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Convergence Results for Non-Conforming hp Methods: The Mortar Finite Element Method

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1. Introduction

In this paper, we present uniform convergence results for the mortar finite element method (which is an example of a non-conforming method), for h, p and hp discretizations over *general* meshes. Our numerical and theoretical results show that the mortar finite element method is a good candidate for hp implementation and also that the optimal rates afforded by the conforming h, p and hp discretizations are preserved when this non-conforming method is used, even over highly non-quasiuniform meshes.

Design over complex domains often requires the concatenation of separately constructed meshes over subdomains. In such cases it is difficult to coordinate the submeshes so that they conform over interfaces. Therefore, non-conforming elements such as the mortar finite element method [**2, 3, 4**] are used to "glue" these submeshes together. Such techniques are also useful in applications where the discretization needs to be selectively increased in localized regions (such as those around corners or other features) which contribute most to the pollution error in any problem. Moreover, different variational problems in different subdomains can also be combined using non-conforming methods.

When p and hp methods are being used, the interface incompatibility may be present not only in the meshes but also in the *degrees* chosen on the elements from the two sides. Hence the concatenating method used must be formulated to accomodate various degrees, and also be stable and optimal *both* in terms of mesh refinement (h version) *and* degree enhancement (p version). Moreover, this stability and optimality should be preserved when highly non-quasiuniform meshes are used around corners (such as the geometrical ones in the hp version).

We present theoretical convergence results for the mortar finite element method from [**7**],[**8**] and extend these in two ways in this paper. First, we show that the stability estimates established for the mortar projection operator (Theorem 2 in

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[**8**]) are **optimal**. Second, we present h, p and hp computations for a Neumann problem, which fills a gap in numerical validation as explained in Section 4.

2. The Mortar Finite Element Method

We begin by defining the mortar finite element method for the following model problem.

$$-\Delta u = f, \;\; u = 0 \;\text{ on }\; \partial\Omega_D, \quad \frac{\partial u}{\partial n} = g \;\text{ on }\; \partial\Omega_N\,. \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with boundary $\partial\Omega = \overline{\partial\Omega}_D \cup \overline{\partial\Omega}_N$ $(\partial\Omega_D \cap \partial\Omega_N = \emptyset)$, and for simplicity it is assumed $\partial\Omega_D \neq \emptyset$. Defining $H^1_D(\Omega) = \{u \in H^1(\Omega) | u = 0 \;\text{ on }\; \partial\Omega_D\}$ (we use Standard Sobolev space notation), we get the variational form of (1) : Find $u \in H^1_D(\Omega)$ satisfying, for all $v \in H^1_D(\Omega)$,

$$a(u,v) \overset{\text{def}}{=} \int_\Omega \nabla u . \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega_N} g v \, ds \overset{\text{def}}{=} F(v)\,. \tag{2}$$

This problem has a unique solution.

We now assume Ω is partitioned into non-overlapping polygonal subdomains $\{\Omega_i\}_{i=1}^K$ assumed to be geometrically conforming for simplicity (though our results also hold for the geometrically non-conforming (see [**2**]) case). The interface set Γ is defined to be the union of the interfaces $\Gamma_{ij} = \Gamma_{ji}$, i.e. $\Gamma = \cup_{i,j}\Gamma_{ij}$ where $\Gamma_{ij} = \partial\Omega_i \cup \partial\Omega_j$. Γ can then be decomposed into a set of disjoint straight line pieces $\gamma_i, i = 1, 2, \ldots, L$. We denote $Z = \{\gamma_1, \ldots, \gamma_L\}$.

Each Ω_i is assumed to be further subdivided into triangles and parallelograms by geometrically conforming, shape regular [**5**] families of meshes $\{\mathcal{T}_h^i\}$. The triangulations over different Ω_i are assumed independent of each other, with no compatibility enforced across interfaces. The meshes do not have to be quasiuniform and can be quite general, with only a mild restriction, Condition(M), imposed below.

For $K \subset \mathbb{R}^n$, let $\mathcal{P}_k(K)$ $(\mathcal{Q}_k(K))$ denote the set of polynomials of total degree (degree in each variable) $\leq k$ on K. We assume we are given families of piecewise polynomial spaces $\{V^i_{h,k}\}$ on the Ω_i,

$$V^i_{h,k} = \{u \in H^1(\Omega_i) \mid \; u|_K \in \mathcal{S}_k(K) \;\text{ for }\; K \in \mathcal{T}^i_h, \;\; u = 0 \;\text{ on }\; \partial\Omega_i \cap \partial\Omega_D\}\,.$$

Here $\mathcal{S}_k(K)$ is $\mathcal{P}_k(K)$ for K a triangle, and $\mathcal{Q}_k(K)$ for K a parallelogram. Note that $V^i_{h,k}$ are *conforming* on Ω_i, i.e. they contain continuous functions that vanish on $\partial\Omega_D$.

We define the space $\tilde{V}_{h,k}$ by,

$$\tilde{V}_{h,k} = \{u \in L_2(\Omega) \mid \; u|_{\Omega_i} \in V^i_{h,k} \;\; \forall i\} \tag{3}$$

and a discrete norm over $\tilde{V}_{h,k} \cup H^1(\Omega)$ by,

$$||u||^2_{1,d} = \sum_{i=1}^K ||u||^2_{H^1(\Omega_i)}. \tag{4}$$

The condition on the mesh, which will be satisfied by almost any kind of mesh used in the h, p or hp version, is given below. Essentially, it says the refinement cannot be stronger than geometric.

Condition(M) *There exist constants α, C_0, ρ, independent of the mesh parameter h and degree k, such that for any trace mesh on $\gamma \in Z$, given by $x_0 <$*

$x_1 < \ldots < x_{N+1}$, *with* $h_j = x_{j+1} - x_j$, *we have* $\dfrac{h_i}{h_j} \le C_0\,\alpha^{|i-j|}$ *where* α *satisfies* $1 \le \alpha < \min\{(k+1)^2, \rho\}$.

To define the "mortaring", let $\gamma \in Z$ be such that $\gamma \subset \Gamma_{ij}$. Since the meshes $\mathcal{T}_h^i$ are not assumed to conform across interfaces, two separate trace meshes can be defined on γ, one from Ω_i and the other from Ω_j. We assume that one of the indices i, j, say i, has been designated to be the *mortar index associated with* γ, $i = M(\gamma)$. The other is then the *non-mortar index*, $j = NM(\gamma)$. We then denote the trace meshes on γ by $\mathcal{T}^h_{M(\gamma)}$ and $\mathcal{T}^h_{NM(\gamma)}$, with the corresponding trace spaces being $V^M(\gamma)$ and $V^{NM}(\gamma)$, where e.g.

$$V^M(\gamma) = V^M_{h,k}(\gamma) = \{u|_\gamma \mid \ u \in V^i_{h,k}\}\,.$$

Given $u \in \tilde{V}_{h,k}$, we denote the mortar and non-mortar traces of u on γ by u^M_γ and u^{NM}_γ respectively. We now restrict the space $\tilde{V}_{h,k}$ by introducing constraints on the differences $u^M_\gamma - u^{NM}_\gamma$. This "mortaring" is accomplished via Lagrange Multiplier spaces $S(\gamma)$ defined on the non-mortar trace meshes $\mathcal{T}^h_{NM(\gamma)}$. Let the subintervals of this mesh on γ be given by $I_i,\ 0 \le i \le N$. Then we set $S(\gamma) = S^{NM}_{h,k}(\gamma)$ defined as,

$$S(\gamma) = \{\chi \in C(\gamma) \mid \chi|_{I_i} \in \mathcal{P}_k(I_i), i = 1, \ldots, N-1\ , \chi|_{I_0} \in \mathcal{P}_{k-1}(I_0), \chi|_{I_N} \in \mathcal{P}_{k-1}(I_N)\}$$

i.e. $S(\gamma)$ consists of piecewise continuous polynomials of degree $\le k$ on the mesh $\mathcal{T}^h_{NM(\gamma)}$ which are one degree less on the first and last subinterval.

We now define $V_{h,k} \subset \tilde{V}_{h,k}$ by,

$$V_{h,k} = \{u \in \tilde{V}_{h,k} |\ \int_\gamma (u^M_\gamma - u^{NM}_\gamma)\chi\, ds = 0 \ \ \forall\, \chi \in S^{NM}_{h,k}(\gamma), \forall\, \gamma \in Z\}\,. \tag{5}$$

Then our discretization to (2) is defined by: Find $u_{h,k} \in V_{h,k}$ satisfying, for all $v \in V_{h,k}$,

$$a_{h,k}(u_{h,k}, v) \overset{\text{def}}{=} \sum_{i=1}^{K} \int_{\Omega_i} \nabla u_{h,k} \,.\, \nabla v\, dx = F(v)\,. \tag{6}$$

THEOREM 1. [**3**] *Problem* (6) *has a unique solution.*

3. Stability and Convergence Estimates

Let $V_0^{NM}(\gamma)$ denote functions in $V^{NM}(\gamma)$ vanishing at the end points of γ. The stability and convergence of the approximate problem depends on the properties of the projection operator $\Pi_\gamma : L_2(\gamma) \to V_0^{NM}(\gamma)$ defined as follows: For $u \in L_2(\gamma),\ \gamma \in Z,\ \Pi_\gamma u = \Pi_\gamma^{h,k} u$ is a function in $V_0^{NM}(\gamma)$ that satisfies,

$$\int_\gamma (\Pi_\gamma^{h,k} u)\ \chi\, ds = \int_\gamma u\ \chi\, ds \quad \forall\, \chi \in S^{NM}_{h,k}(\gamma)\,. \tag{7}$$

Condition(M) imposed in the previous section is sufficient, as shown in [**7, 8**], to ensure the following stability result for the projections Π_γ.

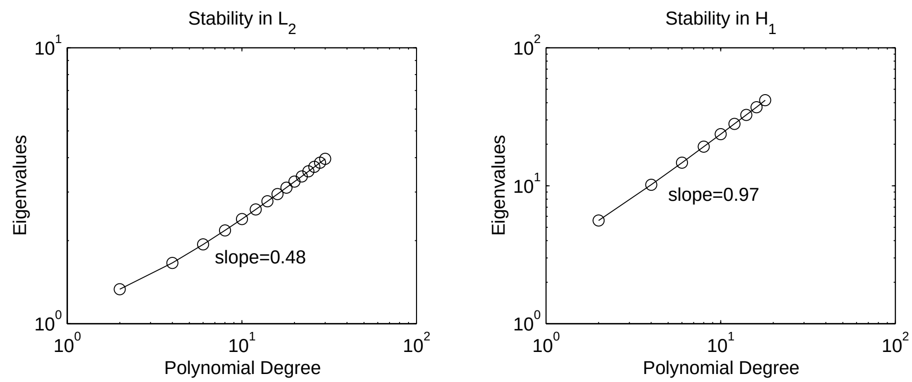

FIGURE 1. (a) Maximum eigenvalue for L_2 (b) Maximum eigenvalue for H^1

THEOREM 2. *Let* $\{V_{h,k}\}$ *be such that Condition(M) holds. Let* $\{\Pi_\gamma^{h,k}, \gamma \in Z\}$ *be defined by* (7). *Then there exists a constant* C, *independent of* h, k *(but depending on* α, C_0, ρ*) such that,*

$$||\Pi_\gamma^{h,k} u||_{0,\gamma} \leq C k^{\frac{1}{2}} ||u||_{0,\gamma} \quad \forall u \in L_2(\gamma) \tag{8}$$

$$||(\Pi_\gamma^{h,k} u)'||_{0,\gamma} \leq C k ||u'||_{0,\gamma} \quad \forall u \in H_0^1(\gamma) \tag{9}$$

A question unanswered in [**8**] was whether (8)–(9) are optimal. Figure 1 shows that the powers of k in (8)–(9) cannot be improved. This is done by approximating the norms of the operator $||\Pi_\gamma^{h,k}||_{\mathcal{L}(L_2(\gamma),L_2(\gamma))}$ and $||\Pi_\gamma^{h,k}||_{\mathcal{L}(H^1(\gamma),H^1(\gamma))}$ (with h fixed), using an eigenvalue analysis. (For details we refer to the thesis [**7**].) It is observed that these norms grow as $O(k^{\frac{1}{2}})$ and $O(k)$ respectively, as predicted by Theorem 2.

Using Theorem 2 and an extension result for *hp* meshes [**8**], we can prove our main theorem, by the argument used in [**3**], Theorem 2 (see [**7, 8**] for details). In the theorem below, $\{N_j\}$ denotes the set of all end points of the segments $\gamma \in Z$.

THEOREM 3. *Let* $\{V_{h,k}\}$ *be such that Condition(M) holds. Then for any* $\epsilon > 0$, *there exists a constant* $C = C(\epsilon)$, *independent of* u, h *and* k *such that,*

$$\begin{aligned} ||u - u_{h,k}||_{1,d} \quad \leq \quad & C \sum_{\gamma \in Z} \inf_{\psi \in S_{h,k}(\gamma)} ||\frac{\partial u}{\partial n} - \psi||_{\left(H^{\frac{1}{2}}(\gamma)\right)'} + \\ & C \inf_{\substack{v \in \tilde{V}_{h,k} \\ v(N_j)=u(N_j)}} \left\{ \sum_i ||u - v||_{1,\Omega_i} + \right. \\ & \left. k^{\frac{3}{4}+\epsilon} \sum_{\gamma \in Z} \left(||u - v_\gamma^M||_{\frac{1}{2}+\epsilon,\gamma} + ||u - v_\gamma^{NM}||_{\frac{1}{2}+\epsilon,\gamma} \right) \right\} \end{aligned} \tag{10}$$

Moreover, for h *or* k *fixed, or for quasiuniform meshes, we may take* $\epsilon = 0$ *if we replace* $||.||_{\frac{1}{2}+\epsilon,\gamma}$ *by* $||.||_{H_{00}^{\frac{1}{2}}(\gamma)}$.

The following estimate for *quasiuniform* meshes follows readily from Theorem 3:

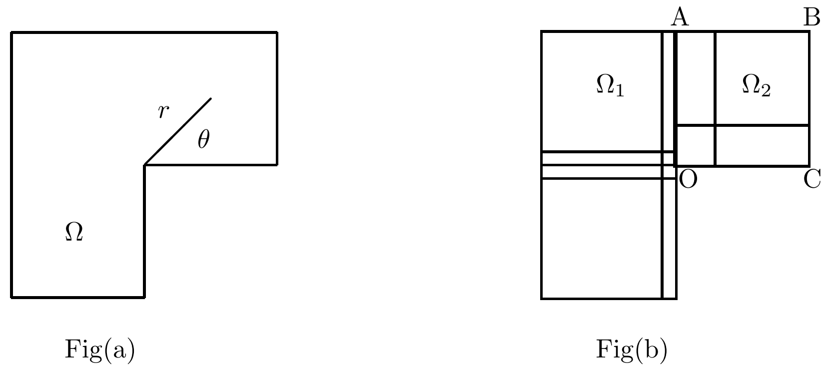

FIGURE 2. (a) L-shaped domain (b) Tensor product mesh for $m = n = 2$

THEOREM 4. *Let the solution u of* (2) *satisfy $u \in H^l(\Omega)$, $l > \frac{3}{2}$ ($l > \frac{7}{4}$ if k varies). For the hp version with* quasiuniform *meshes $\{\mathcal{T}_h^i\}$ on each Ω_i,*

$$||u - u_{h,k}||_{1,d} \leq Ch^{\mu-1}k^{-(l-1)+\frac{3}{4}}||u||_{l,\Omega} \tag{11}$$

where $\mu = \min\{l, k+1\}$ and C is a constant independent of h, k and u.

Theorem 3 also tells us that, using highly non-quasiuniform *radical* meshes in the neighbourhood of singularities (see Section 4 of [**1**]), we can now recover full $O(h^k)$ convergence even when the mortar element method is used. Moreover, *exponential* convergence that is realized when the (conforming) *hp* version is used over *geometrical* meshes will be preserved when the non-conforming mortar finite element is used. We illustrate these results computationally in the next section.

4. Numerical Results

We consider problem (1) on the L-shaped domain shown in Figure 2, which is partitioned into two rectangular subdomains, Ω_1 and Ω_2, by the interface AO. In [**8**], we only considered the case where $\partial\Omega_D = \partial\Omega$. This, however, results in the more restrictive mortar method originally proposed in [**3**], where continuity is enforced at vertices of Ω_i. To implement the method proposed in [**2**] and analyzed here, where the vertex continuity enforcement is removed, we must take Neumann conditions at the ends of AO. We therefore consider here the Neumann case where $\partial\Omega_N = \partial\Omega$, with uniqueness maintained by imposing the condition $u = 0$ at the single point C. Our exact solution is given by,

$$u(r,\theta) = r^{\frac{2}{3}} \cos\left(\frac{2\theta}{3}\right) - 1.$$

where (r, θ) are polar coordinates with origin at O. We use the mixed method to implement the mortar condition. For our computations, we consider tensor product meshes where Ω_2 is divided into n^2 rectangles and Ω_1 is divided into $2m^2$ rectangles (see Figure 2).

It is well-known that this domain will result in a strong $r^{\frac{2}{3}}$ singularity which occurs at the corner O in Figure 2, which limits the convergence to $O(N^{-\frac{1}{3}})$ when the quasiuniform h version is used. Figure 3 shows that this rate is preserved when the mortar finite element is used (graph (1)) with degree $k = 2$ elements. When

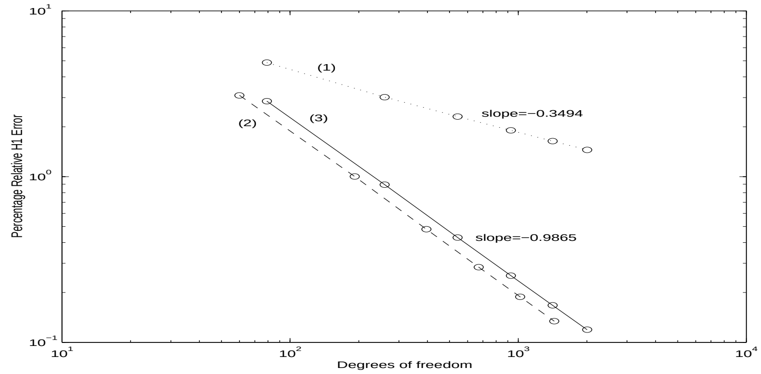

FIGURE 3. The relative error in the energy norm in dependence on h for radical meshes ($k = 2$)

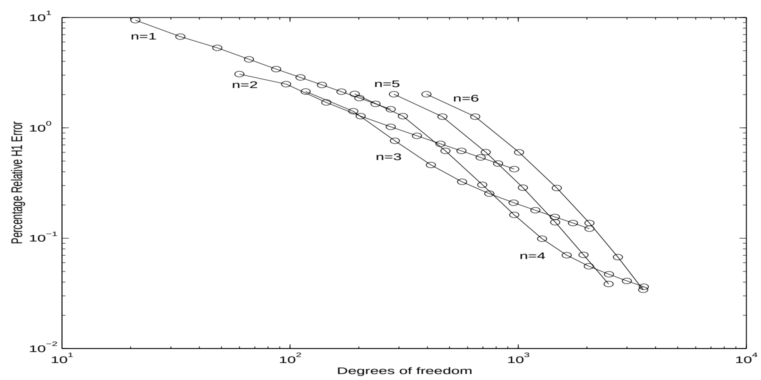

FIGURE 4. The relative error in the energy norm in dependence on N for geometric meshes ($\sigma_1 = 0.17, \quad \sigma_2 = 0.13$)

suitably refined radical meshes are used, then $O(N^{-1})$ convergence is recovered both for the conforming (graph(2)) and mortar (graph(3)) methods.

For the p and hp mortar FEM on geometric meshes, we take $m = n$ and consider the geometric ratio σ (i.e. the ratio of the sides of successive elements, see [**6**]) to vary in each domain Ω_i. The optimal value is 0.15 (see [**6**]), but we take $\sigma_1 = 0.17$ and $\sigma_2 = 0.13$ to make the method non-conforming. We observe in Figure 4, the typical p convergence for increasing degree k for various n. Note that for our problem, at least, we do not see the loss of $O(k^{\frac{3}{4}})$ in the asymptotic rate due to the projection Π_γ not being completely stable (as predicted by Theorem 4 and Figure 1). See Figure 5(a) where we have plotted the case $\sigma_1 = 0.17, \sigma_2 = 0.13$ for $n = 4$ together with the conforming cases $\sigma_1 = \sigma_2 = 0.13$ and 0.17. The results indicate that the p version mortar FEM behaves almost identically to the conforming FEM.

Finally, in Figure 5(b), we plot log(relative error) vs $N^{\frac{1}{4}}$, which gives a straight line, showing the exponential rate of convergence. We also plot log(relative error) vs $N^{\frac{1}{3}}$, which is the theoretical convergence rate for the *optimal* geometric mesh (see [**1**],[**6**]). Since we consider a tensor product mesh here, which contains extra

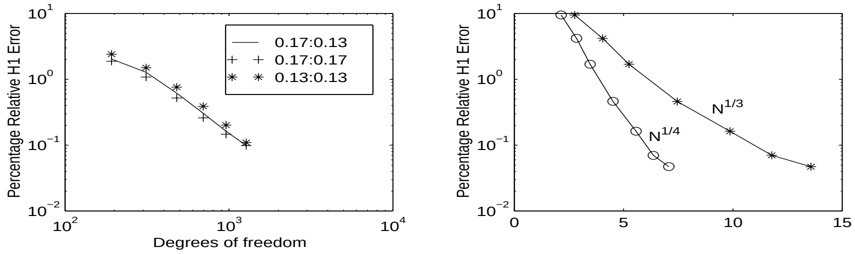

FIGURE 5. (a) Performance of the mortar FEM for n=4 (b) Exponential Convergence for the hp mortar FEM

degrees of freedom, we can only obtain an exponential convergence rate of $Ce^{-\gamma N^{\frac{1}{4}}}$ theoretically.

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