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Overlapping Schwarz Waveform Relaxation for Parabolic Problems

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1. Introduction

We analyze a new domain decomposition algorithm to solve parabolic partial differential equations. Two classical approaches can be found in the literature:

1. Discretizing time and applying domain decomposition to the obtained sequence of elliptic problems, like in [**9, 11, 1**] and references therein.
2. Discretizing space and applying waveform relaxation to the large system of ordinary differential equations, like in [**10, 8, 7**] and references therein.

In contrary to the classical approaches, we formulate a parallel solution algorithm without any discretization. We decompose the domain into overlapping subdomains in space and we consider the parabolic problem on each subdomain over a given time interval. We solve iteratively parabolic problems on subdomains, exchanging boundary information at the interfaces of subdomains. So for a parabolic problem $\frac{\partial u}{\partial t} = \mathcal{L}(u, x, t)$ in the domain Ω with given initial and boundary conditions the algorithm for two overlapping subdomains Ω_0 and Ω_1 would read for $j = 0, 1$

$$\begin{aligned} \frac{\partial u_j^{k+1}}{\partial t} &= \mathcal{L}(u_j^{k+1}, x, t), \quad x, t \in \Omega_j \\ u_j^{k+1}(x, t) &= \begin{cases} u_{1-j}^k(x, t), & x, t \in \Gamma_j := \partial\Omega_j \cap \Omega_{1-j} \\ \text{given bc}, & x, t \in \partial\Omega_j - \Gamma_j \end{cases} \end{aligned}$$

This algorithm is like a classical overlapping additive Schwarz, but on subdomains, a time dependent problem is solved, like in waveform relaxation. Thus the name overlapping Schwarz waveform relaxation.

This algorithm has been considered before in [**4**], where linear convergence on unbounded time intervals was proved for the one dimensional heat equation, and in [**6**] where superlinear convergence for bounded time intervals was shown for a constant coefficient convection diffusion equation.

We study here three model problems which show that the results in [**4**] and [**6**] hold for more general parabolic equations. All the convergence results are in L^∞ with the norm $||u||_\infty := \sup_{x,t} |u(x, t)|$.

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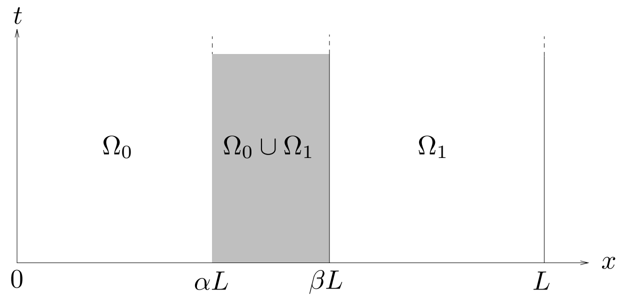

FIGURE 1. Decomposition into two overlapping subdomains

2. Reaction Diffusion Equation

We study the overlapping Schwarz waveform relaxation algorithm for the one dimensional reaction diffusion equation with variable diffusion coefficient

$$\frac{\partial u}{\partial t} = c^2(x,t)\frac{\partial^2 u}{\partial x^2} + f(u) \quad 0 < x < L,\ 0 < t < T$$

with appropriate initial and boundary conditions. We consider only the two subdomain problem given in Figure 1 with nonempty overlap, $0 < \alpha < \beta < 1$. The generalization to N subdomains can be found in [**2**].

2.1. Theoretical Results. Define for $0 < x < L$ and $0 < t < T$

$$\hat{c}^2 := \sup_{x,t} c^2(x,t)\ , \quad \hat{a} := \sup_{x,t,\xi\in\mathbb{R}} \frac{f'(\xi)}{c^2(x,t)}$$

which are assumed to be finite.

THEOREM 1 (Linear convergence for $t \in [0,\infty)$). *If $-\infty < \hat{a} < \left(\frac{\pi}{L}\right)^2$ then the overlapping Schwarz waveform relaxation algorithm converges linearly on unbounded time intervals,*

$$\max_j ||u - u_j^{2k+1}||_\infty \le \gamma^k C \max_j ||u - u_j^0||_\infty$$

with convergence factor

$$\gamma = \frac{\sin(\sqrt{\hat{a}}(1-\beta)L)\cdot\sin(\sqrt{\hat{a}}\alpha L)}{\sin(\sqrt{\hat{a}}(1-\alpha)L)\cdot\sin(\sqrt{\hat{a}}\beta L)} < 1$$

and C a constant depending on the parameters of the problem.

PROOF. The proof can be found in [**3**]. □

Note that the convergence factor γ tends to 1 and convergence is lost as the overlap goes to zero or $\hat{a}$ goes to $(\pi/L)^2$. On the other hand for $\hat{a}$ negative, the sine functions in γ become hyperbolic sine functions and $\gamma \longrightarrow 0$ as $\hat{a} \longrightarrow -\infty$.

THEOREM 2 (Superlinear convergence for $t \in [0,T]$). *The overlapping Schwarz waveform relaxation algorithm converges superlinearly on bounded time intervals,*

$$\max_j ||u - u_j^{2k+1}||_\infty \le \max(1, e^{2\hat{c}^2\hat{a}T})\, erfc\left(\frac{k(\beta-\alpha)L}{\sqrt{\hat{c}^2 T}}\right) \max_j ||u - u_j^0||_\infty.$$

PROOF. The proof can be found in [**3**]. □

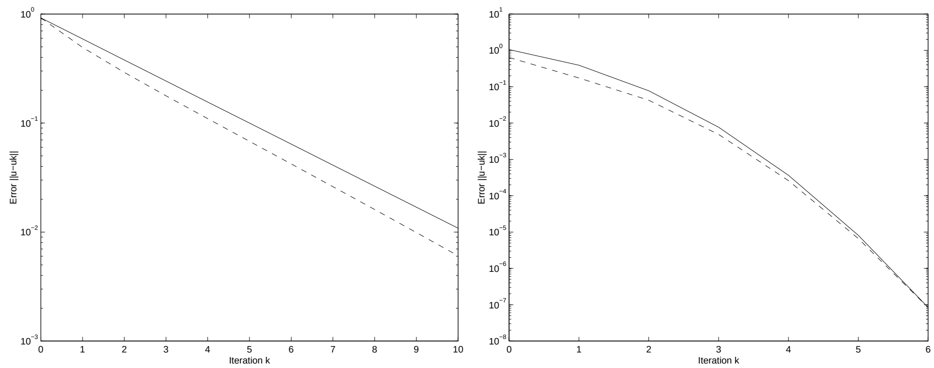

FIGURE 2. Linear convergence on long time intervals on the left and superlinear convergence on short time intervals on the right. Dashed the numerical results and solid the derived upper bounds for the convergence rate

Note that convergence is lost again when the overlap goes to zero ($\alpha = \beta$). But in contrary to the linear result, the superlinear convergence rate does not depend on $\hat{a}$, only the constant in front does. Instead there is a dependence on the diffusion coefficient $\hat{c}^2$ and the length of the time interval: the smaller the product $\hat{c}^2 T$, the faster the convergence.

2.2. Numerical Results. We consider the model problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5(u - u^3), \quad 0 < x < 1, \ 0 < t < T$$

with initial condition $u(x, 0) = x^2$ and boundary conditions $u(0, t) = 0$ and $u(1, t) = e^{-t}$. We chose 20% overlap ($\alpha = 0.4$ and $\beta = 0.6$). On the left of Figure 2 we show the algorithm in the linear convergence regime for a long time interval $T = 3$. Since this is still a bounded time interval, the algorithm would start to exhibit superlinear convergence if the iteration was continued. On the right of Figure 2 we chose a short time interval $T = 0.1$ to see the algorithm directly in the superlinear convergence regime. We used a centered second order finite difference in space with $\Delta x = 0.01$ and backward Euler in time with 300 time steps and the error displayed is the difference between the numerical solution on the whole domain and the iterates on the subdomains.

3. Convection Diffusion Equation

We consider a convection diffusion equation with variable coefficients in one dimension

$$\frac{\partial u}{\partial t} = c^2(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + f(x,t) \quad 0 < x < L, \ 0 < t < T$$

with appropriate initial and boundary conditions.

3.1. Theoretical Results. Define for $0 < x < L$ and $0 < t < T$

$$\hat{c}^2 := \sup_{x,t} c^2(x,t)\ , \quad \hat{b} := \sup_{x,t} \frac{b(x,t)}{c^2(x,t)}, \quad \check{b} := \inf_{x,t} \frac{b(x,t)}{c^2(x,t)}$$

which are assumed to be finite.

THEOREM 3 (Linear convergence for $t \in [0, \infty)$). *The overlapping Schwarz waveform relaxation algorithm converges linearly on unbounded time intervals,*

$$\max_j ||u - u_j^{2k+1}||_\infty \leq \gamma^k \max_j ||u - u_j^0||_\infty$$

with convergence factor

$$\gamma = \frac{1 - e^{-\check{b}(\beta L - L)}}{1 - e^{-\check{b}(\alpha L - L)}} \cdot \frac{1 - e^{-\hat{b}\alpha L}}{1 - e^{-\hat{b}\beta L}}.$$

PROOF. The proof can be found in [**2**]. □

As in the reaction diffusion case convergence is lost as the overlap goes to zero. But a large ratio of convection to diffusion helps the algorithm by decreasing γ.

THEOREM 4 (Superlinear convergence for $t \in [0, T]$). *The overlapping Schwarz waveform relaxation algorithm converges superlinearly on bounded time intervals,*

$$\max_j ||u - u_j^{2k+1}||_\infty \leq e^{(\beta-\alpha)L(\hat{b}-\check{b})k/2} \cdot erfc(\frac{k(\beta - \alpha)L}{\sqrt{\hat{c}^2 T}}) \max_j ||u - u_j^0||_\infty$$

PROOF. The proof can be found in [**2**]. □

Note that the principal convergence rate is the same as in the reaction diffusion case, but there is a lower order dependence on the convection term in the bound. The dependence is lower order, because $\text{erfc}(x) \leq e^{-x^2}$. If the coefficients are constant, the dependence on the convection term disappears completely.

3.2. Numerical Results. We consider the model problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + 5e^{-(t-2)^2-(x-\frac{1}{4})^2} \quad 0 < x < 1,\ 0 < t < T$$

with the same initial and boundary conditions as before. We chose again an overlap of 20% and different convection terms, $b \in \{0, 2.5, 5\}$. Figure 3 shows on the left the algorithm in the linear convergence regime for $T = 3$ and the three different values of b and on the right the algorithm in the superlinear convergence regime for $T = 0.1$ and $b = 2.5$. We used again a finite difference scheme with $\Delta x = 0.01$ and backward Euler in time with 300 time steps.

4. Heat Equation in n-Dimensions

Consider the heat equation in n dimensions

$$\frac{\partial u}{\partial t} = \Delta u + f(\boldsymbol{x}, t) \quad \boldsymbol{x} \in \Omega, 0 < t < T$$

with appropriate initial and boundary conditions. We assume that Ω is a smoothly bounded domain in $\mathbb{R}^n$ and that we have an overlapping decomposition of the domain into a finite number of subdomains [**5**].

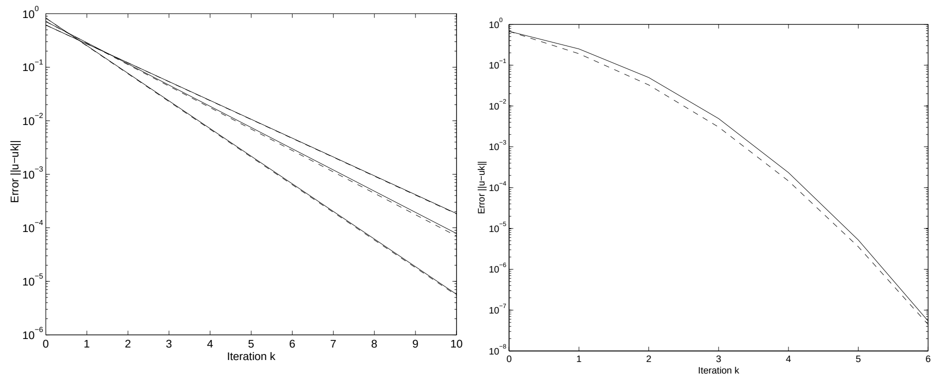

FIGURE 3. Linear convergence on the left and superlinear convergence on the right, dashed the numerical experiment and solid the theoretical bound on the convergence rate.

4.1. Theoretical Results.

THEOREM 5 (Linear convergence for $t \in [0, \infty)$). *The overlapping Schwarz waveform relaxation algorithm converges linearly on unbounded time intervals*

$$\max_j ||u - u_j^{k+m+1}||_\infty \leq \gamma \max_j ||u - u_j^k||_\infty$$

where $\gamma < 1$*, dependent on the geometry but independent of* k*. Note that* m *depends on the number of subdomains.*

PROOF. The proof can be found in [**5**]. □

THEOREM 6 (Superlinear convergence for $t \in [0, T]$). *The overlapping Schwarz waveform relaxation algorithm converges superlinearly on bounded time intervals*

$$\max_j ||u - u_j^k||_\infty \leq (2n)^k \mathit{erfc}(\frac{k\delta}{2\sqrt{nT}}) \max_j ||u - u_j^0||_\infty$$

where δ *is a parameter related to the size of the overlap.*

PROOF. The proof can be found in [**5**]. □

Note that the superlinear convergence rate does not depend on the number of subdomains.

4.2. Numerical Results. We used the homogeneous heat equation in two dimensions with homogeneous initial and boundary conditions on a unit square with the decomposition given in Figure 4 for our numerical experiments. Figure 5 shows for two different sizes of the overlap, $(\alpha, \beta) \in \{(0.3, 0.7), (0.4, 0.6)\}$, on the left the linear convergence behavior of the algorithm for a long time interval, $T = 3$ and on the right the superlinear convergence behavior for a short time interval, $T = 0.1$. We used a centered second order finite difference scheme in space with $\Delta x = 1/30$ and backward Euler in time with 100 time steps.

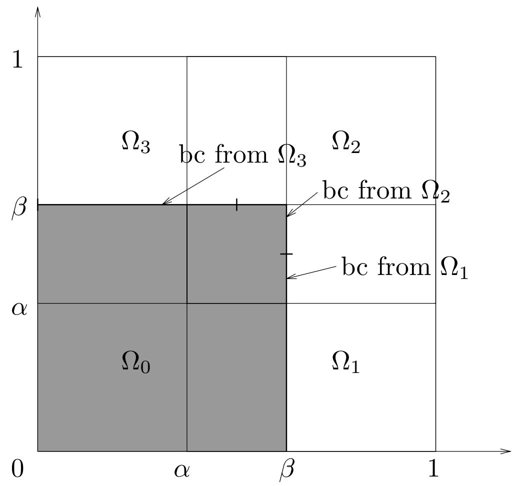

FIGURE 4. Decomposition of the domain for the heat equation model problem

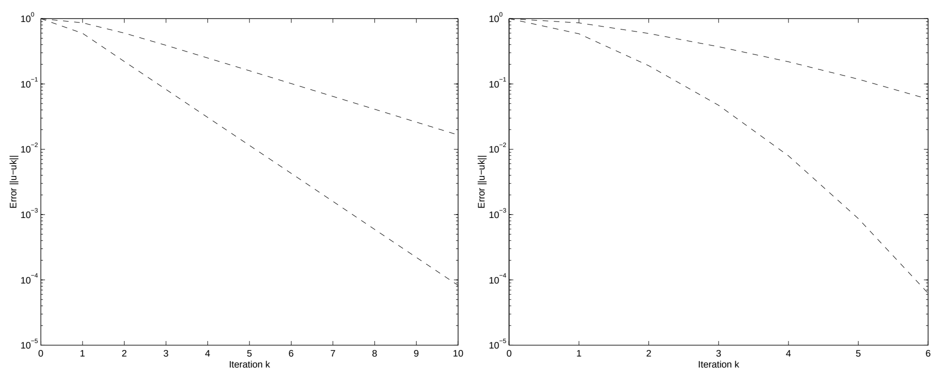

FIGURE 5. Linear convergence on the left and superlinear convergence on the right for the two dimensional heat equation

5. Conclusion

On unbounded time intervals the proposed algorithm converges linearly, provided there is a strong enough dissipation mechanism. Thus the new algorithm inherits the classical convergence behavior of the overlapping Schwarz algorithm for elliptic problems, and also the classical convergence behavior of waveform relaxation on long time windows.

On bounded time intervals the convergence is superlinear, which is a new result for domain decomposition algorithms. The waveform relaxation theory predicts superlinear convergence, but the presence of diffusion leads to a faster superlinear convergence rate than the one predicted by waveform relaxation theory [**6**].

Note that the superlinear convergence rates do not depend on the number of subdomains and thus there is no coarse mesh needed to avoid deterioration of the algorithm [**2**].

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