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Domain Decomposition, Operator Trigonometry, Robin Condition

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1. Introduction

The purpose of this paper is to bring to the domain decomposition community certain implications of a new operator trigonometry and of the Robin boundary condition as they pertain to domain decomposition methods and theory. In Section 2 we recall some basic facts and recent results concerning the new operator trigonometry as it applies to iterative methods. This theory reveals that the convergence rates of many important iterative methods are determined by the operator angle $\phi(A)$ of A: the maximum angle through which A may turn a vector. In Section 3 we bring domain decomposition methods into the operator trigonometric framework. In so doing a new three-way relationship between domain decomposition, operator trigonometry, and the recently developed strengthened C.B.S. constants theory, is established. In Section 4 we examine Robin–Robin boundary conditions as they are currently being used in domain decomposition interface conditions. Because the origins of Robin's boundary condition are so little known, we also take this opportunity to enter into the record here some recently discovered historical facts concerning Robin and the boundary condition now bearing his name.

2. Operator Trigonometry

This author developed an operator trigonometry for use in abstract semigroup operator theory in the period 1966–1970. In 1990 [**9**] the author found that the Kantorovich error bound for gradient methods was trigonometric: $E_A^{1/2}(x_{k+1}) \leqq (\sin A)E_A^{1/2}(x_k)$. Later [**14**] it was shown that Richardson iteration is trigonometric: the optimal spectral radius is $\rho_{\text{opt}} = \sin A$. Many other iterative methods have now been brought into the general operator trigonometric theory: Preconditioned conjugate gradient methods, generalized minimum residual methods, Chebyshev methods, Jacobi, Gauss–Seidel, SOR, SSOR methods, Uzawa methods and AMLI methods. Also wavelet frames have been brought into the operator trigonometric theory, see [**6**]. The model (Dirichlet) problem has been worked out in some detail to illustrate the new operator trigonometryc theory, see [**5**]. There ADI methods are also brought into the operator trigonometric theory. For full information about

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this new operator trigonometry, we refer the reader to [**9, 11, 10, 12, 14, 13, 18**] from which all needed additional details may be obtained.

For our purposes here it is sufficient to recall just a few salient facts. The central notion in the operator trigonometry theory is the angle of an operator, first defined in 1967 through its cosine. Namely, the angle $\phi(A)$ in the operator trigonometry is defined for an arbitrary strongly accretive operator A in a Banach space by

$$\cos A = \inf \frac{\mathrm{Re}\,\langle Ax, x\rangle}{\|Ax\|\|x\|}, \qquad x \in \mathcal{D}(A), \quad Ax \neq 0. \tag{1}$$

For simplicity we may assume in the following that A is a SPD matrix. By an early (1968) min-max theorem the quantity $\sin A = \inf_{\epsilon>0} \|\epsilon A - I\|$ enjoys the property $\cos^2 A + \sin^2 A = 1$. For A a SPD matrix we know that

$$\cos A = \frac{2\lambda_1^{1/2}\lambda_n^{1/2}}{\lambda_n + \lambda_1}, \quad \sin A = \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}. \tag{2}$$

3. Domain Decomposition

Here we will establish the general relationship between domain decomposition methods and the operator trigonometry by direct connection to the treatments of domain decomposition in [**19, 3, 23, 21, 2, 20**], in that order. Proofs and a more complete treatment will be given elsewhere [**4**].

We turn first to the treatment of domain decomposition methods in [**19**, Chapter 11]. Using a similar notation, let $A > 0$, $W_\chi \cong A_\chi > 0$, assume there exist upper and lower bounds C^{upper} and C_{lower} such that the decomposition, $x = \sum_J p_\chi x^\chi$ exists for every $x \in X$ and such that

$$\frac{1}{\Gamma} = C_{\mathrm{lower}} \leqq \frac{\sum_J \langle A p_\chi x^\chi, x^\chi\rangle}{\langle Ax, x\rangle} \equiv \frac{\sum_j \|p_\chi x^\chi\|_A^2}{\|x\|_A^2} \leqq C^{\mathrm{upper}} = \frac{1}{\gamma}. \tag{3}$$

Here γ and Γ are the optimal bounds in $\gamma W^{\mathrm{addSI}} \leqq A \leqq \Gamma W^{\mathrm{addSI}}$ i.e., the condition number $\kappa(W^{-1}A)$ is the ratio Γ/γ. Then it follows for two nonoverlapping domains that the optimal convergence spectral radius is $\rho(M^{\mathrm{addSI}}_{\theta_{\mathrm{optimal}}}) = \|M^{\mathrm{addSI}}_{\theta_{\mathrm{optimal}}}\|_A \leqq \frac{\Gamma-\gamma}{\Gamma+\gamma}$. When conjugate gradient is applied to the additive Schwarz domain decomposition algorithm, in the two level case the asymptotic convergence rate improves to $\rho(CGM^{\mathrm{addSI}}_{\theta_{\mathrm{optimal}}}) = \delta/1 + \sqrt{1-\delta^2}$ where

$$\delta = \sup_{x\in\mathcal{R}(p_1), y\in\mathcal{R}(p_2)} \frac{\langle x, y\rangle_A}{\|x\|_A\|y\|_A} \tag{4}$$

is the C.B.S. constant associated with the two-level decomposition.

THEOREM 1. [**4**]. *Under the stated conditions, the optimal convergence rate of the additive Schwarz domain decomposition algorithm is trigonometric:* $\rho(M_{\theta^{\mathrm{addSI}}_{\mathrm{optimal}}}) = \sin((W^{-1/2}AW^{-1/2})$. *In the two level case with the conjugate gradient scheme applied, the optimal asymptotic convergence rate is also trigonometric:* $\rho(CGM_{\theta^{\mathrm{addSI}}_{\mathrm{optimal}}}) = \sin((W^{-1/2}AW^{-1/2})^{1/2})$.

We may obtain an abstract version of Theorem 1 by following the abstract treatment of [**3**].

THEOREM 2. [**4**]. *With R and R^* the embedding (restriction) and conjugate (prolongation) operators for $\widetilde{V} = V_0 \times V_1 \times \cdots \times V_J$ and B defined by the Fictitious Subspace Lemma as in* [**3**], $\rho^{\mathrm{optimal}}_{\mathrm{addSI}} = \sin(RB^{-1}R^*A)$.

Next we comment on an important connection between the operator trigonometry and the C.B.S. constants which is inherent in the above results. Turning to [**23**], for the preconditioned system BA with $BSPD$ and $\rho \equiv \|I - BA\|_A < 1$, one knows that $\kappa(BA) \leqq \frac{1+\rho}{1-\rho}$. When optimized, the preconditioned Richardson iteration $u^{k+1} = u^k + \omega B(f - Au^k)$ has error reduction rate $(\kappa(BA) - 1)/(\kappa(BA) + 1)$ per iteration. This means the error reduction rate is exactly $\sin(BA)$, [**14**, Theorem 5.1] applying here since BA is SPD in the A inner product. From these considerations we may state

THEOREM 3. [**4**]. *Under the above conditions for the preconditioned system ωBA, the spectral radius $\rho \equiv \|I - \omega BA\|_A$ plays the role of strengthened C.B.S. constant when $\omega = \omega^*$ optimal.*

The principle of Theorem 3 could be applied to the whole abstract theory [**21**], i.e., to additive multilevel preconditionings $BA = \sum_{i=1}^{p} T_i$ and multiplicative preconditionings $BA = I - \sum_{i=0}^{p}(I - T_{p-i})$. The relationships of this principle to the Assumptions 1, 2, 3 of the domain decomposition theory are interesting, inasmuch as the three constants $c_0, \rho(\mathcal{E})$, and ω of those three assumptions are closely related to $\lambda_{\max}(BA)$ and $\lambda_{\min}(BA)$, viz. $c_0^{-2} \leqq \lambda_{\min}(BA) \leqq \cdots \leqq \lambda_{\max}(BA) \leqq \omega[1 + \rho(\mathcal{E})]$.

Next we turn to operator trigonometry related to the FETI (Finite Element Tearing and Interconnecting) algorithm [**2**]. See [**8**] where in an early paper we discussed the potential connections between Kron's tearing theories and those of domain decomposition and FEM and where we utilize graph–theoretic domain decomposition methods to decompose finite element subspaces according to the Weyl–Helmholtz–Hodge parts, which permits the computation of their dimensions. The FETI algorithm is a nonoverlapping domain decomposition with interfacing represented by Lagrange multipliers. To establish a connection of FETI to the operator trigonometry, let us consider the recent [**20**] analysis of convergence of the FETI method. There it is shown that the condition number of the preconditioned conjugate gradient FETI method is bounded independently of the number of subdomains, and in particular, that

$$\kappa = \frac{\lambda_{\max}(P_V M P_V F)}{\lambda_{\min}(P_V M P_V F)} \leqq \frac{c_2 c_4}{c_1 c_3} \leqq C\left(1 + \log\frac{H}{h}\right)^{\gamma} \tag{5}$$

where $P_V F$ is the linear operator of the dual problem of interest and where $P_V M$ is its preconditioner, c_1 and c_2 are lower and upper bounds for F, c_3 and c_4 are lower and upper bounds for M, and where $\gamma = 2$ or 3 depending on assumptions on the FEM triangulation, h being the characteristic element size, H an element-mapping-Jacobian bound.

THEOREM 4. [**4**] *The operator angle of the FETI scheme is bounded above according to* $\sin\phi(P_V M P_V F) \leqq (C(1 + \log\frac{H}{h})^{\gamma} - 1)/2$.

4. Robin Condition

To reduce overlap while maintaining the benefits of parallelism in Schwarz alternating methods, in [**22**] a generalized Schwarz splitting including Robin type interface boundary conditions was investigated. Certain choices of the coefficients in the Robin condition were found to lead to enhanced convergence. In the notation of [**22**] the Robin conditions are written $g_i(u) = \omega_i u + (1 - \omega_i)\frac{\partial u}{\partial n}$, $i = 1, 2$, for two overlapping regions. Later in the discretizations another coefficient α is introduced,

α related to ω by the relation $\omega = (1-\alpha)/(1-\alpha+h\alpha)$, h the usual discretization parameter. The case $\omega = 1$ ($\alpha = 0$) corresponds to Dirichlet interface condition, the case $\omega = 0$ ($\alpha = 1$) corresponds to Neumann interface condition. Optimal convergence rates are found both theoretically and for computational examples for α in values near 0.9, i.e., for ω near $1/(1+9h)$.

Let us convert these coefficients to the standard Robin notation [**7**]

$$\frac{\partial u}{\partial n} + \alpha_R u = f \tag{6}$$

where we have used α_R for the Robin coefficient to avoid confusion with the α of [**22**]. Then in terms of the ω of [**22**] the Robin condition there becomes $\frac{\partial u}{\partial n} + \omega(1-\omega)^{-1}u = f$ where the right hand side has absorbed a factor $(1-\omega)^{-1}$. For the successful $\alpha \approx 0.9$ of [**22**] the Robin constant $\alpha_R \approx 1/9h$. Since h was relatively small in the simulations of [**22**], this means that α_R was relatively large and that the Robin condition employed there was "mostly Dirichlet." This helps intuition, noted in [**22**] as lacking.

Next let us examine the theoretical analysis of [**22**]. Roughly, one wants to determine the spectral radius of a block Jacobi matrix $J = M^{-1}N$. The route to do so travels by similarity transformations from J to $\widetilde{J}$ to G to G' to $HG'H$ to elements of the last columns of T_1^{-1} and T_2^{-1} to four eigenvalues $\lambda_{1,2,3,4}$ and hence to the spectral radius of J. How have the Robin interface conditions affected the spectral radii of J depicted in the figures of [**22**]? To obtain some insight let us consider the following simple example. The one dimensional problem $-u'' = f$ is discretized by centered differences over an interval with five grid points $x_0, \ldots, x_4$, with a Robin boundary condition (6) at x_0 and x_4. For the three interior unknown values u_1, u_2, u_3 we then arrive at the matrix equation

$$\begin{bmatrix} 2-\beta & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2-\beta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} ch\beta + f_1 \\ f_2 \\ dh\beta + f_3 \end{bmatrix} \tag{7}$$

where we have absorbed the left Robin boundary condition $-(u_1-u_0)/h + \alpha_R u_0 = c$ and the right Robin boundary condition $(u_4 - u_3)/h + \alpha_R u_4 = d$, and where β denotes $(1+\alpha_R h)^{-1}$. Then the matrix A of (7) has eigenvalues $\lambda_1 = 2 - \beta/2 - (\beta^2+8)^{1/2}/2$, $\lambda_2 = 2-\beta$, $\lambda_3 = 2 - \beta/2 + (\beta^2+8)^{1/2}/2$. For the successful $\alpha = 0.9$ and assuming $h = 1/45$ (corresponding to [**22**]) we find $\beta = 0.9$ and hence $\lambda_1 = 0.066$, $\lambda_2 = 1.1$, $\lambda_3 = 3.034$. Using Dirichlet rather than Robin boundary conditions corresponds to $\beta = 0$ and $\lambda_1 = 0.586$, $\lambda_2 = 2$, $\lambda_3 = 3.414$. The condition numbers are $\kappa_{\text{Robin}} \cong 45.97$ and $\kappa_{\text{Dirichlet}} \cong 5.83$. The Robin condition moves the spectrum downward and increases condition number. The worst case approaches Neumann and infinite condition number. One needs to stay mostly Dirichlet: this is the intuition. Also the size of the grid parameter h is critical and determines the effective Robin constant.

Turning next to [**1**] and the Robin–Robin preconditioner techniques employed there, we wish to make two comments. First, for the advection–diffusion problems $Lu = cu + \vec{a}\cdot\nabla u - \nu\Delta u = f$ being considered in [**1**], the extracted Robin–Robin interface conditions

$$\left(\nu\frac{\partial}{\partial n_k} - \frac{\vec{a}\cdot\vec{n}_k}{2}\right)v_k = g_k \tag{8}$$

is really just an internal boundary trace of the differential operator and therefore is not independent in any way. A better terminology for (8) might be Peclet–Peclet, corresponding to the well-known Peclet number $P = \frac{aL}{\nu}$ over a fluid length L. Second, when the sign of $\vec{a}$ may change with upwinding or downwinding, the coefficient α_R in (6) may become an eigenvalue in the boundary operator. Solution behavior can then become quite different. Such an internal Steklov–Steklov preconditioner would permit interior "flap" of solutions.

Robin was (Victor) Gustave Robin (1955–1897) who lectured at the Sorbonne at the end of the previous century. Our interest in Robin began twenty years ago when writing the book [**7**]. The results of the subsequent twenty year search will be published now in [**15, 16**] to commemorate the 100th anniversary of Robin's death in 1897. Little is known about Robin personally. However we have uncovered all of his works (they are relatively few). Nowhere have we found him using the Robin boundary condition. Robin wrote a nice thesis in potential theory and also worked in thermodynamics. We have concluded that it is neither inappropriate nor especially appropriate that the third boundary condition now bears his name.

5. Conclusion

In the first domain decomposition conference [**17**] we presented new applications of domain decomposition to fluid dynamics. Here in the tenth domain decomposition conference we have presented new theory from linear algebra and differential equations applied to domain decomposition.

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