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Interface Conditions and Non–overlapping Domain Decomposition Methods for a Fluid–Solid Interaction Problem

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1. Introduction

We present some non–overlapping domain decomposition methods for an inviscid fluid–solid interaction model which was proposed in [**8**] for modeling elastic wave propagation through fluid–filled borehole environments. Mathematically, the model is described by the coupled system of the elastic wave equations and the acoustic wave equation. First, we give a rigorous mathematical derivation of the interface conditions used in the model and to introduce some new variants of the interface conditions. The new interface conditions are mathematically equivalent to the original interface conditions, however, they can be more conveniently used to construct effective non–overlapping domain decomposition methods for the fluid-solid interaction problem. Then, we construct and analyze some parallelizable non–overlapping domain decomposition iterative methods for the fluid–solid interaction model based on the proposed new interface conditions.

The problems of wave propagation in composite media have long been subjects of both theoretical and practical studies, important applications of such problems are found in inverse scattering, elastoacoustics, geosciences, oceanography. For some recent developments on modeling, mathematical and numerical analysis, and computational simulations, we refer to [**2, 3, 8, 7, 10, 12**] and the references therein.

The non–overlapping domain decomposition iterative methods developed in this paper are based on the idea of using the convex combinations of the interface conditions in place of the original interface conditions to pass the information between subdomains, see [**9, 1, 4, 6**] for the expositions and discussions on this approach for uncoupled homogeneous problems. On the other hand, for the heterogeneous fluid–solid interaction problem, it is more delicate to employ the idea because using straightforward combinations of the original interface conditions as the transmission conditions may lead to divergent iterative procedures. So the domain decomposition methods of this paper may be regarded as the generalizations

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of the methods proposed in [**1, 4, 6, 9**] to the time–dependent heterogeneous problems. For more discussions on heterogeneous domain decomposition methods, we refer to [**11**] and the references therein.

The organization of this paper is as follows. In Section 2, the fluid–solid interaction model is introduced. In Section 3, the interface conditions, which are part of the model, are first derived using the physical arguments and then proved rigorously using vanishing shear modulus approximation. In Section 4, some parallelizable non–overlapping domain decomposition algorithms are proposed for solving the fluid–solid interaction problem. It is proved that these algorithms converge strongly in the energy spaces of the underlying fluid–solid interaction problem.

2. Description of the problem

We consider the propagation of waves in a composite medium Ω which consists of a fluid part Ω_f and a solid part Ω_s, that is, $\Omega = \Omega_f \cup \Omega_s$. Ω will be identified with a domain in $\mathbb{R}^N$ for $N = 2, 3$, and will be taken to be of unit thickness when $N = 2$. Let $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ denote the interface between two media, and let $\Gamma_f = \partial\Omega_f \setminus \Gamma$ and $\Gamma_s = \partial\Omega_s \setminus \Gamma$. Suppose that the solid is a pure elastic medium and the fluid is a pure acoustic media (inviscid fluids). Then the wave propagation is described by the following systems of partial differential equations

$$\frac{1}{c^2} p_{tt} - \Delta p = g_f, \qquad \text{in } \Omega_f, \tag{1}$$

$$\rho_s \mathbf{u}_{tt} - \operatorname{div}(\sigma(\mathbf{u})) = \mathbf{g}_s, \qquad \text{in } \Omega_s, \tag{2}$$

$$\frac{\partial p}{\partial n_f} - \rho_f \mathbf{u}_{tt} \cdot n_s = 0, \qquad \text{on } \Gamma, \tag{3}$$

$$\sigma(\mathbf{u}) n_s - p n_f = 0, \qquad \text{on } \Gamma, \tag{4}$$

$$\frac{1}{c} p_t + \frac{\partial p}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{5}$$

$$\rho_s \mathcal{A}_s \mathbf{u}_t + \sigma(\mathbf{u}) n_s = 0, \qquad \text{on } \Gamma_s, \tag{6}$$

$$p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x), \qquad \text{in } \Omega_f, \tag{7}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{u}_t(x,0) = \mathbf{u}_1(x), \qquad \text{in } \Omega_s, \tag{8}$$

where

$$\sigma(\mathbf{u}) = \lambda_s \operatorname{div} \mathbf{u} I + 2\mu_s \epsilon(\mathbf{u}), \qquad \varepsilon(\mathbf{u}) = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]. \tag{9}$$

In the above description, p is the pressure function in Ω_f and $\mathbf{u}$ is the displacement vector in Ω_s. ρ_i $(i = f, s)$ denotes the density of Ω_i, n_i $(i = f, s)$ denotes the unit outward normal to $\partial\Omega_i$. $\lambda_s > 0$ and $\mu_s \geq 0$ are the Lamé constants of Ω_s. Equation (9) is the constitutive relation for Ω_s. I stands for the $N \times N$ identity matrix. The boundary conditions in (5) and (6) are the first order absorbing boundary conditions for acoustic and the elastic waves, respectively. These boundary conditions are transparent to waves arriving normally at the boundary (cf. [**5**]). Finally, equations (3) and (4) are the interface conditions which describe the interaction between the fluid and the solid.

A derivation of the above model can be found in [**8**], where a detailed mathematical analysis concerning the existence, uniqueness and regularity of the solutions was also presented. The finite element approximations of the model were studied in [**7**], both semi–discrete and fully–discrete finite element methods were proposed and optimal order error estimates were obtained in both cases for the fluid-solid interaction model.

3. Interface conditions

The purpose of this section is to present a rigorous mathematical justification and interpretation for the interface conditions (3) and (4), which describe the interaction between the fluid and the solid. These interface conditions were originally derived in [**8**] using the heuristic physical arguments. For the sake of completeness, we start this section with a brief review of the heuristic derivation.

3.1. Derivation of interface conditions by physical arguments. Since an acoustic media can be regarded as an elastic media with zero shear modulus, the motion of each part of the composite media is described by a system of the elastic wave equations

$$\rho_i(\mathbf{u}_i)_{tt} = \operatorname{div}\sigma(\mathbf{u}_i) + \mathbf{g}_i, \qquad \text{in } \Omega_i, \tag{10}$$

$$\sigma(\mathbf{u}_i) = \lambda_i \operatorname{div}\mathbf{u}_i I + 2\mu_i\varepsilon(\mathbf{u}_i), \tag{11}$$

$$\varepsilon(\mathbf{u}_i) = \frac{1}{2}[\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^T], \tag{12}$$

where $i = f, s$. $\mathbf{u}_i$ denotes the displacement vector in Ω_i, and $\mu_f = 0$. Notice that the displacement is used as the primitive variable in both fluid and solid region.

Physically, as a system, the following two conditions must be satisfied on the interface between the fluid and the solid (cf. [**3**] and references therein).

- No relative movements occur in the normal direction of the interface.
- The stress must be continuous across the interface.

Mathematically, these conditions can be formulated as

$$\mathbf{u}_s \cdot n_s + \mathbf{u}_f \cdot n_f = 0, \qquad \text{on } \Gamma, \tag{13}$$

$$\sigma(\mathbf{u}_s)n_s + \sigma(\mathbf{u}_f)n_f = 0, \qquad \text{on } \Gamma. \tag{14}$$

In practice, it is more convenient to use the pressure field in the acoustic medium, so it is necessary to rewrite the interface conditions (13) and (14) using the pressure–displacement formulation. To this end, we introduce the pressure of the fluid, $p = -\lambda_f \operatorname{div}\mathbf{u}_f$. Since $\mu_f = 0$, (14) can be rewritten as

$$\sigma(\mathbf{u}_s)n_s = -\sigma(\mathbf{u}_f)n_f = pn_f, \quad \text{on } \Gamma.$$

which gives the interface condition (4). To convert (13), differentiating it twice with respective to t and using (10) we get

$$\rho_f(\mathbf{u}_s \cdot n_s)_{tt} = -\rho_f(\mathbf{u}_f \cdot n_f)_{tt} = -\operatorname{div}(\sigma(\mathbf{u}_f)) \cdot n_f = \frac{\partial p}{\partial n_f}, \quad \text{on } \Gamma,$$

so we get (13). Here we have used the fact that the source $\mathbf{g}_f$ vanishes on the interface Γ.

Finally, to get the full model (1)–(9) we also need to transform the interior equation in the fluid region from the displacement formulation into the pressure formulation. For a detailed derivation of this transformation, we refer to [**8**].

3.2. Validation of interface conditions by vanishing shear modulus approximation. The goal of this subsection is to show the interface conditions (13) and (14) are proper conditions to describe the interaction between the fluid region and solid region. In the same time, our derivation also reveals that in what sense the conditions (13) and (14) hold. To do this, we regularize the constitutive equation of the fluid by introducing a small (artificial) shear modulus $\mu_f = \delta > 0$, and then to look for the limiting model of the regularized problem as δ goes to zero.

The regularized constitutive equation of the fluid reads as

$$\sigma^\delta(\mathbf{u}_f) = \lambda_f(\operatorname{div}\mathbf{u}_f)I + 2\delta\varepsilon(\mathbf{u}_f). \tag{15}$$

Let $(\mathbf{u}_f^\delta, \mathbf{u}_s^\delta)$ be the solution of the regularized problem (10)–(12) and (15) with prescribed boundary conditions (say, first order absorbing boundary conditions) and initial conditions. It is well–known that the interface conditions for second order elliptic problems are the continuity of the function value and the continuity of the normal flux across the interface (cf. [**9**]). For the regularized problem (10)–(12) and (15), this means that

$$\mathbf{u}_f^\delta = \mathbf{u}_s^\delta, \qquad \text{on } \Gamma, \tag{16}$$

$$\sigma^\delta(\mathbf{u}_f^\delta)n_f = -\sigma(\mathbf{u}_s^\delta)n_s, \qquad \text{on } \Gamma. \tag{17}$$

The main result of this section is the following convergence theorem.

THEOREM 1. *There exist* $\mathbf{u}_f \in H(div,\Omega_f)$ *and* $\mathbf{u}_s \in H^1(\Omega_s)$ *such that*

(i) $\mathbf{u}_f^\delta$ *converges to* $\mathbf{u}_f$ *weakly in* $H(div,\Omega_f)$.

(ii) $\mathbf{u}_s^\delta$ *converges to* $\mathbf{u}_s$ *weakly in* $H^1(\Omega_s)$.

(iii) $(\mathbf{u}_f, \mathbf{u}_s)$ *satisfies the interface conditions (13) and (14) in* $(H_{00}^{\frac{1}{2}}(\Gamma))'$.

PROOF. Due to the page limitation, we only sketch the idea of the proof. To see a detailed proof of similar type, we refer to [**8, 11**].

The idea of the proof is to use the energy method. To apply the energy method, the key step is to get the uniform (in δ) estimates for the solution $(\mathbf{u}_f^\delta, \mathbf{u}_s^\delta)$. This can be done by testing the interior equations of the fluid and solid against $(\mathbf{u}_f^\delta)_t$ and $(\mathbf{u}_s^\delta)_t$, respectively. Finally, the proof is completed by using a compactness argument and taking limit in (10)–(12) and (15)–(17) as δ goes to zero. □

4. Non–overlapping domain decomposition methods

Because of the existence of the physical interface, it is very nature to use non–overlapping domain decomposition method to solve the fluid–solid interaction problem. In fact, Non–overlapping domain decomposition methods have been effectively used to solve several coupled boundary value problems from scientific applications, see [**11**] and the references therein.

In this section we first propose a family of new interface conditions which are equivalent to the original interface conditions (3) and (4). From a mathematical point of view, this is the key step towards developing non–overlapping domain decomposition methods for the problem. Based on these new interface conditions, we introduce two types of parallelizable non–overlapping domain decomposition iterative algorithms for solving the system (1)–(9) and establish the usefulness of these algorithms by proving their strong convergence in the energy spaces of the underlying fluid–solid interaction problem.

Due to the page limitation, the algorithms and the analyses are only given at the differential level in this paper. Following the ideas of [**1, 4, 6**], it is not very hard rather technical and tedious to construct and analyze the finite element discrete analogues of the differential domain decomposition algorithms. Another point which is worth mentioning is that the domain decomposition algorithms of this paper can be used for solving the discrete systems of (1)–(9) which arise from using other discretization methods such as finite difference and spectral methods, as well as hybrid methods of using different discretization methods in different media (subdomains).

4.1. Algorithms. Recall the interface conditions on the fluid–solid contact surface are

$$\frac{\partial p}{\partial n_f} = \rho_f \mathbf{u}_{tt} \cdot n_s, \qquad p n_f = \sigma(\mathbf{u}) n_s, \quad \text{on } \Gamma. \tag{18}$$

Rewrite the second equation in (18) as

$$-p_t = \sigma(\mathbf{u}_t) n_s \cdot n_s, \qquad 0 = \sigma(\mathbf{u}_t) n_s \cdot \tau_s, \quad \text{on } \Gamma, \tag{19}$$

where τ_s denotes the unit tangential vector on $\partial\Omega_s$. The equivalence of $(18)_2$ and (19) holds if the initial conditions satisfy some compatibility conditions (cf. [**8**]).

Lemma 2. *The interface conditions in (18) are equivalent to*

$$\tfrac{\partial p}{\partial n_f} + \alpha p_t = \rho_f \mathbf{u}_{tt} \cdot n_s - \alpha \sigma(\mathbf{u}_t) n_s \cdot n_s, \qquad \textit{on } \Gamma, \tag{20}$$

$$\rho_f \mathbf{u}_{tt} + \beta \sigma(\mathbf{u}_t) n_s = \tfrac{\partial p}{\partial n_f} n_s - \beta p_t n_s, \qquad \textit{on } \Gamma, \tag{21}$$

$$\sigma(\mathbf{u}_t) n_s \tau_s = 0, \qquad \textit{on } \Gamma, \tag{22}$$

for any pair of constants α and β such that $\alpha + \beta \neq 0$.

Based on the above new form of the interface conditions we propose the following two types of iterative algorithms. The first one resembles to Jacobi type iteration and the other resembles to Gauss–Seidel type iteration.

Algorithm 1

Step 1 $\forall p^0 \in P_f, \quad \forall \mathbf{u}^0 \in \mathbf{V}_s$.
Step 2 Generate $\{(p^n, \mathbf{u}^n)\}_{n\geq 1}$ iteratively by solving

$$\tfrac{1}{c^2} p^n_{tt} - \Delta p^n = g_f, \qquad \text{in } \Omega_f, \tag{23}$$

$$\tfrac{1}{c} p^n_t + \tfrac{\partial p^n}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{24}$$

$$\tfrac{\partial p^n}{\partial n_f} + \alpha p^n_t = \rho_f \mathbf{u}^{n-1}_{tt} \cdot n_s - \alpha \sigma(\mathbf{u}^{n-1}_t) n_s \cdot n_s, \qquad \text{on } \Gamma; \tag{25}$$

$$\rho_s \mathbf{u}^n_{tt} - \operatorname{div} \sigma(\mathbf{u}^n) = \mathbf{g}_s, \qquad \text{in } \Omega_s, \tag{26}$$

$$\rho_s \mathcal{A}_s \mathbf{u}^n_t + \sigma(\mathbf{u}^n) n_s = 0, \qquad \text{on } \Gamma_s, \tag{27}$$

$$\rho_f \mathbf{u}^n_{tt} + \beta \sigma(\mathbf{u}^n_t) n_s = \tfrac{\partial p^{n-1}}{\partial n_f} n_s - \beta p^{n-1}_t n_s, \qquad \text{on } \Gamma, \tag{28}$$

$$\sigma(\mathbf{u}^n_t) n_s \cdot \tau_s = 0, \qquad \text{on } \Gamma. \tag{29}$$

Algorithm 2

Step 1 $\forall \mathbf{u}^0 \in \mathbf{V}_s$.
Step 2 Generate $\{p^n\}_{n\geq 0}$ and $\{\mathbf{u}^n\}_{n\geq 1}$ iteratively by solving

$$\frac{1}{c^2}p^n_{tt} - \Delta p^n = g_f, \qquad \text{in } \Omega_f, \tag{30}$$
$$\frac{1}{c}p^n_t + \frac{\partial p^n}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{31}$$
$$\frac{\partial p^n}{\partial n_f} + \alpha p^n_t = \rho_f \mathbf{u}^n_{tt}\cdot n_s - \alpha\sigma(\mathbf{u}^n_t)n_s\cdot n_s, \qquad \text{on } \Gamma; \tag{32}$$

$$\rho_s \mathbf{u}^{n+1}_{tt} - \operatorname{div}\sigma(\mathbf{u}^{n+1}) = \mathbf{g}_s, \qquad \text{in } \Omega_s, \tag{33}$$
$$\rho_s \mathcal{A}_s \mathbf{u}^{n+1}_t + \sigma(\mathbf{u}^{n+1})n_s = 0, \qquad \text{on } \Gamma_s, \tag{34}$$
$$\rho_f \mathbf{u}^{n+1}_{tt} + \beta\sigma(\mathbf{u}^{n+1}_t)n_s = \frac{\partial p^n}{\partial n_f}n_s - \beta p^n_t n_s, \qquad \text{on } \Gamma, \tag{35}$$
$$\sigma(\mathbf{u}^{n+1}_t)n_s\cdot\tau_s = 0, \qquad \text{on } \Gamma. \tag{36}$$

REMARK 3. Appropriate initial conditions must be provided in the above algorithms. We omit these conditions for notation brevity.

4.2. Convergence Analysis. In this subsection we shall establish the utility of Algorithms 1 and 2 by proving their convergence. Because the convergence proof for Algorithm 2 is almost same as the proof of Algorithm 1, we only give a proof for Algorithm 1 in the following.

Introduce the error functions at the nth iteration

$$r^n = p - p^n, \quad \mathbf{e}^n = \mathbf{u} - \mathbf{u}^n.$$

It is easy to check that $(r^n, \mathbf{e}^n)$ satisfies the error equations

$$\frac{1}{c^2}r^n_{tt} - \Delta r^n = 0, \qquad \text{in } \Omega_f, \tag{37}$$
$$\frac{1}{c}r^n_t + \frac{\partial r^n}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{38}$$
$$\frac{\partial r^n}{\partial n_f} + \alpha r^n_t = \rho_f \mathbf{e}^{n-1}_{tt}\cdot n_s - \alpha\sigma(\mathbf{e}^{n-1}_t)n_s\cdot n_s, \qquad \text{on } \Gamma; \tag{39}$$
$$\rho_s \mathbf{e}^n_{tt} - \operatorname{div}\sigma(\mathbf{e}^n) = 0, \qquad \text{in } \Omega_s, \tag{40}$$
$$\rho_s \mathcal{A}_s \mathbf{e}^n_t + \sigma(\mathbf{e}^n)n_s = 0, \qquad \text{on } \Gamma_s, \tag{41}$$
$$\rho_f \mathbf{e}^n_{tt} + \beta\sigma(\mathbf{e}^n_t)n_s = \frac{\partial r^{n-1}}{\partial n_f}n_s - \beta r^{n-1}_t n_s, \qquad \text{on } \Gamma, \tag{42}$$
$$\sigma(\mathbf{e}^n_t)n_s\cdot\tau_s = 0, \qquad \text{on } \Gamma. \tag{43}$$

Define the "pseudo–energy"

$$E_n = E(\{r^n, \mathbf{e}^n\}) = \left\| \frac{\partial r^n}{\partial n_f} + \alpha r^n_t \right\|^2_{L^2(\lg)} + \left\| \rho_f \mathbf{e}^n_{tt} + \beta\sigma(\mathbf{e}^n)n_s \right\|^2_{L^2(\lg)}.$$

LEMMA 4. *There holds the following inequality*

$$E_{n+1}(\tau) \leq E_n(\tau) - R_n(\tau), \tag{44}$$

where

$$R_n(\tau) = 4\int_0^\tau \int_\Gamma \left[\alpha \frac{\partial r^n}{\partial n_f} r^n_t + \beta\sigma(\mathbf{e}^n_t)n_s\cdot \mathbf{e}^n_{tt} \right] dxdt.$$

To make the estimate (44) be useful, we need to find a lower bound for $R_n(\tau)$. Testing (37) against r_t^n to get

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{1}{c}r_t^n\right\|_{0,\Omega_f}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla r^n\|_{0,\Omega_f}^2 + \left|\frac{1}{\sqrt{c}}r_t^n\right|_{0,\Gamma_f}^2 = \int_\Gamma \frac{\partial r^n}{\partial n_f} r_t^n dx,$$

which implies that

(45)

$$\int_0^\tau \int_\Gamma \frac{\partial r^n}{\partial n_f} r_t^n dxdt = \frac{1}{2}\left\|\frac{1}{c}r_t^n(\tau)\right\|_{0,\Omega_f}^2 + \frac{1}{2}\|\nabla r^n(\tau)\|_{0,\Omega_f}^2 + \left\|\frac{1}{\sqrt{c}}r_t^n\right\|_{L^2(L^2(\Gamma_f))}.$$

Here we have implicitly assumed that $r^n(0) = r_t^n(0) = 0$.

Differentiating (40) with respect to t and testing it against $\mathbf{e}_{tt}^n$ give us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_s}\mathbf{e}_{tt}^n\|_{0,\Omega_s}^2 \quad &+ \quad \frac{d}{dt}\|\sqrt{\mu_s}\varepsilon(\mathbf{e}_t^n)\|_{0,\Omega_s}^2 + \frac{1}{2}\frac{d}{dt}\left\|\sqrt{\lambda_s}\,\mathrm{div}(\mathbf{e}_t^n)\right\|_{0,\Omega_s}^2 \\ &+ \quad c_0\,|\sqrt{\rho_s}\mathbf{e}_t^n|_{0,\Gamma_s}^2 \le \int_\Gamma \sigma(\mathbf{e}_t^n)\cdot n_s\mathbf{e}_{tt}^n dx,\end{aligned}$$

which implies that

$$\begin{aligned}\text{(46)}\qquad \int_0^\tau\int_\Gamma \rho_s\sigma(\mathbf{e}_t^n)n_s\cdot\mathbf{e}_{tt}^n dxdt \ge\ & \frac{1}{2}\|\sqrt{\rho_s}\mathbf{e}_{tt}^n(\tau)\|_{0,\Omega_s}^2 + \|\sqrt{\mu_s}\varepsilon(\mathbf{e}_t^n(\tau))\|_{0,\Omega_s}^2 \\ &+\frac{1}{2}\left\|\sqrt{\lambda_s}\,\mathrm{div}(\mathbf{e}_t^n(\tau))\right\|_{0,\Omega_s}^2 + c_0||\sqrt{\rho_s}\mathbf{e}_t^n||_{L^2(L^2(\Gamma_s))}^2 \\ &-\frac{1}{2}\|\sqrt{\rho_s}\mathbf{e}_{tt}^n(0)\|_{0,\Omega_s}^2.\end{aligned}$$

Since $\mathbf{e}^n(0) = \mathbf{e}_t^n(0) = 0$, it follows from (37) that

$$\|\sqrt{\rho_s}\mathbf{e}_{tt}^n(0)\|_{0,\Omega_s} = \left\|\frac{1}{\sqrt{\rho_s}}\mathrm{div}(\mathbf{e}^n(0))\right\|_{0,\Omega_s} = 0.$$

Combining (45) and (46) we get the following lemma.

LEMMA 5. *$R_n(\tau)$ satisfies the following inequality*

$$\begin{aligned}R_n(\tau) \quad \ge \quad & 2\alpha\left[\left\|\frac{1}{c}r_t^n(\tau)\right\|_{0,\Omega_f}^2 + \|\nabla r^n(\tau)\|_{0,\Omega_f}^2 + 2\left\|\frac{1}{\sqrt{c}}r_t^n\right\|_{L^2(L^2(\Gamma_f))}^2\right] \\ &+2\beta\left[\|\sqrt{\rho_s}\mathbf{e}_{tt}^n(\tau)\|_{0,\Omega_s}^2 + \|\sqrt{\mu_s}\varepsilon(\mathbf{e}_t^n(\tau))\|_{0,\Omega_s}^2\right. \\ &\left.+\|\sqrt{\lambda_s}\,div(\mathbf{e}_t^n(\tau)\|_{0,\Omega_s}^2 + 2c_0\,\|\sqrt{\rho_s}\mathbf{e}_t^n\|_{L^2(L^2(\Gamma_s))}^2\right].\end{aligned}$$

Finally, from Lemma 4 and 5 we get the following convergence theorem.

THEOREM 6. *Let $\{(p^k, \mathbf{u}^k)\}$ be generated by Algorithm 1 or Algorithm 3. For $\alpha > 0$ and $\beta > 0$, we have*

(1) *$p^k \longrightarrow p$ strongly in $L^\infty(H^1(\Omega_f)) \cap W^{1,\infty}(L^2(\Omega_f))$,*
(2) *$\mathbf{u}^k \longrightarrow \mathbf{u}$ strongly in $W^{1,\infty}(\mathbf{H}^1(\Omega_s)) \cap W^{2,\infty}(\mathbf{L}^2(\Omega_s))$.*

REMARK 7. If we choose $\alpha = 0$, $\beta = \infty$, Algorithms 1 and 2 become N–N alternating type algorithms. In addition, at the end of each N–N iteration one can add the following relaxation step to speed up the convergence

$$\begin{aligned}p^n \quad &:= \quad \mu p^n + (1-\mu)p^{n-1}, \\ \mathbf{u}_{tt}^n \quad &:= \quad \mu\mathbf{u}_{tt}^n + (1-\mu)\mathbf{u}_{tt}^{n-1},\end{aligned}$$

where μ is any constant satisfying $0 < \mu < 1$.

5. Conclusions

In this paper we have presented a mathematically rigorous derivation of the interface conditions for the *inviscid* fluid–solid interaction model proposed in [**8**], and developed two families of non–overlapping domain decomposition iterative methods for solving the governing partial differential equations. Our analysis demonstrated that it is crucial and delicate to choose the *right* transmission conditions for constructing domain decomposition methods for the problem, since trivial use of the physical interface conditions as the transmission conditions may result in slowly convergent even divergent iterative methods. Finally, we believe that the ideas and methods presented in this paper can be extended to other heterogeneous problems, in particular, the *viscid* fluid–solid interaction problems in which the Navier–Stokes equations should be used in the fluid medium. This work is currently in progress and will be reported in the near future.

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