) and (15) correspond to the discretization of the Robin boundary condition $\nu \frac{\partial}{\partial n} - \frac{\vec{a}.\vec{n}}{2}$. Considering the previous 1D example, we have

$$h(A_0^{-1} v_{-1}^1 + \frac{A_0^0}{2} v_0^1) = (\nu + ah/2) \frac{v_0^1 - v_{-1}^1}{h} - \frac{a}{2} v_{-1}^1 = \nu \frac{v_0^1 - v_{-1}^1}{h} - \frac{a}{2}(2v_{-1}^1 - v_0^1),$$

and

$$h(A_j^{1l} v_1^2 + \frac{A_0^0}{2} v_0^2) = \nu \frac{v_0^2 - v_1^2}{h} + \frac{a}{2} v_0^2. \tag{16}$$

Another discretization of $\nu \frac{\partial}{\partial n} - \frac{\vec{a}.\vec{n}}{2}$ would not give (16). This the reason why the approximate inverse is directly defined at the algebraic level. The discretization of the Robin boundary condition is in some sense adaptive with respect to the discretization of the operator (SUPG, upwind finite difference scheme or finite volume scheme). In the previous 1D example, a straight forward discretization of the Robin boundary condition would give for domain 1

$$\nu \frac{v_0^1 - v_{-1}^1}{h} - \frac{a}{2} v_0^1.$$

When $ah \gg \nu$, which is usually the case, it is quite different from (14).

4. Adaption to the Mortar Method

The mortar method was first introduced by C. Bernardi, Y. Maday and T. Patera ([**3**]). It has been extended to advection-diffusion problems by Y. Achdou ([**1**]). It enables to take nonmatching grids at the interfaces of the subdomains without loss of accuracy compared to matching grids. In our case, the additional difficulty lies in the equations (14) and (15) which are no longer defined. Indeed, the coefficients A_{0j}^{0l} are defined only for matching grids where they correspond to coefficients of the matrix before the domain decomposition. Only the coefficients B_j^{ml} are available. Then, the trick is to take for A_{0j}^{0l} in (14) and (15), the matrix entries at the nearest interior points of the subdomains. Therefore, the equations (14) and (15) are replaced by

$$A_{ij}^{kl} v_{kl}^1 = 0,\ i < 0, j \in \mathbb{Z},\quad A_{0j}^{-1l} v_{-1l}^1 + \frac{A_{-1j}^{-1l}}{2} v_{0l}^1 = g_j$$

and

$$A_{ij}^{kl} v_{kl}^2 = 0,\ i > 0, j \in \mathbb{Z},\quad A_{0j}^{1l} v_{1l}^2 + \frac{A_{1j}^{1l}}{2} v_{0l}^2 = g_j.$$

TABLE 1. Number of iterations for different domain decompositions ($\vec{a} = \min(300y^2, 3)\, e_1$)

	Precond.	40-20-40	40-40-40	40-60-40	60-60-60	60-60-60(geo)	60-60-60-60-60
$\delta t = 1$	R-R	11	12	13	13	12	11
$\nu = 0.001$	N-N	31	37	38	43	67	60
	–	21	33	37	>100	66	>100
$\delta t = 0.1$	R-R	13	13	12	11	11	10
$\nu = 0.01$	N-N	22	23	25	26	31	33
	–	16	22	24	27	>100	19

TABLE 2. Number of iterations for different velocity fields, a three-domain decomposition and 40 points on each interface.

Precond.	normal	parallel	rotating	oblique
R-R	10	2	11	11
N-N	25	2	13	27
–	21	>50	45	12

5. Numerical Results

The advection-diffusion is discretized on a Cartesian grid by a Q1-streamline-diffusion method ([**6**]). Nonmatching grids at the interfaces are handled by the mortar method ([**1**]). The interface problem (11) is solved by a preconditioned GMRES algorithm. The preconditioners are either of the type Robin-Robin (R-R), Neumann-Neumann (N-N) or the identity (–). In the test presented below, all the subdomains are squares of side 0.5. The figures in Tables 1 and 2 are the number of iterations for reducing the initial residual by a factor 10^{-10}.

In Table 1, the first five columns correspond to a three-domain decomposition and the last one to a five-domains partition. The grid in each subdomain is a $N \times N$ Cartesian grid, not necessarily uniform. The first line indicates the parameters N. For instance $40 - 20 - 40$ means that the first and third subdomain have a 40×40 grid whereas the second subdomain has a 20×20 grid. In this case, the grids do not match at the interfaces. The grids are uniform except for the last but one column: in this case, the grid is geometrically refined in the y-direction with a ratio of 1.2. The velocity which is not varied, has a boundary layer in the y-direction.

In Table 2, the velocity field has been varied:

normal (to the interfaces): $\vec{a} = \min(300 * y^2, 3)e_1$, parallel (to the interfaces): $\vec{a} = e_2$, rotating: $\vec{a} = (y - y_0)e_1 - (x - x_0)e_2$ where (x_0, y_0) is the center of the computational domain and oblique: $\vec{a} = 3e_1 + e_2$. The mesh is fixed, the viscosity is $\nu = 0.01$ and the time step is $\delta t = \frac{1}{c} = 1$.

Tables 1 and 2 show that the proposed Robin-Robin preconditioner is very stable with respect to the mesh refinement, the number of subdomains, the aspect ratio of the meshes and the velocity field. More complete tests as well as the complete description of the solver will be given in [**2**].

6. Conclusion

We have proposed a preconditionner for the non symmetric advection-diffusion equation which generalizes the Neumann-Neumann preconditionner [**5**] in the sense that:

- It is exact for a two-domain decomposition.
- In the symmetric case, it reduces to the Neumann-Neumann preconditioner.

The tests have been performed on a decomposition into strips with various velocities and time steps. The results prove promising. In a forthcoming paper [**2**], we shall consider more general decompositions and the addition of a coarse level preconditioner.

References

1. Yves Achdou, *The mortar method for convection diffusion problems*, C.R.Acad. Sciences Paris, serie I **321** (1995), 117–123.
2. Yves Achdou, Patrick Letallec, and Frédéric Nataf, *The Robin-Robin preconditioner for transport equations*, In preparation.
3. C. Bernardi, Y. Maday, and A. T. Patera, *A new nonconforming approach to domain decomposition: the mortar element method*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), Pitman Res. Notes Math. Ser., vol. 299, Longman Sci. Tech., Harlow, 1994, pp. 13–51.
4. F.Gastaldi, L.Gastaldi, and A.Quarteroni, *Adaptative domain decomposition methods for advection dominated equations*, East-West J. Numer. Math **4** (1996), 165–206.
5. J.F.Bourgat, R. Glowinski, P. Le Tallec, and M. Vidrascu, *Variational formulation and algorithm for trace operator in domain decomposition calculations*, Proceedings of DDM 2, AMS, 1988, pp. 3–16.
6. C. Johnson, U. Navert, and J. Pitkaranta, *Finite element method for linear hyperbolic problems*, Comp. Meth. in Appl. Eng. **45** (1984), 285–312.
7. Frédéric Nataf and François Rogier, *Factorization of the convection-diffusion operator and the Schwarz algorithm*, Math. Models Appl. Sci. **5** (1995), 67–93.

INSA Rennes, 20 Av des Buttes de Coesmes, 35043 Rennes, France
E-mail address: `yves.achdou@insa-rennes.fr`

CMAP UMR7641 CNRS, Ecole Polytechnique, 91128, Palaiseau Cedex, France
E-mail address: `nataf@cmapx.polytechnique.fr`