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On Schwarz Alternating Methods for Nonlinear Elliptic Problems

Shiu Hong Lui

1. Introduction

The Schwarz Alternating Method is a method devised by H. A. Schwarz more than one hundred years ago to solve linear boundary value problems. It has garnered interest recently because of its potential as a very efficient algorithm for parallel computers. See the fundamental work of Lions in [**7**] and [**8**]. The literature on this method for the boundary value problem is huge, see the recent reviews of Chan and Mathew [**5**] and Le Tallec [**14**], and the book of Smith, Bjorstad and Gropp [**11**]. The literature for nonlinear problems is rather sparse. Besides Lions' works, see also Cai and Dryja [**3**], Tai [**12**], Xu [**15**], Dryja and Hackbusch [**6**], Cai, Keyes and Venkatakrishnan [**4**], Tai and Espedal [**13**], and references therein. Other papers can be found in the proceedings of the annual domain decomposition conferences. In this paper, we prove the convergence of the Schwarz sequence for some 2nd-order nonlinear elliptic partial differential equations. We do not attempt to define the largest possible class of problems or give the weakest condition under which the Schwarz Alternating Method converges. The main aim is rather to illustrate that this remarkable method works for a very wide variety of nonlinear elliptic PDEs.

Let Ω be a bounded domain in $\mathbf{R}^N$ with a smooth boundary. Suppose $\Omega = \Omega_1 \cup \Omega_2$, where the subdomains Ω_i have smooth boundaries and are overlapping. We assume the nontrivial case where both subdomains are proper subsets of Ω. Let (u, v) denote the usual $L^2(\Omega)$ inner product and $\|u\|^2 = (u, u)$. Denote the energy inner product in the Sobolev space $H_0^1(\Omega)$ by $[u, v] = \int_\Omega \nabla u \cdot \nabla v$ and let $\|u\|_1 = [u, u]^{1/2}$. Denote the norm on $H^{-1}(\Omega)$ by $\|\cdot\|_{-1}$ with

$$\|u\|_{-1} = \sup_{\|v\|_1=1} |[u, v]|.$$

Let $\triangle_i$ be the Laplacian operator considered as an operator from $H_0^1(\Omega_i)$ onto $H^{-1}(\Omega_i)$, $i = 1, 2$. The smallest eigenvalue of $-\triangle$ on Ω is denoted by λ_1 while the smallest eigenvalue of $-\triangle_i$ is denoted by $\lambda_1(\Omega_i)$, $i = 1, 2$. The collection of eigenvalues on Ω is denoted by $\{\lambda_j\}_{j=1}^{\infty}$. For notational convenience, we define $\lambda_0 = -\infty$. We take overlapping to mean that $H_0^1(\Omega) = H_0^1(\Omega_1) + H_0^1(\Omega_2)$. In

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this paper, a function in $H_0^1(\Omega_i)$ is considered as a function defined on the whole domain by extension by zero. Let P_i denote the orthogonal (with respect to the energy inner product) projection onto $H_0^1(\Omega_i)$, $i = 1, 2$. It is well known that

$$d \equiv \max(\|(I-P_2)(I-P_1)\|_1,\ \|(I-P_1)(I-P_2)\|_1) < 1.$$

See Lions [**7**] and Bramble et. al. [**2**]. Throughout this paper, C will be used to denote a (not necessarily the same) positive constant.

We shall consider two classes of Schwarz methods. The first class, nonlinear Schwarz method, denotes a method where a sequence of nonlinear problems is solved one subdomain after another one. The second class, linear Schwarz method, is devoted to the method where a linear problem is solved in each subdomain.

The first Schwarz method for nonlinear problems is due to Lions [**7**]. He considers a functional $I \in C^1(H_0^1(\Omega), \mathbf{R})$ which is coercive, weakly lower semicontinuous, uniformly convex and bounded below. By making a correction alternately in each subdomain which minimizes the functional, he shows that the sequence converges to the unique minimizer of the functional.

2. Nonlinear Schwarz Method

In this section, we use the Schwarz method in conjunction with the methods of Banach and Schauder fixed points and of Global Inversion. The first result is an adaptation of the variational approach of Lions [**7**] for linear problems to nonlinear problems. We assume the nonlinearity satisfies a certain Lipschitz condition with a sufficiently small Lipschitz constant so that the method of proof for the linear problem still applies. See Lui [**9**] for a proof.

Theorem 1. *Consider the equation*

$$-\triangle u = f(x, u, \nabla u) + g \ \text{on}\ \Omega \tag{1}$$

with homogeneous Dirichlet boundary conditions. Assume for every $u, v \in H_0^1(\Omega)$,

$$\|f(x, u, \nabla u) - f(x, v, \nabla v)\| \le c\sqrt{\lambda_1}\|u - v\|_1,$$

where c *is a constant such that* $c < 1$ *and*

$$d < \sqrt{1-c^2} - c. \tag{2}$$

Assume $g \in L^2(\Omega)$. *For* $n = 0, 1, 2, \cdots$ *and some* $u^{(0)} \in H_0^1(\Omega)$, *define the Schwarz sequence as:*

$$-\triangle u^{(n+\frac{1}{2})} = f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) + g \ \text{on}\ \Omega_1,\ u^{(n+\frac{1}{2})} = u^{(n)} \ \text{on}\ \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = f(x, u^{(n+1)}, \nabla u^{(n+1)}) + g \ \text{on}\ \Omega_2,\ u^{(n+1)} = u^{(n+\frac{1}{2})} \ \text{on}\ \partial\Omega_2.$$

Then, the Schwarz sequence converges geometrically to the solution of (1) in the energy norm. Here, $u^{(n+\frac{1}{2})}$ *is considered as a function in* $H_0^1(\Omega)$ *by defining it to be* $u^{(n)}$ *on* $\Omega \setminus \Omega_1$ *and* $u^{(n+1)}$ *is defined as* $u^{(n+\frac{1}{2})}$ *on* $\Omega \setminus \Omega_2$.

It is an open problem to determine whether the Schwarz sequence converges geometrically with just the condition $c < 1$.

Next, we give a similar result for an equation whose solution is shown to exist by the Schauder/Schaeffer fixed point theorem. See Nirenberg [**10**] for instance.

THEOREM 2. *Consider the equation*

$$-\triangle u = f(x, u, \nabla u) + g \text{ on } \Omega \tag{3}$$

with homogeneous Dirichlet boundary conditions. Assume that $f \in C^1(\overline{\Omega}\times\mathbf{R}\times\mathbf{R}^N)$ *and for every* $x \in \Omega$, $a \in \mathbf{R}$, $\xi \in \mathbf{R}^N$, $\partial f(x,a,\xi)/\partial u \leq 0$ *and* $|f(x,a,\xi)| \leq C(1+|\xi|^\gamma)$, *where* C, γ *are positive constants with* $\gamma < 1$. *Assume* $g \in H^1(\Omega)$, $u^{(0)} \in H_0^1(\Omega)$ *and sufficiently smooth* ($g \in H^{[N/2]+1}(\Omega)$ *and* $u^{(0)} \in H^{[N/2]+3}(\Omega)$). *For* $n = 0, 1, 2, \cdots$, *define the Schwarz sequence as:*

$$-\triangle u^{(n+\frac{1}{2})} = f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) + g \text{ on } \Omega_1,\ u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = f(x, u^{(n+1)}, \nabla u^{(n+1)}) + g \text{ on } \Omega_2,\ u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Then, the Schwarz sequence converges geometrically to the solution of (3) in the L^∞ *norm.*

The proof can be found in Lui [**9**]. It can be divided into four steps. The first step is to show that the Schwarz sequence is well defined. Next, we show that the sequence is bounded in the H^1 norm so that there exists a weak limit. Then, we show that the sequence actually converges strongly (in H^1) to this limit. Finally, we use the maximum principle to show that this limit is in fact the unique solution to the differential equation. Note that geometric convergence results from the strong maximum principle which is used to show that the ratio of successive errors in the L^∞ norm is bounded by some constant less than one.

It is natural to inquire whether the rather strong condition on the nonlinearity, $\partial f/\partial u \leq 0$, is really necessary. We believe that any restriction on f leading to a unique solution would also do. However, without any conditions on f, the quasi-linear equation may have multiple solutions and some numerical evidence suggests that the Schwarz sequence does not converge. We tried several examples for which there are at least two distinct solutions. We monitor $\|u^{(n+\frac{1}{2})} - u^{(n)}\|$ in $\Omega_1 \cap \Omega_2$ and find that it oscillates.

Next, we show that the Schwarz method can be applied to a certain class of semilinear elliptic problem whose solution can be shown to be unique using the Global Inversion Theorem. See Ambrosetti and Prodi [**1**].

THEOREM 3. *Consider the semilinear elliptic equation*

$$-\triangle u = \lambda u + f(x, u) + g \text{ on } \Omega \tag{4}$$

with homogeneous Dirichlet boundary conditions. Here $\lambda \in \mathbf{R}$ *is given with* $\lambda \neq \lambda_j$ *for all* j *and such that there exist positive integers* j, l *so that for every* $t \in \mathbf{R}$,

$$\lambda_{j-1}(\Omega_1) < \lambda + f_u(x,t) \leq \lambda < \lambda_j(\Omega_1),\ \forall x \in \Omega_1,$$

and

$$\lambda_{l-1}(\Omega_2) < \lambda + f_u(x,t) \leq \lambda < \lambda_l(\Omega_2),\ \forall x \in \Omega_2.$$

Assume $f \in C^1(\overline{\Omega}, \mathbf{R})$ *and satisfies the conditions*

$$\frac{\|f(x, v_n)\|_{-1}}{\|v_n\|_1} \to 0 \quad \text{whenever } \|v_n\|_1 \to \infty \tag{5}$$

and

$$\lambda_{k-1} < \lambda + f_u(x,t) < \lambda_k$$

for every $x \in \Omega$ and $t \in \mathbf{R}$ and for some $k \in \mathbf{N}$. The function g is assumed to be in $H^{-1}(\Omega)$. For $n = 0, 1, 2, \cdots$ and any $u^{(0)} \in H_0^1(\Omega)$, define the Schwarz sequence as:

$$-\triangle u^{(n+\frac{1}{2})} = \lambda u^{(n+\frac{1}{2})} + f(x, u^{(n+\frac{1}{2})}) + g \text{ on } \Omega_1, \; u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = \lambda u^{(n+1)} + f(x, u^{(n+1)}) + g \text{ on } \Omega_2, \; u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Then the Schwarz sequence converges geometrically to the unique solution of the semilinear elliptic equation (4) in the L^∞ norm.

We note that (5) is satisfied when, for instance, f is a bounded function.

For the above semilinear equation, we made use of the property $f_u \le 0$ in the final step of the proof to show that the limit of the Schwarz sequence is the unique solution to the original problem. It is unknown whether this assumption is really necessary.

Next we consider the resonance problem for the above semilinear equation.

THEOREM 4. *Consider the semilinear equation*

$$-\triangle u = \lambda_1 u + f(x, u) + g \text{ on } \Omega \tag{6}$$

with homogeneous Dirichlet boundary conditions. Here $f \in C^1(\overline{\Omega}, \mathbf{R})$ and satisfies the following conditions:

1. $\exists M$ *such that* $|f(x, s)| \le M, \; \forall x \in \Omega, \; s \in \mathbf{R}$.
2. $\lim_{s \to \pm\infty} f(x, s) = f_\pm, \; \forall x \in \Omega$.
3. $f_- \cdot \int_\Omega \phi_1 < -\int_\Omega g\phi_1 < f_+ \cdot \int_\Omega \phi_1$, *where ϕ_1 is the positive eigenfunction of $-\triangle$ corresponding to the principal eigenvalue λ_1.*
4. $f_u(x, s) \le 0, \; \forall x \in \Omega, \; s \in \mathbf{R}$.

The function g is assumed to be in $H^{-1}(\Omega)$. For $n = 0, 1, 2, \cdots$ and any $u^{(0)} \in H_0^1(\Omega)$, define the Schwarz sequence as:

$$-\triangle u^{(n+\frac{1}{2})} = \lambda_1 u^{(n+\frac{1}{2})} + f(x, u^{(n+\frac{1}{2})}) + g \text{ on } \Omega_1, \; u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = \lambda_1 u^{(n+1)} + f(x, u^{(n+1)}) + g \text{ on } \Omega_2, \; u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Then the Schwarz sequence converges geometrically to the unique solution of the semilinear elliptic equation (6) in the L^∞ norm.

3. Linear Schwarz Method

In the last section, each subdomain problem is still a nonlinear problem. We now consider iterations where linear problems are solved in each subdomain. This is of great importance because in practice, we always like to avoid solving nonlinear problems. One way is in the framework of Newton's method. Write a model semilinear problem as $G(u) = u - \triangle^{-1} f(x, u)$ for $u \in H_0^1(\Omega)$. Suppose it has a solution u and suppose that $\|\triangle^{-1} f_u(x, u)\| < 1$, then for initial guess $u^{(0)}$ sufficiently close to u, the Newton iterates $u^{(n)}$ defined by

$$u^{(n+1)} = u^{(n)} - G_u(u^{(n)})^{-1} G(u^{(n)}) \tag{7}$$

converge to u. Note that the assumption means that $G_u = I - \triangle^{-1} f_u$ has a bounded inverse in a neighborhood of u. Now each linear problem (7) can be solved using the classical Schwarz Alternating Method. We take a different approach.

THEOREM 5. *Consider the equation*

$$-\triangle u = f(x, u, \nabla u) + g \text{ on } \Omega \tag{8}$$

with homogeneous Dirichlet boundary conditions. Assume for every $u, v \in H_0^1(\Omega)$,

$$\|f(x, u, \nabla u) - f(x, v, \nabla v)\| \leq c\sqrt{\lambda_1}\|u - v\|_1,$$

where c *is a constant such that* $c < 1$ *and*

$$d < \sqrt{1 - c^2} - c.$$

Assume $g \in L^2(\Omega)$. *For* $n = 0, 1, 2, \cdots$ *and any* $u^{(0)} \in H_0^1(\Omega)$, *define the Schwarz sequence by,*

$$-\triangle u^{(n+\frac{1}{2})} = f(x, u^{(n)}, \nabla u^{(n)}) + g \text{ on } \Omega_1, \; u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) + g \text{ on } \Omega_2, \; u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Then, the Schwarz sequence converges to the solution of (8) in the energy norm.

Note that each subdomain problem is a linear one.

4. Work in Progress and Conclusion

We now report some recent progress on Schwarz Alternating Methods for the two-dimensional, steady, incompressible, viscous Navier Stokes equations. In the stream function formulation, these equations reduce to the 4th-order nonlinear elliptic PDE

$$\triangle^2\psi = RK(\triangle\psi, \psi) + f,$$

where ψ is the stream function, R is the Reynolds number, K is a skew-symmetric bilinear form defined by $K(u, v) = v_y u_x - v_x u_y$, and f is a forcing term. We have constructed three different Schwarz sequences, nonlinear, linear and parallel sequences and have been able to show global convergence of the sequences in the H^2 norm to the true solution provided the Reynolds number is sufficiently small. Here, nonlinear and linear sequences refer to whether nonlinear or linear problems are solved in each subdomain, and parallel sequence refers to the independence of the problems in each subdomain. We give some further details for the nonlinear Schwarz sequence below.

In the general case, the boundary conditions are inhomogeneous. We make a simple change of variable so that the boundary conditions become homogeneous. The problem now becomes $\triangle^2\phi = RG(\phi)$ where $\phi \in H_0^2(\Omega)$ and G is an appropriate nonlinear term. Let $\phi^{(0)} \in H_0^2(\Omega)$. For $n = 0, 1, 2, \cdots$, define the nonlinear Schwarz sequence as

$$\begin{aligned} \triangle^2\phi^{(n+\frac{1}{2})} &= RG(\phi^{(n+\frac{1}{2})}) \text{ on } \Omega_1 \\ \left(\phi^{(n+\frac{1}{2})}, \frac{\partial\phi^{(n+\frac{1}{2})}}{\partial n}\right) &= \left(\phi^{(n)}, \frac{\partial\phi^{(n)}}{\partial n}\right) \text{ on } \partial\Omega_1 \end{aligned}$$

and

$$\begin{aligned} \triangle^2\phi^{(n+1)} &= RG(\phi^{(n+1)}) \text{ on } \Omega_2 \\ \left(\phi^{(n+1)}, \frac{\partial\phi^{(n+1)}}{\partial n}\right) &= \left(\phi^{(n+\frac{1}{2})}, \frac{\partial\phi^{(n+\frac{1}{2})}}{\partial n}\right) \text{ on } \partial\Omega_2. \end{aligned}$$

The result is that this sequence converges in H^2 to the exact solution provided $R < C/(M+1)$, where C is a constant depending on the geometry and is less than one, and $M = \|\phi^{(0)} - \phi\|_{H^2}$. We are now attempting to show a local convergence result for Reynolds numbers larger than one.

In this paper, we showed how Schwarz Alternating Methods can be imbedded within the framework of Banach and Schauder fixed point theories and Global Inversion theory to construct solutions of 2nd-order nonlinear elliptic PDEs. Future work include Schwarz methods for multiple subdomains, nonlinear parabolic and hyperbolic PDEs and the consideration of Schwarz methods on nonoverlapping subdomains.

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HKUST, Department of Mathematics, Clear Water Bay, Kowloon, Hong Kong
E-mail address: shlui@uxmail.ust.hk