Additive Schwarz Methods for Hyperbolic Equations

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1. Introduction

In recent years, there has been gratifying progress in the development of domain decomposition algorithms for symmetric and nonsymmetric elliptic problems and even some indefinite problems. Many methods possess the attractive property that the convergence rate is optimal, i.e., independent of the size of the discrete problem and of the number of subdomains, or within a polylog factor of optimal. There is, in comparison, relatively little in the domain decomposition literature on hyperbolic problems. Quarteroni [**8, 9**] used nonoverlapping domain decomposition methods based on the spectral collocation approximation on systems of conservation laws. Gastaldi and Gastaldi [**5, 6**] set up a nonoverlapping domain decomposition scheme based on the finite element approximation for the transport equation. These contributions establish the boundary operators that lead to well-posed decoupled problems, which can then be discretized and solved by standard means.

Our interests in this paper are rather different. We examine overlapping domain decomposition preconditioners, and leave the original global discretization fully in tact. Rather than deriving interface conditions that lead to decomposed solutions that are mathematically equivalent (to within some specified discretization tolerance) to the solutions of the undecomposed problem, we derive an approximate inverse that can be applied in a concurrent manner, subdomain-by-subdomain, and that effectively preconditions the original undecomposed operator, whose action is already trivial to apply in the same concurrent manner. There seem to have been to date no such additive or multiplicative Schwarz preconditioners leading to optimal convergence rates for hyperbolic equations.

Based on the standard Galerkin method [**4**] an ASM algorithm is formulated. The preconditioned problems are solved by the GMRES method. The convergence

rate is shown to be asymptotically independent of the time and space mesh parameters and the number of subdomains, provided that the time step is fine enough, namely of such a size as would be typical for temporal stability reasons in an explicit discretization. As these limits are exceeded, numerical experiments based on a Galerkin discretization show a rapid deterioration in convergence rate. (Upwinded discretizations permit explicit stability limits to be exceeded, in the sense that the resulting preconditioned iterations on each time step can converge sufficiently rapidly to be cost-effective in comparison with explicit methods, as discussed in a forthcoming sequel.) Convergence rate is experimentally observed to be relatively independent of overlap.

Just as in the parabolic case, but in contrast to the elliptic case, no coarse-level mesh is required in forming an optimal preconditioner. Good speedups are available on a distributed-memory machine, as would be expected of a problem with a purely local preconditioner.

2. Model problem

We consider for convenience the constant-coefficient linear scalar hyperbolic equation:

$$(1) \qquad \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0, \qquad \text{in} \quad \Omega \times I,$$

together with proper boundary and initial conditions, where Ω is a bounded domain in R^2 with boundary Γ and $I = (0, T)$ is a time interval.

All results in this paper extend without difficulty to the more general linear hyperbolic problem:

$$\frac{\partial u}{\partial t} + \text{div}(bu) + cu = f, \qquad \text{in} \quad (x, t) \in \Omega \times I,$$

where Ω is a bounded domain in R^d ($d = 2$ or 3), the coefficients $b = (b_1, \cdots, b_d)$ and c depend smoothly on (x, t), and $\frac{1}{2}\text{div}b + c \geq c_0 \geq 0$ in $\Omega \times I$, for stability.

By implicit temporal finite differencing, we obtain the following problem:

$$(2) \qquad \left\{ \begin{array}{ll} -\tau_k(\frac{\partial u_k}{\partial x} + \frac{\partial u_k}{\partial y}) + u_k = f, & \text{in} \quad \Omega \\ u_k = g, & \text{on} \quad \Gamma_- \end{array} \right\}, k = 1, 2, \cdots, K,$$

where τ_k is the kth time step, K is the number of steps, $\sum_{k=1}^{K} \tau_k = T$, $f = u_{k-1}$, and Γ_- is the inflow boundary defined by

$$\Gamma_- = \{(x, y) \in \Gamma : n(x, y) \cdot \beta < 0\},$$

where $n(x, y)$ is the outward unit normal to Γ at the point $(x, y) \in \Gamma$, and $\beta = (-\tau_k, -\tau_k)$. Any implicit multistep time-integration method leads to a system like (2), in which f more generally contains a linear combination of the solution at earlier time steps.

The following notation will be used throughout this chapter:

$$< u, v >_- = \int_{\Gamma_-} uv(n \cdot \beta)ds, \quad < u, v >_+ = \int_{\Gamma_+} uv(n \cdot \beta)ds,$$
$$< u, v > = \int_{\Gamma} uv(n \cdot \beta)ds, \qquad |u|_\beta = (\int_{\Gamma} u^2 |n \cdot \beta| ds)^{\frac{1}{2}},$$
$$\|u\| = \|u\|_{L_2(\Omega)}, \qquad\qquad |u| = (\int_{\Gamma} u^2 ds)^{\frac{1}{2}},$$

where $\Gamma_+ = \Gamma \backslash \Gamma_- = \{(x, y) \in \Gamma : n(x, y) \cdot \beta \geq 0\}$.

3. Standard Galerkin method

Let us consider the standard Galerkin method for the problem (2), which can be given the following variational formulation: Find $u \in H^1(\Omega)$, such that

(3) $\qquad (u_\beta + u, v)- < u, v >_- = (f, v)- < g, v >_-, \quad \forall v \in H^1(\Omega),$

where we omit the subscript k, and where $u_\beta = -\tau(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})$. By Green's formula, it is easy to show that

$$(u_\beta, v) = < u, v > -(u, v_\beta).$$

The stability of (3) is a consequence of the following property of the bilinear form $B_\beta(u, v) \equiv (u_\beta + u, v)- < u, v >_-$:

$$B_\beta(u, u) = \|u\|^2 + \frac{1}{2}|u|_\beta^2.$$

The symmetric part of $B_\beta(u, v)$ is

$$A_\beta(u, v) = (u, v) + \frac{1}{2}(< u, v >_+ - < u, v >_-),$$

and the skew-symmetric part is

$$S_\beta(u, v) = \frac{1}{2} < u, v > -(u, v_\beta).$$

Define the β-norm as $\| \cdot \|_\beta = \sqrt{B_\beta(\cdot, \cdot)}$.

We choose $V^h \subset H^1(\Omega)$ as a finite element space of continuous piecewise polynomial functions of degree one or higher on a mesh of quasi-uniform element size h. We discretize equation (3) in space by the Galerkin finite element method and have the approximation: Find $u^h \in V^h$ at each time step k, such that

(4) $\qquad B_\beta(u^h, v^h) = (f, v^h)- < g, v^h >_-, \quad \forall v^h \in V^h.$

We require the following assumption for the theoretical analysis:

ASSUMPTION 1. *The relation between τ and h is*

$$\tau \leq Ch^{1+s},$$

where $s \geq 0$.

In the case of velocity magnitudes different from unity in (1), Assumption 1 becomes a CFL condition, and the allowable time step must be reduced in inverse proportion to the global maximum of the velocity.

We have some lemmas pertaining to B_β, A_β, and S_β as follows.

LEMMA 2. *There exist positive constants c_1 and c_2, independent of τ, such that*

$$\begin{aligned} |B_\beta(u, v)| &\leq c_1\|u\|_\beta \cdot \|v\|_\beta, \quad \forall u, v \in V^h(\Omega), \\ B_\beta(u, u) &\geq c_2\|u\|_\beta^2, \qquad \forall u \in V^h(\Omega). \end{aligned}$$

LEMMA 3. *There exist positive constants c_3 and c_4, independent of τ, such that*

$$\begin{aligned} |A_\beta(u, v)| &\leq c_3\|u\|_\beta \cdot \|v\|_\beta, \quad \forall u, v \in V^h(\Omega), \\ A_\beta(u, u) &\geq c_4\|u\|_\beta^2, \qquad \forall u \in V^h(\Omega). \end{aligned}$$

LEMMA 4. *There exists a constant $c_5 > 0$, independent of τ, such that*

$$|S_\beta(u, v)| \leq c_5\tau(\frac{1}{h}\|u\|\|v\| + |u||v|) \leq c_5(h^s\|u\|\|v\| + |u||v|), \quad \forall u, v \in V^h(\Omega).$$

An additive Schwarz algorithm for (4) is formulated following [**2**]. Let $\Omega_i, i = 1, \cdots, N$, be nonoverlapping subregions of Ω with quasi-uniform diameters H, such that $\bigcup \bar{\Omega}_i = \bar{\Omega}$. The vertices of any Ω_i not on $\partial\Omega$ coincide with the fine-grid mesh vertices. We define an overlapping decomposition of Ω, denoted by $\{\Omega_i', i = 1, \cdots, N\}$, by extending each Ω_i to a larger region Ω_i', which is cut off at the physical boundary of Ω. The overlap is generous in the sense that there exists a constant $\alpha > 0$ such that $\mathrm{dist}(\partial\Omega_i' \bigcap \Omega, \partial\Omega_i' \bigcap \Omega) \geq \alpha H, \forall i$.

Corresponding to the domain decomposition, we decompose the finite element space V^h at each time step k in the customary manner [**2**], i.e., $V^h = V_1^h + \cdots + V_N^h$, where V_k^h is a discrete space whose support is confined to the extended subdomain Ω_i.

The basic building blocks of the algorithm, projection operators $Q_i : V^h \to V_i^h, i = 1, \cdots, N$, are defined by

$$(5) \qquad B_\beta(Q_i u^h, v^h) = B_\beta(u^h, v^h), \quad \forall v^h \in V_i^h.$$

The subproblems have homogeneous Dirichlet boundary conditions for the interior boundary. We can introduce the operator $T = Q_1 + \cdots + Q_N$ and form the transformed linear system

$$(6) \qquad Tu^h = b,$$

where the right-hand side is defined by $b \equiv Tu^h = \sum_{i=1}^N Q_i u^h$, which can be computed without the knowledge of u^h by solving the subproblems (5).

If T is invertible, we show below that equation (6) has the same, unique solution as (4). The operator T is inconvenient to obtain explicitly, but the action of T on a function in V^h is straightforward to compute, consisting of independent problems in subdomains. Thus the preconditioned form (6) can be solved by a Krylov iterative method, such as GMRES [**10**].

With Assumption 1 and the inverse inequalities

$$(7) \qquad \|u\|_1 \leq \frac{c}{h}\|u\|,$$

and (from [**7**])

$$(8) \qquad |u| \leq C\sqrt{\|u\| \cdot \|u\|_1},$$

we have

$$
\begin{aligned}
\|u\|_\beta^2 &\leq \|u\|^2 + \frac{1}{2}\tau C|u|^2 \\
&\leq \|u\|^2 + C\tau\|u\| \cdot \|u\|_1 \\
&\leq \|u\|^2 + Ch^s\|u\|^2 \\
&\leq C\|u\|^2.
\end{aligned}
$$

(assuming that $h \leq 1$). On the other hand, we obtain,

$$(9) \qquad \|u\|^2 \leq \|u\|_\beta^2,$$

which leads to:

LEMMA 5. *The β-norm is equivalent to the L_2 norm.*

Therefore, following [**2**], we come to the conclusion that:

LEMMA 6. *There exists a constant $C_0 > 0$, independent of h and H such that, for all $u^h \in V^h$, there exist $u_i^h \in V_i^h$ with $u^h = \sum_{i=1}^N u_i^h$, and $\sum_{i=1}^N \|u_i^h\|_\beta^2 \leq C_0^2 \|u^h\|_\beta^2$. C_0 generally depends upon the subdomain overlap α.*

We give an estimate in the following lemma for the skew-symmetric part $S_\beta(\cdot, \cdot)$, which shows that the skew-symmetric part is a lower order term compared with the symmetric part, and can therefore be controlled.

LEMMA 7. *There exists a constant δ, $0 < \delta < 1$, independent of τ, h, and H, such that*

$$|S_\beta(u^h, Tu^h)| \leq \delta B_\beta(u^h, Tu^h), \forall u^h \in V^h.$$

PROOF. We use the inequalities (7) and (8) throughout the proof.
By the definition of Q_i, $i = 1, \ldots, N$, we have

$$B_\beta(Q_i u^h, Q_i u^h) = B_\beta(u^h, Q_i u^h),$$

and furthermore

$$\|Q_i u^h\|_{\beta(\Omega_i')} \leq C \|u^h\|_{\beta(\Omega_i')}.$$

Following Lemma 4, Lemma 5, and Assumption 1, we can show

$$|S_\beta(Q_i u^h, u^h - Q_i u^h)| \leq C h^s \|u^h\|_{\beta(\Omega_i')}^2.$$

Using Lemma 2, Lemma 5, and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|u^h\|_\beta^2 &\leq \sum_{i=1}^N B_\beta(Q_i u^h, u_i^h) \\
&\leq c \sum_{i=1}^N \|Q_i u^h\|_\beta \cdot \|u_i^h\|_\beta \\
&\leq c \sqrt{\sum_{i=1}^N \|Q_i u^h\|_\beta^2} \cdot \sqrt{\sum_{i=1}^N \|u_i^h\|_\beta^2} \\
&\leq c C_0 \sqrt{B_\beta(u^h, Tu^h)} \cdot \|u^h\|_\beta,
\end{aligned}
$$

and hence we obtain

$$\|u^h\|_\beta^2 \leq C B_\beta(u^h, Tu^h),$$

which finally leads to the conclusion. $\square$

We can summarize the following main result:

THEOREM 8. *(a) There exist constants $c > 0$ and $C > 0$, independent of τ, h and H, such that*

$$C \|u^h\|_\beta \geq \|Tu^h\|_\beta \geq c \|u^h\|_\beta, \quad \forall u^h \in V^h.$$

(b) There exists a constant $C(\delta) > 0$, such that $\forall u^h \in V^h$

$$A_\beta(u^h, Tu^h) \geq C(\delta) \|u^h\|_\beta^2.$$

Since the symmetric part of the preconditioned linear system is positive definite, GMRES will converge at a rate that is asymptotically independent of h, H, and τ.

TABLE 1. Convergence rate dependence on time-step exponent s

s	It.	Time
0.5	4.0	16.71s
0.1	7.4	26.25s
0.0	10.0	33.95s
-0.1	44.1	146.95s

4. Numerical Results

The preceding theorems are useful in motivating effective algorithms but leave unanswered quantitative questions about the magnitudes of constants in part (a) of Theorem 8 about the extent of dependence of $C(\delta)$ on the size of the overlap in parts (b) of the same theorems, and about the sensitivity of results to inexact solutions in the subdomains. The latter is important since inexactness is usually a practical requirement. For these reasons, we include some numerical experiments, whose purpose is to quantify the dependence of the convergence rate on potentially "bad" parameters, including time step exponent, subdomain overlap, inexactness, overall problem size, and number of subdomains into which the problem is decomposed.

We first vary s between the very conservative $s = \frac{1}{2}$, down to the Courant limit of $s = 0$, and a little beyond into negative values. We solve model problem (1) with backward Euler time-stepping on a uniform grid with central-differencing. We hold the problem size fixed at $h^{-1} = 512$, implying approximately one-quarter of a million degrees of freedom overall, and the the number of subdomains at $p = 16$, arranged in a 4×4 decomposition, with 128×128 grid cells owned by each subdomain. The overlap between subdomains is one mesh cell. We demand a reduction of 10^{-5} in relative residual norm at each time step, accomplished by linear subiterations of GMRES with a subdomain preconditioner of ILU(0).

In Table 1, we tabulate the number of linear iterations per time step, averaged over 10 consecutive steps, and also the execution time for these ten time steps, as measured on the Intel Paragon, with one processor per subdomain. It is evident that the theoretical restriction on the time step to the Courant limit is necessary for reasonable conditioning of the linear iterations.

In Table 2 we vary the subdomain overlap in the preceding example, using two different subdomain preconditioners, exact solvers (indicated by "LU"), and inexact solvers of zero-fill incomplete LU-type ("ILU"). For ILU, three different values of s are tried, hovering around the Courant limit. Convergence criteria and iteration counts are as before. The overlap is tabulated in terms of the thickness of the overlap region in number of cells all around each subdomain, except where cut off at the boundary. We observe that increasing overlap has a slightly beneficial effect when it alone is the bottleneck to better convergence, as in the LU situation. In the practical ILU case, overlap beyond a minimum of one has little to no effect on the convergence rate, provided reasonable values of s are employed. In the case of negative s, increasing the overlap actually causes the convergence rate to deteriorate.

Comparing the first and third result columns, we see that inexactness has a price of approximately a factor of two in convergence rate. In practice, this does not

TABLE 2. Convergence rate dependence on subdomain overlap

overlap	LU,$s = 0$	ILU,$s = 0.1$	ILU,$s = 0$	ILU,$s = -0.1$
1	5.0	7.4	10.0	44.1
2	3.0	7.7	10.0	45.2
4	3.0	7.4	10.0	48.5

TABLE 3. Convergence rate dependence on number of subdomains, and fixed-size parallel scalability

p	per node	It.	Time	Time/It.	Rel. Sp.	Rel. Sp./It.
4	256×256	10.0	171.54s	1.715s		
16	128×128	10.0	33.95s	0.339s	5.05	5.05
64	64×64	10.2	10.41s	0.102s	16.48	16.81

translate into any advantage for exact solvers since the convergence criterion at each time step would usually be commensurate with the temporal truncation error, and looser than that employed here, and the cost for computing an exact factorization of a coefficient matrix on each time step cannot be amortized in practical time-dependent problems (though it could be in (1)).

For Table 3, we fix $s = 0$, the overlap at 1, and the subdomain preconditioner as ILU(0). We perform a problem-size-fixed scaling analysis at $h^{-1} = 512$ by employing successively more subdomains, in going from 4 to 16 to 64 processors. Note that the problem size on each processor decreases by a factor of 2 in each of the x and y directions in this scaling. As before we tabulate the average number of iterations per time step averaged over 10 steps, and the execution time for first ten time steps. The execution time is also presented per iteration, and the speedups (relative to four processors) are presented for both overall time and for time per iteration. This allows for separate measurement of "numerical scalability" of the algorithm and "implementation scalability" of the software/hardware system, with any deterioration of convergence rate at highly granular decompositions factored out.

Our main observations are the virtual independence of convergence rate on the number of subdomains p, for s at the Courant limit, as predicted by the theory, and the better than linear parallel scalability. The latter phenomenon is due to the increasingly good reuse of data in the working set required by the subdomain solvers as the problem-per-processor shrinks. This is a well-known effect in memory-limited machines. Because of the insensitivity of the convergence rate to decomposition, the two speedup measurements are nearly identical.

Table 4 is similar to Table 3; in fact, the last line of each tabulates the same execution, and both run over the same number of processors, except that Table 4 runs a problem small enough to fit on one processor, which grows in size as the number of processors grows. This is known as a Gustafson scaling analysis. It is a practical scaling for large-scale applications and it has the advantage of keeping the workingset per node constant over a range of problem size and processor number.

TABLE 4. Convergence rate dependence on number of subdomains, and Gustafson parallel scalability

p	h^{-1}	It.	Time	Time/It.	Rel. Eff.	Rel. Eff./It.
1	64	8.0	6.60s	0.083s		
4	128	10.1	8.94s	0.089s	0.74	0.93
16	256	10.2	10.00s	0.098s	0.66	0.84
64	512	10.2	10.41s	0.102s	0.63	0.81

The one-subdomain case is special (and would have converged in one iteration had we employed an LU solver). In tabulating efficiency, we take the ratio of the execution times on the successively scaled problems. The efficiency can be viewed as the incremental efficiency of the last processor added, when loaded with the same work per processor. Presenting the relative efficiency per iteration is more important in this case, since the iteration count does degrade in going from one to many subdomains.

Our main observation is that the efficiency remains very high, almost explicit-like. There is no coarse grid to bottleneck this method. On the other hand the frequent global inner products are minor bottlenecks.

We employed the Portable Extensible Toolkit for Scientific Computing (PETSc) [1] from Argonne National Laboratory for the numerical studies.

5. Conclusions

We have used the standard Galerkin method and to formulate an optimal additive Schwarz method for general scalar linear hyperbolic equations. The same techniques leading to optimal convergence rates for the parabolic and elliptic cases have been are used here, after identification of the proper norm. The method of proof does not permit evaluation of the key constants in the theory.

The theoretical techniques employed here may be applicable to other equations, e.g., linearized Euler equations and hyperbolic systems of conservation laws, after transformation to canonical form and operator splitting. We are currently pursuing such extensions.

Because of Assumption 1 limiting the size of τ, the implicit method described herein might not appear to offer any advantage relative to the correspondingly spatially discretized temporally explicit method, which has equally good or better parallelization properties, and would not require iteration on each time step to solve a linear system. On the other hand, temporal truncation accuracy limits the algebraic accuracy required in the solution of the implicit system to just a few matrix-vector products, and the implicit form may be thought of as a defect-correction solver. Two practical applications of the results of this paper may be to: (1) problems with multiple scales, with some scales finer than the explicit stability limit, all of which could be treated implicitly with this method, and (2) problems with embedded hyperbolic regions, for which a uniform Schwarz preconditioned framework is desired. We mention [3] as an example.

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