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Some Results on Schwarz Methods for a Low–Frequency Approximation of Time–Dependent Maxwell's Equations in Conductive Media

Andrea Toselli

1. Introduction

In this paper, we recall some recent theoretical results on two–level overlapping Schwarz methods for finite element approximations of Maxwell's equations and present some numerical results to compare the performances of different overlapping Schwarz methods, when varying the number of subregions, the mesh size of the coarse space and the time step of the implicit finite difference scheme employed.

When studying low-frequency electromagnetic fields in conductive media, the displacement current term in Maxwell's equations is generally neglected and a parabolic partial differential equation is solved. The electric field $\mathbf{u}$ satisfies the equation

$$\mathbf{curl}\,\left(\mu^{-1}\mathbf{curl}\,\mathbf{u}\right) + \sigma\frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{J}}{\partial t}, \text{ in } \Omega, \tag{1}$$

where $\mathbf{J}(\mathbf{x}, t)$ is the current density and μ and σ are the magnetic permeability and the electric conductivity of the medium. For their meaning and for a general discussion of Maxwell's equations, see [**3**]. Here Ω is a bounded, three–dimensional polyhedron, with boundary Γ and outside normal $\mathbf{n}$. For a perfect conducting boundary, the electric field satisfies the essential boundary condition

$$\mathbf{u} \times \mathbf{n}|_{\Gamma} = 0. \tag{2}$$

For the analysis and solution of Maxwell's equations suitable Sobolev spaces must be introduced. The space $H(\mathrm{curl}, \Omega)$, of square integrable vectors, with square integrable curls, is a Hilbert space with the scalar product

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{u}, \mathbf{v}), \tag{3}$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$. The subspace of $H(\mathrm{curl}, \Omega)$ of vectors with vanishing tangential component on Γ is denoted by $H_0(\mathrm{curl}, \Omega)$. For the properties of $H(\mathrm{curl}, \Omega)$ and $H_0(\mathrm{curl}, \Omega)$, see [**5**].

In the following section, we recall the variational problem associated with (1) and (2), and the finite element spaces employed for its approximation. In Section 3,

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we describe the two–level Schwarz method studied and state some theoretical results about its convergence properties. Section 4 is devoted to the numerical results.

2. Discrete problem

When an implicit FD scheme is employed and a finite element space $V \subset H_0(\mathrm{curl}, \Omega)$ is introduced, equations (1) and (3) can be approximated by:

Find $\mathbf{u} \in V$ such that

$$a_\eta(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) + \eta\,(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \forall\,\mathbf{v} \in V, \tag{4}$$

at each time step; η is a positive quantity, proportional to the time step Δt, and f depends on the solution at the previous steps, as well as on the right hand side of (1). See [**1**] for the finite element approximation of Maxwell's equations, and [**10**] for the time approximation of parabolic problems.

For the finite element approximation, we first consider a shape–regular triangulation $\mathcal{T}_h$ of the domain Ω, consisting of tetrahedra. Here h is the maximum diameter of $\mathcal{T}_h$. We employ the Nédélec spaces of the first kind, of degree $k > 0$, which were introduced in [**8**]; see also [**5**]. Other choices of finite element spaces are also possible, see [**9**], as well as triangulations made of hexahedra and prisms, see [**8**], [**9**].

Let $V \subset H_0(\mathrm{curl}, \Omega)$ be the Nédélec space of degree k, built on $\mathcal{T}_h$, of vectors with vanishing tangential component on the boundary. We recall that vectors in V are not continuous, in general, but that only the continuity of their tangential component is preserved across the faces of the tetrahedra, as it is physically required for the electric field. The degrees of freedom associated to this finite element space involve integrals of the tangential components over the edges and the faces, as well as moments computed over each tetrahedron. See [**8**], [**9**], [**5**] and the references in [**12**] and [**7**] for further details on Nédélec spaces.

3. Additive two–level method

In order to build a two–level overlapping preconditioner for (4), we have to introduce an additional coarse triangulation and a decomposition of Ω into overlapping subregions. This can be done in a standard way; see [**11**] as a general reference for this section.

We suppose that the triangulation $\mathcal{T}_h$ is obtained by refining a shape-regular coarse triangulation $\mathcal{T}_H$, $H > h$, made of tetrahedra $\{\Omega_i\}_{i=1}^J$. Let us consider a covering of Ω, say $\{\Omega_i'\}_{i=1}^J$, such that each open set Ω_i' is the union of tetrahedra of $\mathcal{T}_h$ and contains Ω_i. Define the overlapping parameter δ by

$$\delta = \min_i\{\mathrm{dist}(\partial\Omega_i', \Omega_i)\}.$$

We suppose that $\delta \geq \alpha H$, for a constant $\alpha > 0$ (generous overlap), and that every point $P \in \Omega$ belongs to at most N_c subregions (finite covering).

The coarse space V_0 is defined as the Nédélec space over the coarse triangulation $\mathcal{T}_H$, $H > h$, and the local spaces $V_i \subset V$, associated to the subregions $\{\Omega_i'\}_{i=1}^J$, are obtained by setting the degrees of freedom outside Ω_i' to zero. Since the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ are nested, the coarse space V_H is contained in V; see [**6**]. The space V admits the decomposition $V = \sum_{i=0}^J V_i$.

Let us now define the following projections for $i = 0, \cdots, J$:

$$T_i : V \longrightarrow V_i, \tag{5}$$

$$a_\eta (T_i \mathbf{u}, \mathbf{v}) = a_\eta (\mathbf{u}, \mathbf{v}), \; \forall \mathbf{v} \in V_i, \tag{6}$$

where $a_\eta(\cdot, \cdot)$ is defined in (4) and let us introduce the additive Schwarz operator

$$T = \sum_{j=0}^{J} T_j : V \longrightarrow V. \tag{7}$$

We solve the equation

$$T \, \mathbf{u} = \mathbf{g},$$

with the conjugate gradient method, without any further preconditioner, employing $a_\eta(\cdot, \cdot)$ as the inner product and a suitable right hand side $\mathbf{g}$; see [**4**], [**11**].

Two-level multiplicative schemes can also be designed; see [**4**],[**11**]. The error $\mathbf{e}_n$ at the n-th step satisfies the equation

$$\mathbf{e}_{n+1} = E \, \mathbf{e}_n = (I - T_J) \cdots (I - T_0) \, \mathbf{e}_n, \; \forall n \geq 0. \tag{8}$$

Different choices of multiplicative and hybrid operators are also possible and Krylov subspace methods, such as GMRES, can be employed as accelerators; see [**11**] for a more detailed discussion. A hybrid method will be considered in the next section.

The main result concerning the convergence properties of the standard additive and multiplicative algorithms is contained in the following theorem:

THEOREM 1. *If the domain Ω is convex, and the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ shape-regular and quasiuniform, then the condition number of the additive algorithm and the norm of the error operator E are bounded uniformly with respect to h, the number of subregions and η. The bounds increase with the inverse of the relative overlap H/δ and the finite covering parameter N_c.*

A proof of this theorem can be found in [**12**], for the case $\eta = 1$, and in [**7**] for the general case. We remark that the proofs employ the discrete Helmholtz decomposition of Nédélec spaces and suitable projections onto the spaces of discrete divergence–free functions; see [**5**] and [**6**]. The convexity of the domain Ω and the quasiuniformity of the meshes appear to be necessary for the error bounds of discrete divergence–free vectors.

4. Numerical results

In this section we present some numerical results to analyze the dependence of some Schwarz algorithms on the overlap, the number of subdomains and the time step. We will also show some results for a non–convex domain, for which the analysis carried out in [**12**] and [**7**] is not valid. We consider the Dirichlet problem (4) and use hexahedral Nédélec elements.

We have tested the following Schwarz algorithms:

(i) The Conjugate gradient method applied to the *additive one–level* operator

$$T_{as1} = \sum_{i=1}^{J} T_i,$$

where the T_i are the projections onto the local subspaces, defined in (5), (6).

TABLE 1. Estimated condition number and N_c (in parenthesis), versus H/δ and the number of subregions: additive one–level algorithm, $\Omega = (0,1)^3$, $\eta = 1$, $16 \times 16 \times 16$–element fine mesh (13,872 unknowns).

	$2 \times 2 \times 2$	$4 \times 4 \times 4$	$8 \times 8 \times 8$
$H/\delta = 8$	28.7 (8)	-	-
$H/\delta = 4$	14.3 (8)	40.0 (8)	-
$H/\delta = 8/3$	10.1 (8)	-	-
$H/\delta = 2$	8.62 (8)	13.5 (8)	46.4 (8)
$H/\delta = 4/3$	8.02 (8)	27.0 (27)	-
$H/\delta = 1$	-	5.19 (27)	15.5 (27)

TABLE 2. Estimated condition number and N_c (in parenthesis), versus H/δ and the number of subregions: additive two–level algorithm, $\Omega = (0,1)^3$, $\eta = 1$, $16 \times 16 \times 16$–element fine mesh (13,872 unknowns).

	$2 \times 2 \times 2$	$4 \times 4 \times 4$	$8 \times 8 \times 8$
$H/\delta = 8$	15.7 (8)	-	-
$H/\delta = 4$	9.67 (8)	10.3 (8)	-
$H/\delta = 8/3$	8.48 (8)	-	-
$H/\delta = 2$	8.42 (8)	8.91 (8)	9.23 (8)
$H/\delta = 4/3$	8.73 (8)	27.0 (27)	-
$H/\delta = 1$	-	18.2 (27)	26.3 (27)

(ii) The Conjugate gradient method applied to the *additive two–level* operator

$$T_{as2} = T_0 + \sum_{i=1}^{J} T_i,$$

where the T_0 the projection onto the coarse space V_0.

(iii) The GMRES method applied to the non–symmetric *two–level hybrid* operator

$$T_{hy} = I - (I - T_0)\left(I - \omega \sum_{i=1}^{J} T_i\right),$$

where ω is a scaling parameter.

Our results have been obtained on a SUN Ultra1, using the PETSc 2.0 library; see [**2**].

The independence of the condition number on the diameter h of the fine mesh is observed, when the number of subdomains and the relative overlap δ/H are fixed; the results are not presented here.

We first consider a unit cube Ω, with a fixed value of $\eta = 1$. Tables 1 and 2 show the estimated condition numbers for algorithms (i) and (ii), as functions of the relative overlap and the number of subregions; in parenthesis, we also show the value of N_c defined in the previous section. We observe:

Table 3. Number of iterations versus H/δ and the number of subregions, to reduce the residual error by a factor 10^{-6}: additive two–level algorithm (ii), $\Omega = (0,1)^3$, $\eta = 1$, $16 \times 16 \times 16$–element fine mesh (13,872 unknowns).

	8	64
$H/\delta = 8$	26	-
$H/\delta = 4$	24	22
$H/\delta = 8/3$	23	-
$H/\delta = 2$	21	21
$H/\delta = 4/3$	21	33
$H/\delta = 1$	-	27

Table 4. Number of iterations and residual error (in parenthesis), versus H/δ and the number of subregions, to reduce the residual error of the preconditioned system by a factor 10^{-9}: hybrid two–level algorithm (iii) with optimal scaling, $\Omega = (0,1)^3$, $\eta = 1$, $16 \times 16 \times 16$–element fine mesh (13,872 unknowns).

	8	64
$H/\delta = 8$	28 (4.5×10^{-3})	-
$H/\delta = 4$	26 (1.6×10^{-3})	26 (9.8×10^{-5})
$H/\delta = 8/3$	24 (7.3×10^{-5})	-
$H/\delta = 2$	23 (1.0×10^{-5})	24 (8.6×10^{-5})
$H/\delta = 4/3$	20 (2.8×10^{-5})	50 (1.3×10^{-7})
$H/\delta = 1$	-	20 (4.8×10^{-6})

- For both algorithms, the condition number decreases with H/δ decreasing, when the number of subregions and N_c are fixed. In accordance with the analysis in [**7**], the condition number increases with N_c; the bound of the largest eigenvalue of the Schwarz operators grows linearly with N_c.
- For a fixed value of the relative overlap and N_c, the condition number grows rapidly with the number of subregions for the one–level algorithm, while it grows slowly for the two-level case.
- The two–level algorithm behaves better than the one–level method when the overlap is small, but seems more sensitive to N_c.

For the same domain $\Omega = (0,1)^3$ and the same value of $\eta = 1$, Tables 3 and 4 show the number iterations for algorithms (ii) and (iii), as functions of H/δ, for the cases of 8 and 64 subregions. In order to compare the two methods, a reduction by a factor 10^{-6} of the residual error of the unpreconditioned system was chosen for the CG algorithm in (ii), while a reduction by a factor 10^{-9} of the residual error of the preconditioned system was considered in (iii). In Table 4, we also show the residual error of the unpreconditioned system in parenthesis. GMRES was restarted each 30 iterations.

The results show that algorithm (iii) gives a number of iterations that is comparable to the ones of (ii). According to our numerical tests, the optimal value of the scaling factor ω depends on the overlap and the number of subregions. The results

TABLE 5. Estimated condition number versus H/δ and the number of subregions: additive one–level algorithm, $\Omega = (0,1)^3 \setminus [0,1/2]^3$, $16 \times 16 \times 16$–element fine mesh (12,336 unknowns).

	$2\times2\times2$	$4\times4\times4$	$8\times8\times8$
$H/\delta = 8$	31.5	-	-
$H/\delta = 4$	15.0	41.2	-
$H/\delta = 2$	8.65	13.6	48.9
$H/\delta = 1$	-	5.84	18.5

TABLE 6. Estimated condition number versus H/δ and the number of subregions: additive two–level algorithm, $\Omega = (0,1)^3 \setminus [0,1/2]^3$, $\eta = 1$, $16\times16\times16$–element fine mesh (12,336 unknowns).

	$2\times2\times2$	$4\times4\times4$	$8\times8\times8$
$H/\delta = 8$	52.2	-	-
$H/\delta = 4$	43.4	17.3	-
$H/\delta = 2$	36.2	17.1	10.2
$H/\delta = 1$	-	24.8	27.2

in Table 4 were obtained with optimal scaling. We also ran some tests with a symmetrized version of the hybrid algorithm, obtained by adding another smoothing step on the subdomains, see [**11**], but the results are not so good as in (iii) and are not presented here. The full multiplicative preconditioner was not implemented in the PETSc version that we used.

We have also considered the case $\Omega = (0,1)^3 \setminus [0,1/2]^3$, in which Ω is not convex and for which the analysis carried out in [**12**] and [**7**] is not valid. Tables 5 and 6 show the estimated condition numbers for algorithms (i) and (ii), as functions of the relative overlap and the number of subregions; in order to facilitate the comparison, we have shown the number of subregions and elements corresponding to the discretization of the whole $(0,1)^3$. We observe:

- All the remarks made for the case $\Omega = (0,1)^3$ are still valid, in general, except that the one–level algorithm behaves better than the two-level one, unless the number of subregions is large.
- The one–level algorithm shows a slight increase of the condition number, compared to the convex case, while a considerable deterioration of the performances of the two–level algorithm is observed, unless the number of subregions is large.
- Theorem 1 seems to be valid for this particular choice of non convex domain, even if the constants are found to be larger than the ones in the convex case.

Finally, we have tested algorithms (i) and (ii) for different values of η. We recall that η is proportional to the time step of the FD scheme employed for the approximation of (1). According to the analysis in [**7**] the condition number of algorithm (ii) can be bounded independently of η. For methods (i) and (ii), Figures 1 and 2 show the estimated condition numbers as functions of η, for different values of the overlap and 8 subregions. Figures 3 and 4 show the results for 64 subdomains.

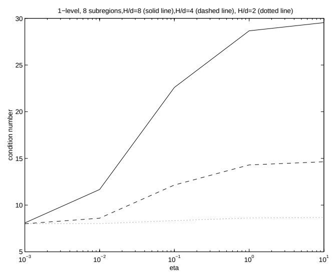

FIGURE 1. Estimated condition number versus η ($H/\delta = 8$, solid line; $H/\delta = 4$, dashed line; $H/\delta = 2$, dotted line): additive 1–level algorithm, $\Omega = (0,1)^3$, $16\times16\times16$–element fine mesh, 8 subregions.

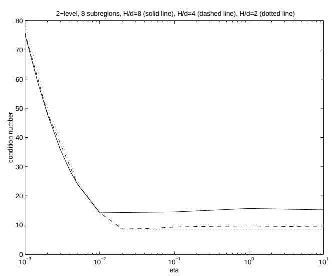

FIGURE 2. Estimated condition number versus η ($H/\delta = 8$, solid line; $H/\delta = 4$, dashed line; $H/\delta = 2$, dotted line): additive 2–level algorithm, $\Omega = (0,1)^3$, $16\times16\times16$–element fine mesh, 8 subregions.

For values of η larger than 0.1, the same remarks made for the case $\eta = 1$ hold. But, in practice, for smaller values of η, the performances of the two–level method deteriorate and the algorithm becomes inefficient for very small η. The coarse space correction is not effective if $\eta < c(\delta)\,\delta^2$, where $c(\delta)$ is found to be close to one. On the contrary the condition number of method (i) is found to be decreasing with the time step and, as η tends to zero, tends to N_c. For $\eta < c(\delta)\,\delta^2$, the latter algorithm has better performances than the two–level one, or, in other words, for a fixed value of the time step, the overlap should not be too large or the coarse space correction will not be effective.

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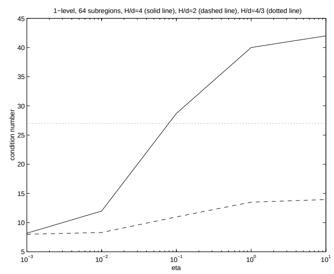

FIGURE 3. Estimated condition number versus η ($H/\delta = 4$, solid line; $H/\delta = 2$, dashed line; $H/\delta = 4/3$, dotted line): additive 1–level algorithm, $\Omega = (0,1)^3$, $16 \times 16 \times 16$–element fine mesh, 64 subregions.

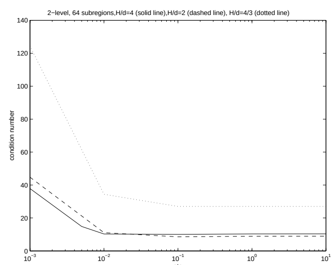

FIGURE 4. Estimated condition number versus η ($H/\delta = 4$, solid line; $H/\delta = 2$, dashed line; $H/\delta = 4/3$, dotted line): additive 2–level algorithm, $\Omega = (0,1)^3$, $16 \times 16 \times 16$–element fine mesh, 64 subregions.

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, 251 MERCER ST, NEW YORK, NY 10012
E-mail address: `toselli@cims.nyu.edu`