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Nonclassical Shocks and the Cauchy Problem: General Conservation Laws

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ABSTRACT. In this paper we establish the existence of *nonclassical* entropy solutions for the Cauchy problem associated with a conservation law having a *nonconvex* flux-function. Instead of the classical Oleinik entropy criterion, we use a single *entropy inequality* supplemented with a *kinetic relation.* We prove that these two conditions characterize a unique *nonclassical Riemann solver.* Then we apply the wave-front tracking method to the Cauchy problem. By introducing a new total variation functional, we can prove that the corresponding approximate solutions converge strongly to a nonclassical entropy solution.

1. Introduction

In this paper we establish a new existence theorem for weak solutions of the Cauchy problem associated with a nonlinear hyperbolic conservation law,

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,t) \in \boldsymbol{R} \quad x \in \boldsymbol{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \boldsymbol{R}. \tag{1.2}$$

The flux-function $f : \boldsymbol{R} \to \boldsymbol{R}$ is *nonconvex* and the initial data $u_0 : \boldsymbol{R} \to \boldsymbol{R}$ is a function with bounded total variation. We are interested in weak solutions that are of bounded total variation and additionally satisfy the fundamental *entropy inequality*

$$\partial_t U(u) + \partial_x F(u) \leq 0 \tag{1.3}$$

for a (fixed) strictly convex entropy $U : \boldsymbol{R} \to \boldsymbol{R}$. As usual, the entropy-flux is defined by $F'(u) = U'(u) f'(u)$. We refer to Lax [**21, 22**] for these fundamental notions.

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This self-contained paper is part of a series [**3, 5, 6**] devoted to proving the existence of nonclassical solutions for the *Cauchy problem* (1.1)–(1.2) supplemented with a single entropy inequality, (1.3), and a "kinetic relation" (see below). The paper [**3**] treated the case of a cubic flux $f(u) = u^3$ and placed a rather strong assumption on the kinetic function. Our purpose here is to provide an existence result for a *large class* of fluxes and kinetic relations covering all the examples arising in the applications. We will also provide examples where the total variation blows up when our assumptions are violated.

It is well-known since the works of Kružkov [**20**] and Volpert [**33**] that the problem (1.1)-(1.2) admits a unique (classical) entropy solution satisfying *all* of the entropy inequalities (1.3). In the present work we are interested in weak solutions constrained by a *single* entropy inequality. This question is motivated by zero diffusion-dispersion limits like

$$\partial_t u + \partial_x f(u) = \epsilon\, u_{xx} + \gamma \varepsilon^2\, u_{xxx}, \qquad \varepsilon \to 0 \;\text{ with }\; \gamma \;\text{ fixed.} \tag{1.4}$$

Hayes and LeFloch [**13, 14, 15**] observed that limiting solutions given by (1.4) and many similar *continuous* or *discrete* models satisfy the single entropy inequality (1.3) for a *particular choice* of entropy U, induced by the regularization terms. As is well-known, when the flux is convex the entropy inequality (1.3) singles out a *unique* weak solution of (1.1)-(1.2). However when the flux lacks convexity, this is no longer true and there is room for an additional selection criterion. It appears that weak solutions of the Cauchy problem (1.1)–(1.3) may exhibit *undercompressive, nonclassical shocks* which are the source of *non-uniqueness.* In [**13, 14**] it was proposed to further constrain the *entropy dissipation* of a nonclassical shock in order to uniquely determine its propagation speed. The corresponding relation is called a *kinetic relation.*

Jacobs, McKinney and Shearer [**17**] and then Hayes and LeFloch [**13**] (also [**16**]) observed that limits of diffusive-dispersive regularizations like (1.4) depend on the parameter γ and may fail to coincide with the classical entropy solutions of Kružkov-Volpert's theory. The sign of the parameter γ turns out to be critical. The corresponding kinetic function has been determined for several examples analytically and numerically.

The concept of a kinetic relation was introduced earlier in the material science literature, in the context of propagating phase transitions in solids undergoing phase transformations. James [**18**] recognized that weak solutions satisfying the standard entropy inequality were not unique. Abeyaratne and Knowles [**1, 2**] and Truskinovsky [**31, 32**] were pioneers in studying the Riemann problem and the properties of shock waves in phase dynamics. The kinetic relation was placed in a mathematical perspective by LeFloch in [**23**]. Earlier works on the Riemann problem with phase transitions include the papers by Slemrod [**30**] (where a model like (1.4) was introduced) and Shearer [**29**] (where the Riemann problem was solved using Lax entropy inequalities).

The papers [**13, 16, 17**] are concerned with the existence and properties of the traveling wave solutions associated with nonclassical shocks. The implications of a single entropy inequality for nonconvex equations and for non-genuinely nonlinear systems were discovered in [**13, 14**]. The numerical computation of nonclassical shocks via finite difference schemes was tackled in [**15, 25**]. Finally, for a review of these recent results we refer the reader to [**24**].

In [**3**], where the cubic case $f(u) = u^3$ is considered, it is proved that starting from a nonclassical Riemann solver, a front-tracking algorithm (Dafermos [**8**], DiPerna [**9**], Bressan [**7**], Risebro [**28**], Baiti and Jenssen [**4**]) applied to the Cauchy problem (1.1)-(1.2) converges to a weak solution satisfying the entropy condition (1.3), provided the initial data have bounded total variation.

The main difficulty in [**3**] was to derive a *uniform bound* on the total variation of the approximate solutions since nonclassical solutions do not satisfy the standard Total Variation Diminishing (TVD) property. Due to the presence of nonclassical shocks one was forced to introduce a *new functional*, equivalent to the total variation, which was decreasing in time for approximate solutions. This was achieved by estimating the strengths of waves across each type of interaction.

In the present paper we generalize [**3**] in two different directions: on one hand we consider general fluxes having one inflection point. The study of this case is required before tackling the harder case of systems [**5,6**]. On the other hand we relax the hypotheses imposed in [**3**] on the kinetic function, especially the somehow restrictive assumption that shocks with small strength were always classical.

As already pointed out, the difficult part in the convergence proof is finding a *modified measure of total variation.* In the cubic case [**3**] elementary properties of the (cubic) flux were used, in particular its symmetry with respect to 0. In the case of nonsymmetric fluxes it happens that an explicit form of the modified total variation can not be easily derived. To accomplish the same purpose here, we use a fixed-point argument on a suitable function space (see Sections 4 and 5). This approach should also clarify the choices made in [**3**] (see Section 6).

The paper is organized as follows. In Section 2 we start by listing our hypotheses and in Section 3 investigate how to solve the Riemann problem in the class of nonclassical solutions. In particular we prove that, under mild assumptions, every Riemann solver generating an $\mathbf{L}^1$-continuous semigroup of entropy solutions must be of the form considered here. Sections 4 to 6 are devoted to the definition and construction of the modified total variation. Finally, in Section 7 we present examples of blow-up of the total variation in cases when our hypotheses fail.

We also mention two companion papers which treat the uniqueness of nonclassical solutions [**5**] and the existence of nonclassical solutions for systems [**6**], respectively.

2. Assumptions

This section displays the assumptions required on the flux-function f and on the kinetic function φ. We assume that f is a smooth function of the variable u and admits a *single non-degenerate inflection point.* In other words, with obvious normalization, we make the following two assumptions:

(A1) $f(0) = 0,\ \ f'(u) > 0,\ \ u\, f''(u) > 0$ for all $u \neq 0$.

(A2) For some $p \geq 1$, f has the following Taylor expansion at $u = 0$

$$f(u) = H\, u^{2p+1} + o(u^{2p+1}) \qquad \text{for some } H \neq 0.$$

The results of this paper extends to the case where $u\, f''(u) < 0$ holds. Note that (A1) implies

$$\lim_{u \to \pm\infty} f(u) = \pm\infty.$$

Consider the graph of the function f in the (u, f)-plane. For any $u \neq 0$ there exists a unique line that passes through the point with coordinates $(u, f(u))$ and is tangent to the graph at a point $\big(\tau(u), f(\tau(u))\big)$ with $\tau(u) \neq u$. In other words

$$f'\big(\tau(u)\big) = \frac{f(u) - f\big(\tau(u)\big)}{u - \tau(u)}. \tag{2.1}$$

Note that $u\,\tau(u) < 0$ and set also $\tau(0) = 0$. Thanks to the assumption (A1) on f, the map $\tau : \boldsymbol{R} \to \boldsymbol{R}$ is monotone decreasing and onto, and so is invertible. The inverse function satisfies

$$f'(u) = \frac{f(u) - f\big(\tau^{-1}(u)\big)}{u - \tau^{-1}(u)} \qquad \text{for all} \quad u \neq 0. \tag{2.2}$$

For any $u \neq 0$, define the point $\varphi^*(u) \neq u$ by the relation

$$\frac{f(u)}{u} = \frac{f\big(\varphi^*(u)\big)}{\varphi^*(u)}, \tag{2.3}$$

so that the points with coordinates

$$\big(\varphi^*(u), f(\varphi^*(u))\big), \qquad (0,0), \qquad \big(u, f(u)\big)$$

are aligned. Again from the assumptions (A1) above, it follows that $\varphi^* : \boldsymbol{R} \to \boldsymbol{R}$ is monotone decreasing and onto. Finally observe that

$$u\,\tau^{-1}(u) \leq u\,\varphi^*(u) \leq u\,\tau(u) \qquad \text{for all } u. \tag{2.4}$$

In Section 3 we shall prove that, in order to have uniqueness for the Riemann problem, for every left state u one has to single out a unique right state $\varphi(u)$ that can be connected to u with a nonclassical shock. The function $\varphi : \boldsymbol{R} \mapsto \boldsymbol{R}$ is called a *kinetic function* and depends on the regularization adopted for (1.1).
Given φ, we define the function $\alpha : \boldsymbol{R} \mapsto \boldsymbol{R}$ by the relation

$$\frac{f(u) - f\big(\alpha(u)\big)}{u - \alpha(u)} = \frac{f(u) - f\big(\varphi(u)\big)}{u - \varphi(u)}, \tag{2.5}$$

so that the points with coordinates

$$\big(\varphi(u), f(\varphi(u))\big), \qquad \big(\alpha(u), f(\alpha(u))\big), \qquad \big(u, f(u)\big)$$

are aligned.

In the whole of this paper a strictly convex entropy-entropy flux pair (U, F) is fixed to serve in the entropy inequality (1.3). In Proposition 3.1 we shall prove that for any $u_l \neq 0$ there exists a point $\varphi^\sharp(u_l)$ (depending on u_l and on the choice of (U, F)) such that the discontinuity (u_l, u_r) is admissible with respect to (1.3) iff $u_l\,\varphi^\sharp(u_l) \leq u_r u_l \leq u_l^2$. Finally, we shall denote by $g^{[k]}$ the k-th iterate of a map g.

Now select a kinetic function $\varphi : \boldsymbol{R} \mapsto \boldsymbol{R}$ satisfying the following set of properties:

[H1] $u\,\varphi^\sharp(u) \leq u\,\varphi(u) \leq u\,\tau(u)$ for all u;

[H2] φ is monotone decreasing;

[H3] φ is Lipschitz continuous;

[H4] $u\,\alpha(u) \leq 0$ for all u;

[H5] there exists $\varepsilon_0 > 0$ such that the Lipschitz constant η of the function $\varphi^{[2]}$ on the interval $I_0 := [-\varepsilon_0, \varepsilon_0]$ is less than 1. Moreover

$$\sup_{u \neq 0} \frac{\varphi^{[2]}(u)}{u} < 1. \tag{2.6}$$

The kinetic function describes the set of all *admissible nonclassical shock waves* to be used shortly in Section 3. In the rest of the present section we discuss each of the above assumptions and demonstrate that they are *"almost optimal."*

The condition [H1] means that the jump connecting u to $\varphi(u)$ is a nonclassical shock satisfying the entropy inequality (1.3) (cfr. Proposition 3.1 in Section 3). See Figure 2.1. The regularity properties [H2]-[H3] are basic, having here in mind the examples arising in the applications [**13, 17**].

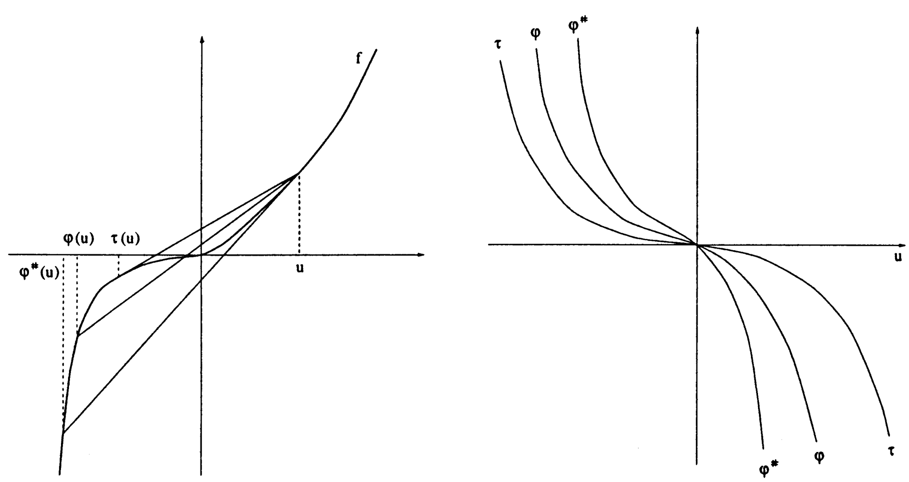

FIGURE 2.1

Interestingly the entropy inequality [H1] *implies* that

$$\alpha\big(\varphi(u)\big) \geq \alpha(u) \tag{2.7}$$

or equivalently

$$0 < \operatorname{sgn}(u)\, \varphi\big(\varphi(u)\big) \leq |u| \quad \text{for all } u \neq 0. \tag{2.8}$$

See (3.12)-(3.15). This will guarantee the solvability of every Riemann problem using *at most two waves.*

Our requirement [H4] is somewhat stronger than (2.7) and will ensure that the solution of the Riemann problem is *classical* as long as the left and the right state have the *same sign,* that is, lie in the same region of convexity. Note that the condition [H4] also forces φ to take its values on a smaller interval:

$$u\, \varphi^*(u) \leq u\, \varphi(u) \leq u\, \tau(u) \qquad \text{for all } u \neq 0. \tag{2.9}$$

Using (2.9) for u and also for $\varphi(u)$ evidently implies that φ satisfies (2.8).

Finally [H5] restricts the behavior of $\varphi^{[2]}$ (hence of φ) close to 0. It is worth pointing out that (2.6) is simply a strengthened version of (2.8) in which we are just excluding the case of equality. Moreover [H5] excludes only the case of equality in d) of Lemma 2.1 below.

For concreteness, in the case where φ is smooth, then [H5] is equivalent to saying $\varphi'(0) > -1$ and $|\varphi^{[2]}(u)| < |u|$ for all $u \neq 0$. If φ is only Lipschitz continuous [H5] is indeed more general than these two conditions.

Let us derive some properties for the above functions near the inflection point.

LEMMA 2.1. *Under the assumptions* (A1)-(A2) *made on on the flux f, the functions τ and φ^* satisfy*

a) $\tau'(0) \in (-1, 0)$.

b) $|\tau(u)| < |u|$ *for small* u.

c) $\big(\varphi^*\big)'(0) = -1$.

d) *If* (2.8) *holds and φ is differentiable at $u = 0$ then $\varphi'(0) \in [-1, 1]$.*

PROOF. By hypothesis we have $f(u) = Hu^{2p+1} + o(u^{2p+1})$. By the definition (2.1), $\tau = \tau(u)$ satisfies

$$\big(H(2p+1)\tau^{2p} + o(\tau^{2p})\big)(u - \tau) = Hu^{2p+1} + o(u^{2p+1}) - H\tau^{2p+1} - o(\tau^{2p+1}).$$

By a bifurcation analysis it follows that τ is differentiable at $u = 0$. So, if we expand $\tau(u) = Cu + o(u)$, then it follows

$$Hu^{2p+1}\Big(2pC^{2p+1} - (2p+1)C^{2p} + 1\Big) + o(u^{2p+1}) = 0,$$

hence

$$h(C) := 2pC^{2p+1} - (2p+1)C^{2p} + 1 = 0.$$

By studying the zeroes of the function h, it follows that $\tau'(0) = C \in (-1, 0)$. (To illustrate this, note that for $f(u) = u^3$ we have $\tau(u) = -u/2$ and $\tau'(0) = -1/2$.) Hence a) holds as well as b).

By our hypotheses on the flux and the definition (2.3) of φ^* it follows that

$$Hu^{2p} = H\big(\varphi^*(u)\big)^{2p} + o\left(u^{2p}\right) + o\left(\big(\varphi^*(u)\big)^{2p}\right). \tag{2.10}$$

Writing $\varphi^*(u) = C'u + o(u)$, (2.10) yields

$$Hu^{2p} = H(C'u)^{2p} + o(u^{2p}),$$

hence $(C')^{2p} = 1$ which, together with $u\varphi^*(u) < 0$, implies $C' = -1$ and c) is proven.

Finally, assume that φ is differentiable so $\varphi(u) = C''\,u + o(u)$. In view of (2.8)

$$\operatorname{sgn}(u)\,\varphi\big(\varphi(u)\big) = (C'')^2\,|u| + o(u) \leq |u|,$$

thus $C'' \in [-1, 1]$. Hence d) follows. □

3. General Nonclassical Riemann Solver

A nonclassical Riemann solver is now defined from the kinetic function φ given in Section 2. The classical entropy solutions (Oleinik [**27**], Liu [**26**]) are recovered with the trivial choice $\varphi = \tau$. We also prove that our construction is essentially the unique possible one as long as the fundamental entropy inequality (1.3) is enforced (Assumption [H1]).

It is well-known that the Oleinik entropy criterion [**27**] states that a shock connecting u_- to u_+ is (Oleinik)-admissible iff

$$\frac{f(w) - f(u_-)}{w - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \tag{3.1}$$

for all w between u_- and u_+. An easy consequence of (3.1) is that the chord connecting the points $\big(u_-, f(u_-)\big)$ and $\big(u_+, f(u_+)\big)$ does not cross the graph of the flux f.

Proposition 3.1. *Consider the conservation law* (1.1) *in the class of weak solutions satisfying the entropy inequality* (1.3) *for some strictly convex entropy U.*

Then for every u there exists a point $\varphi^\sharp(u)$ such that a shock wave connecting a left state u_- to a right state u_+ satisfies the entropy inequality iff

$$u_- \, \varphi^\sharp(u_-) \leq u_- \, u_+ \leq u_-^2. \tag{3.2}$$

Moreover we have

$$u_- \, \tau^{-1}(u_-) < u_- \, \varphi^\sharp(u_-). \tag{3.3}$$

Proof. Let $\lambda = \lambda(u_-, u_+)$ be the shock speed and consider the entropy dissipation

$$D(u_-, u_+) := -\lambda \, \big(U(u_+) - U(u_-)\big) + F(u_+) - F(u_-).$$

We easily calculate that

$$\begin{aligned} D(u_-, u_+) &= \int_{u_-}^{u_+} \big(f'(m) - \lambda\big) \, U'(m) \, dm \\ &= -\int_{u_-}^{u_+} \big(f(m) - f(u_-) - \lambda \, (m - u_-)\big) \, U''(m) \, dm. \end{aligned} \tag{3.4}$$

The Rankine-Hugoniot relation for (1.1) yields λ:

$$\lambda = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Suppose for definiteness that $u_- > 0$. When $u_+ > u_-$, since f is convex in the region $m \in (u_-, u_+)$ we have

$$f(m) - f(u_-) - \lambda \, (m - u_-) < 0 \tag{3.5}$$

and therefore $D(u_-, u_+) > 0$. Moreover, it follows from (3.4) and the concavity/convexity properties of f, that the entropy dissipation $u \mapsto D(u_-, u)$ achieves a minimum negative value at $u = \tau(u_-)$ and vanishes at exactly two points (see an argument in [**14**]):

$$\begin{aligned} &D(u_-, \cdot) \ \text{ is monotone decreasing for } u < \tau(u_-), \\ &D(u_-, \cdot) \ \text{ is monotone increasing for } u > \tau(u_-), \\ &D\big(u_-, \tau(u_-)\big) < 0, \\ &D(u_-, u_-) = 0, \\ &D\big(u_-, \varphi^\sharp(u_-)\big) = 0. \end{aligned} \tag{3.6}$$

Hence (3.2) follows. On the other hand when $u_+ \leq \tau^{-1}(u_-)$ it is geometrically clear that the part of the graph of f corresponding to $m \in (u_+, u_-)$ lies *above* the chord connecting the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$. This means that the opposite sign holds now in (3.5). But since $u_+ < u_-$ we again obtain that $D(u_-, u_+) > 0$. This implies that $u_-\, \tau^{-1}(u_-) < u_-\, \varphi^\sharp(u_-)$. □

The shocks satisfying

$$u_-\, \tau(u_-) \le u_-\, u_+ \le u_-^2 \tag{3.7}$$

are Oleinik-admissible and will be referred to as *classical shocks.* On the other hand for entropy admissible *nonclassical shocks*, (3.1) is violated, i.e.,

$$u_-\, \varphi^\sharp(u_-) \le u_-\, u_+ \le u_-\, \tau(u_-). \tag{3.8}$$

This establishes that the condition [H1] in Section 2 is in fact a *consequence* of the entropy inequality (1.3).

From now on we rely on the kinetic function φ selected in Section 2 and we solve the Riemann problem (1.1),

$$u(x,0) = u_0(x) = \begin{cases} u_l, & \text{for } x < 0, \\ u_r, & \text{for } x > 0, \end{cases} \tag{3.9}$$

where u_l and u_r are constants. We restrict attention to the case $u_l > 0$, the other case being completely similar. To define the *nonclassical Riemann solver* we distinguish between four cases:

(i) If $u_r \ge u_l$, the solution u is a (Lipschitz continuous) rarefaction wave connecting monotonically u_l to u_r.

(ii) If $u_r \in \big[\alpha(u_l), u_l\big)$, the solution is a classical shock wave connecting u_l to u_r.

(iii) If $u_r \in \big(\varphi(u_l), \alpha(u_l)\big)$, the solution contains a (slower) nonclassical shock connecting u_l to $\varphi(u_l)$ followed by a (faster) classical shock connecting to u_r.

(iv) If $u_r \le \varphi(u_l)$, the solution contains a nonclassical shock connecting u_l to $\varphi(u_l)$ followed by a rarefaction connecting to u_r.

For $u_l = 0$, the Riemann problem is a single rarefaction wave, connecting monotonically u_l to u_r. The function u will be called the *φ-admissible nonclassical solution* of the Riemann problem. Clearly different choices for φ yield different weak solutions u. This is natural as we already pointed out that limits given by (1.4) and similar models do depend on the parameter γ.

The above construction is essentially unique, as we show with the following two theorems.

THEOREM 3.2. *Consider the Riemann problem* (1.1)-(3.9) *in the class of piecewise smooth solutions satisfying the entropy inequality* (1.3) *for some strictly convex entropy U.*

Then either the Riemann problem admits a unique solution or else there exists a one-parameter family of solutions containing at most two (shock or rarefaction) waves.

Next for any nonclassical shock connecting some states u_- and u_+ with the speed λ, we impose the kinetic relation

$$D(u_-, u_+) = \begin{cases} \Phi^-(\lambda) & \text{if } u_+ < u_-, \\ \Phi^+(\lambda) & \text{if } u_+ > u_-, \end{cases} \tag{3.10}$$

where the kinetic functions are Lipschitz continuous and satisfy

$$\begin{aligned} &\Phi^\pm(0) = 0, \\ &\Phi^\pm \text{ is monotone decreasing,} \\ &\Phi^\pm(\lambda) \geq D^\pm(\lambda). \end{aligned} \tag{3.11}$$

In the latter condition the lower bound $D^\pm$ is the maximum negative value of the entropy dissipation

$$D^\pm(\lambda) := D\big(\tau^{-1}(u), u\big), \quad \lambda = f'(u) \qquad \text{for } \pm u \geq 0.$$

Then (3.10) *selects a unique nonclassical solution in the one-parameter family of solutions.*

Observe that given $\lambda > 0$ there are exactly one positive value and one negative value u such that $\lambda = f'(u)$. This property led us to define kinetic functions $\Phi^\pm$ for nonclassical shocks corresponding to decreasing and to increasing jumps.

Proof. The inequalities in Proposition 3.1 restrict the range of values taken by nonclassical shocks. First of all we show here that *at most two waves* can be combined together.

We now claim that

$$\varphi^\sharp\big(\varphi^\sharp(u_-)\big) = u_- \quad \text{for all } u_-. \tag{3.12}$$

Indeed we have by definition

$$D\big(u_-, \varphi^\sharp(u_-)\big) = 0, \quad u_- \neq \varphi^\sharp(u_-)$$

and

$$D\left(\varphi^\sharp(u_-), \varphi^\sharp\big(\varphi^\sharp(u_-)\big)\right) = 0, \quad \varphi^\sharp(u_-) \neq \varphi^\sharp\big(\varphi^\sharp(u_-)\big).$$

The conclusion follows immediately from the fact that the entropy dissipation has a single "nontrivial" zero; see (3.6).

We want to prove that the function $u \mapsto \varphi^\sharp(u)$ is decreasing. Again, by a bifurcation argument it follows that $\varphi^\sharp$ is differentiable. Now notice that $D(u, v) = -D(v, u)$ hence

$$\partial_u D\big(u_-, \varphi^\sharp(u_-)\big) = -\partial_v D\big(\varphi^\sharp(u_-), u_-\big). \tag{3.13}$$

From (3.6) we have that $\text{sgn}\left(\partial_v D\big(\varphi^\sharp(u_-), u_-\big)\right) = -\text{sgn}\left(\partial_v D\big(u_-, \varphi^\sharp(u_-)\big)\right)$, hence it follows that

$$\text{sgn}\left(\partial_u D\big(u_-, \varphi^\sharp(u_-)\big)\right) = \text{sgn}\left(\partial_v D\big(u_-, \varphi^\sharp(u_-)\big)\right). \tag{3.14}$$

Taking the total differential of the identity $D\big(u_-, \varphi^\sharp(u_-)\big) = 0$ with respect to u_- and using (3.14) gives $d\varphi^\sharp/du_- < 0$ for all u_-.

Consider a nonclassical shock connecting u_- to $\varphi(u_-)$. By hypothesis $u_-\varphi^\sharp(u_-) \leq u_-\varphi(u_-)$ hence by the monotonicity of $\varphi^\sharp$ and (3.12) it follows that $u_-\varphi^\sharp\big(\varphi(u_-)\big) \leq u_-\varphi^\sharp\big(\varphi^\sharp(u_-)\big) = (u_-)^2$. This prevents us to combine together more than two waves. Indeed since the speeds of the (rarefaction or shock) must be ordered (increasing) along a combination of waves, it is easily checked geometrically that the only possible wave patterns are:

1. a rarefaction wave,
2. a classical shock wave,
3. a nonclassical shock followed by a classical shock,
4. or else a nonclassical shock followed by a rarefaction.

Finally we discuss the selection of nonclassical shocks. It is enough to prove that for each fixed u_- there is a unique nonclassical connection to a state u_+ satisfying both the jump relation and the kinetic relation.

Suppose $u_- > 0$ is *fixed* and regard the entropy dissipation as a function of the speed λ:

$$\Psi(\lambda) = D\big(u_-, u_+(\lambda)\big), \quad \lambda = \frac{f\big(u_+(\lambda)\big) - f(u_-)}{u_+(\lambda) - u_-}.$$

It is not hard to see that

$$\begin{aligned}
&\Psi \text{ is increasing for } \lambda \in \big[f'(\tau(u_-)), f'(u_-)\big],\\
&\Psi\big(f'(\tau(u_-))\big) = D^+\big(f'(\tau(u_-))\big) \leq \Phi\big(f'(\tau(u_-))\big),\\
&\Psi\big(f'(u_-)\big) = 0 \geq \Phi\big(f'(u_-)\big).
\end{aligned}$$

In view of the assumptions made on Ψ it is clear that the equation

$$\Psi(\lambda) = \Phi(\lambda)$$

admits exactly one solution. This completes the proof that the nonclassical wave is unique. □

The property (3.12) implies that

$$0 < \operatorname{sgn}(u)\, \varphi\big(\varphi(u)\big) \leq |u| \quad \text{for all } u \neq 0, \tag{3.15}$$

which is (2.8).

We have already seen that a kinetic relation is sufficient to select a unique way of solving the Riemann Problem and the solution was described earlier. Now we want to prove that this is essentially the *unique expression* a Riemann Solver can have.

More precisely, assume the following are given:

- a set $\mathcal{A}$ of admissible waves satisfying the entropy inequality (1.3) for a fixed, strictly convex pair (U, F);
- for every pair of states (u_l, u_r), a way of solving the associated Riemann problem, using only admissible waves in $\mathcal{A}$. Denote by $\mathcal{R}(u_l, u_r)$ the Riemann solution;
- an $\mathbf{L}^1$-continuous semigroup of solution for (1.1)-(1.2), compatible with the above Riemann solutions. (Note that in [**5**] it is proven that, if such a semigroup exists, then there is a unique way of solving the Riemann problem associated with any pair of states u_l, u_r.)

Any collection of $\{\mathcal{R}(u_l, u_r); u_l, u_r \in \boldsymbol{R}\}$ satisfying the above assumptions will be called here a *basic $\mathcal{A}$-admissible Riemann Solver*. We are going to prove that

$\mathcal{R}(u_l, u_r)$ coincides with (i)-(iv) for some choice of the function φ. This completely justifies our study of the Nonclassical Riemann Solver made in the present paper.

The admissibility criterion imposed by $\mathcal{A}$ could be recovered by the analysis of the limits of some regularizations of (1.1) like (1.4), or by a kinetic relation as in this paper (see also [**13,14**]). But it could also be given *a priori* by some physical or mathematical argument.

THEOREM 3.3. *Every basic $\mathcal{A}$-admissible Riemann Solver coincides with a Nonclassical Riemann Solver for a suitable choice of the function φ.*

PROOF. In the previous discussion it was observed that there are only four possible wave patterns, namely a single shock, a single rarefaction wave or else a nonclassical shock followed by either a shock or a rarefaction. Without loss of generality, assume $u_l > 0$. Any state $u_r > u_l$ can be connected to the right of u_l only by a rarefaction wave, hence $\mathcal{R}(u_l, u_r)$ must coincide with this rarefaction.

In the following we shall consider all the shocks connecting u_l to $\tau(u_l)$ to be nonclassical. Since u_l can be connected by a single classical wave only to points $u_r > \tau(u_l)$, then u_l must be connected by a nonclassical shock to at least one right state $u_r \leq \tau(u_l)$.

Let us see that this right point is unique. By contradiction, assume there exist points $\tilde{u} < \bar{u} < 0$ such that u_l can be connected to both of them by a nonclassical shock. By hypothesis $\bar{u}$ and $\tilde{u}$ are connected by an (admissible) rarefaction. Hence the Riemann problem $(u_l, \tilde{u})$ can be solved either by a single nonclassical shock or by a nonclassical shock to $\bar{u}$ followed by a rarefaction to $\tilde{u}$. This contradicts the uniqueness of the Riemann solver $\mathcal{R}(u_l, u_r)$. It follows that u_l can be connected with a nonclassical shock to exactly one right state, call it $\varphi(u_l)$.

By uniqueness, this implies immediately that all the states $u_r < \varphi(u_l)$ are connected to the right of u_l by the nonclassical shock to $\varphi(u_l)$ followed by a rarefaction to u_r.

Introduce now the point $\alpha(u_l)$ as in (2.5). The points in the interval $[\alpha(u_l), u_l)$ can not be reached neither by a rarefaction, nor by a wave pattern containing a (single) nonclassical shock. Hence they must be reached by a classical shock. Now, if $\varphi(u_l) = \alpha(u_l) = \tau(u_l)$ then we are done and the Riemann solution $\mathcal{R}(u_l, u_r)$ coincides with the Liu solution. Otherwise $\varphi(u_l) < \tau(u_l) < \alpha(u_l)$ and the points u_r in the interval $\big[\varphi(u_l), \tau(u_l)\big)$ are reached by the nonclassical shock followed by a classical shock, since this is the only way to connect u_l and u_r. It remains to cover $\big[\tau(u_l), \alpha(u_l)\big)$. The points in this interval can be reached either by a single classical shock or by the nonclassical shock followed by a classical one. So, let $\bar{u}_l := \sup\{u_r \geq \varphi(u_l)$ that are connected to the left of u_l by the nonclassical shock followed by a classical one $\}$. Then $\bar{u}_l \leq \alpha(u_l)$ and every $u > \bar{u}_l$ is connected to left of u_l by a single classical shock. By the $\mathbf{L}^1$-continuity property and an analysis of the wave-speeds it follows that the solution of the Riemann problem $(u_l, \bar{u}_l)$ with a nonclassical shock followed by a classical shock and the one with a single classical shock must coincide, hence $\bar{u}_l = \alpha(u_l)$. It follows that $\mathcal{R}(u_l, u_r)$ coincides with the nonclassical Riemann solver for this choice of φ. □

4. New Total Variation Functional

A classical way to prove convergence of approximate schemes for conservation laws is to give uniform bounds on the $\mathbf{L}^\infty$ and $\mathbf{BV}$ norms of the approximate solutions and then pass to the limit by using Helly's compactness theorem. Unfortunately, in contrast to the classical case, the total variation of the approximate solutions can increase across interactions due to the creation or interaction of nonclassical shocks. Hence a careful analysis is needed, of how the strengths of waves change across interactions. In the classical case of systems [**12**] the so-called interaction potential Q is used to compensate a (possible) increase of the total variation. In our case, however, it appears that if two fronts of strength σ and σ' interact at time t (here strength means the size of the jump in the discontinuity) then there are cases in which the variation of the total variation is linear in the strength of the incoming waves, i.e. $\Delta \mathbf{TV}(t) \sim C\big(|\sigma| + |\sigma'|\big)$. This implies that we cannot use the potential Q to control the increase in the total variation since Q is a quadratic functional (see [**12**]).

Our approach is to construct a modified total variation functional which decreases in time along suitable wave-front tracking approximations of (1.1)-(1.2), and which is equivalent to the usual total variation, i.e. we are looking for a functional $\mathbf{V}$ such that for every piecewise constant approximate solution $v(t,x)$ constructed by front-tracking we have $\Delta \mathbf{V}\big(v(t,\cdot)\big) \le 0$ for every $t > 0$ and there exist positive constants C_1, C_2, depending only on the $\mathbf{L}^\infty$ and $\mathbf{BV}$ norms of the initial data u_0, such that $C_1\, \mathbf{V}(v) \le \mathbf{TV}(v) \le C_2\, \mathbf{V}(v)$ (see [**3**]). The definition of $\mathbf{V}$ can be regarded as a generalization of the standard distance $|u_r - u_l|$.

Now, let $u : \boldsymbol{R} \mapsto \boldsymbol{R}$ be a piecewise constant function and let x_α, $\alpha = 1, \dots, N$, be the points of discontinuity of u. Define

$$\mathbf{V}(u) := \sum_{\alpha=1}^{N} \sigma\big(u(x_\alpha-), u(x_\alpha+)\big), \tag{4.1}$$

where $\sigma(u_l, u_r)$ measures the strength of the wave connecting the left state u_l to the right state u_r. Notice that if $\sigma(u_l, u_r) = |u_r - u_l|$, then $\mathbf{V}(u) = \mathbf{TV}(u)$. So, a new definition of the strength $\sigma(u_l, u_r)$ is necessary. More precisely, we set

$$\sigma(u_l, u_r) := \begin{cases} \big(\psi(u_r) - \psi(u_l)\big)\, \mathrm{sgn}\,(u_r - u_l)\, \mathrm{sgn}\,(u_l) & \text{if } \big(u_r - \varphi(u_l)\big)\, \mathrm{sgn}\,(u_l) \ge 0, \\ \psi(u_r) + \psi(u_l) - 2\,\psi\big(\varphi(u_l)\big) & \text{if } \big(u_r - \varphi(u_l)\big)\, \mathrm{sgn}\,(u_l) \le 0. \end{cases} \tag{4.2}$$

where $\psi : \boldsymbol{R} \mapsto \boldsymbol{R}$ is a continuous function that is increasing (resp. decreasing) for u positive (resp. negative). It is also assumed that $\psi(0) = 0$.

The wave strength σ depends on the kinetic function φ as well as on the function ψ to be determined in Section 5. Observe that the function $u_r \mapsto \sigma(u_l, u_r)$ is a piecewise linear function in term of $\psi(u_r)$ resembling the letter W. It achieves a *local minimum value* at $u_r = u_l$ and at $u_r = \varphi(u_l)$, the latter corresponding of course to the nonclassical shock. Therefore the strength of the nonclassical shock is counted *less* than what it would be with the standard total variation. This choice is made to *compensate for the increase* of the standard total variation that arises in certain wave interactions involving nonclassical shocks.

Let u_ν be the sequence of piecewise constant solutions of (1.1)-(1.2) constructed via wave-front tracking from an approximation of the initial data u_0, following [**3**]. We replace the data u_0 with a piecewise constant approximation $u_\nu(0)$ such that

$$u_\nu(0) \to u_0 \quad \text{in the } \mathbf{L}^1 \text{ norm}, \qquad \mathbf{TV}(u_\nu(0)) \to \mathbf{TV}(u_0). \tag{4.3}$$

Based on the nonclassical Riemann solver of Section 3, we approximately solve the corresponding Cauchy problem for small time. Let δ_ν be a sequence of positive numbers converging to zero. For each ν, the approximate solution u_ν is constructed as follows. Solve approximatively the Riemann problem at each discontinuity point of u_ν. This is obtained by approximating the solution given by the nonclassical Riemann solver: every shock or nonclassical shock travels with the correct shock speed, while the rarefaction fans are approximated by rarefaction fronts. More precisely, every rarefaction wave connecting the states u_l and u_r, say, with $\sigma(u_l, u_r) > \delta_\nu$ is approximated by a finite number of small jumps traveling with speed equal to the right characteristic speed and with strength less than or equal to δ_ν.

When two wave-front meet, we again use the nonclassical approximate Riemann solver and continue inductively in time. The main aim is to estimate the total variation, that is to prove that there exists a positive constant C such that

$$\mathbf{TV}\big(u_\nu(t)\big) \le C, \quad t \ge 0 \tag{4.4}$$

uniformly in ν.

From now on we assume that a kinetic function satisfying [H1]-[H5] is fixed. First of all notice that under these hypotheses the interaction patterns for all couples of waves are analogous to those considered and listed in Section 2 of [**3**]. We shall rely on this classification in the rest of the present section. To prove that u_ν is well-defined, it is sufficient to show that the above construction can be carried on for all positive times.

Proposition 4.1. *Assume that the function*

$$u \mapsto \operatorname{sgn}(u)\big(\psi(u) - \psi\big(\varphi(u)\big)\big)$$

is monotone increasing. Then the approximate solutions $u_\nu(t)$ are well-defined for all times $t \ge 0$ and satisfy

$$\|u_\nu(t)\|_{\mathbf{L}^\infty(\boldsymbol{R})} \le \max\big\{c, |\varphi(c)|\big\}, \qquad c := \|u_\nu(0)\|_{\mathbf{L}^\infty(\boldsymbol{R})}. \tag{4.5}$$

Proof. As in [**3**] it is sufficient to prove that the total number of waves does not increase in time, so it can be bounded uniformly in t (for fixed ν). Since only two waves may leave after the interaction of two waves, it is sufficient to prove that the rarefactions do not increase their strength across interaction. Denote by σ the strength of rarefactions and $\Delta\sigma$ the change across the interaction. Referring to the cases of wave interactions listed in [**3**], we have (recalling that we assume $u_l > 0$):

Case 1. Trivial case: $\Delta\sigma < 0$.

Case 4. The variation of the strength across the interaction is computed by

$$\begin{aligned}\Delta\sigma &= \big(\psi(u_r) - \psi\big(\varphi(u_l)\big)\big) - \big(\psi(u_m) - \psi(u_l)\big)\\ &\le \psi\big(\varphi(u_m)\big) - \psi(u_m) - \big(\psi\big(\varphi(u_l)\big) - \psi(u_l)\big) \le 0.\end{aligned}$$

Case 6. This is a limiting case of Case 4.

$$\Delta\sigma = \psi\big(\varphi(u_m)\big) - \psi(u_m) - \psi\big(\varphi(u_l)\big) + \psi(u_l) \le 0.$$

Case 17. Now the variation is given by

$$\Delta\sigma = \big(\psi\big(\varphi(u_l)\big) - \psi(u_r)\big) - \big(\psi(u_m) - \psi(u_r)\big) < 0.$$

So the approximate solutions are well-defined for all positive times.

We now prove (4.5). It is obvious that the only interactions that can increase the $\mathbf{L}^\infty$-norm are those in which a nonclassical shock is involved. Let $\mathcal{R}(u)$ be the range of a piecewise constant function u. For every approximate solution u_ν, across an interaction at time t we have

$$\mathcal{R}\big(u_\nu(t+,\cdot)\big) \subseteq \mathcal{R}\big(u_\nu(t-,\cdot)\big) \cup \mathcal{R}\Big(\varphi\big(u_\nu(t-,\cdot)\big)\Big), \tag{4.6}$$

as follows from the definition of the Riemann solver in Section 3. It is clear that (4.5) holds for $t = 0+$. Now fix ν and assume that for a positive time t we have

$$M(t) := \|u_\nu(t,\cdot)\|_{\mathbf{L}^\infty} > \|u_\nu(0,\cdot)\|_{\mathbf{L}^\infty}.$$

Then by (4.6), there exists $\tilde u \in \mathcal{R}\big(u_\nu(0,\cdot)\big)$ and a positive integer n such that

$$M(t) = \big|\varphi^{[n]}(\tilde u)\big|.$$

Recall that $\big|\varphi^{[2]}(u)\big| \le |u|$. Hence n must be odd, otherwise by induction

$$M(t) = \big|\varphi^{[n]}(\tilde u)\big| \le |\tilde u| \le \|u_\nu(0,\cdot)\|_{\mathbf{L}^\infty},$$

which is a contradiction. So $n = 2q+1$ and again by induction it follows that

$$M(t) = \Big|\varphi^{[2q]}\big(\varphi(\tilde u)\big)\Big| \le \big|\varphi(\tilde u)\big| \le \Big|\varphi\big(\|u_\nu(0,\cdot)\|_{\mathbf{L}^\infty}\big)\Big|.$$

Hence (4.5) follows. This completes the proof of Proposition 4.1. □

Assuming now that the approximate initial data satisfy

$$\|u_\nu(0)\|_{\mathbf{L}^\infty(\boldsymbol{R})} \le C \ \ \|u_0\|_{\mathbf{L}^\infty(\boldsymbol{R})}, \tag{4.7}$$

we conclude from (4.5) that

$$\|u_\nu(t)\|_{\mathbf{L}^\infty(\boldsymbol{R})} \le C' \quad \text{for all } t \ge 0, \tag{4.8}$$

uniformly in ν.

We next derive a uniform **BV** bound or, more precisely, we prove that **V** decreases along approximate solutions.

Proposition 4.2. *Assume that the function*

$$u \mapsto \operatorname{sgn}(u)\Big(\psi(u) - \psi\big(\varphi(u)\big)\Big) \quad \textit{is monotone increasing.} \tag{4.9}$$

Then for the approximate solutions,

$$t \mapsto \mathbf{V}\big(u_\nu(t)\big) \quad \textit{is monotone decreasing.} \tag{4.10}$$

PROOF. The function $t \mapsto \mathbf{V}\big(u_\nu(t)\big)$ is piecewise constant with discontinuities located only at interaction times. Hence it suffices to show that $\mathbf{V}$ decreases across every collision. Assume that the three states u_l, u_m and u_r, are separated by two interacting wave fronts of strength σ_1^- and σ_2^-, each being generated by a (nonclassical) Riemann solution. Each front can be either a classical, nonclassical shock or a rarefaction wave.

The complete list of interaction patterns can be found in Section 2 of [**3**], but Cases 5 and 12 therein never occur because of our assumption [H2]. In [**3**] a case by case analysis was developed. Here thanks to the general definition (4.1)-(4.2) we have some simplifications.

Any outgoing pattern is made of at most two waves, with strength σ_1^+ and (possibly) σ_2^+. Hence the variation of $\mathbf{V}$ across the interaction is given by $\Delta\mathbf{V} = (\sigma_1^+ + \sigma_2^+) - (\sigma_1^- + \sigma_2^-) := \Sigma^+ - \Sigma^-$.

The function σ introduced in (4.2) satisfies the following key properties:

(i) σ is *additive on ordered waves*, in the sense that if u_1, u_2, u_3 are three states such that $u_1 < u_2 < u_3$ and such that $\operatorname{sgn}(u_i)\big(u_j - \varphi(u_i)\big) \geq 0$ for all $i > j$, then
$$\sigma(u_1, u_3) = \sigma(u_1, u_2) + \sigma(u_2, u_3),$$
$$\sigma(u_3, u_1) = \sigma(u_3, u_2) + \sigma(u_2, u_1);$$

(ii) if $\operatorname{sgn}(u_1) = \operatorname{sgn}(u_2)$ then $\sigma(u_1, u_2) = \sigma(u_2, u_1)$;

(iii) for every outgoing pattern we have $\Sigma^+ = \sigma(u_l, u_r)$.

These properties can be checked from the definition of σ. In particular (iii) implies that $\Delta\mathbf{V} \leq 0$ iff $\Sigma^- \geq \sigma(u_l, u_r)$.

The interaction cases can be split in four families.

CASES 8, 9, 10, 11, 15, 16, 17. The states before the interaction are ordered. Hence by (i) it follows that $\Sigma^- = \sigma(u_l, u_r)$ and so $\Delta\mathbf{V} = 0$.

CASES 1, 2, 3, 4, 7. The states before the interaction are not ordered. Then a cancellation takes place, but the canceled wave lives on one region of convexity. Using (i)-(ii) it follows that $\Sigma^- > \sigma(u_l, u_r)$, hence $\Delta\mathbf{V} < 0$.

CASES 13, 14, 18, 19. As in the previous case but now the canceled wave cross the state 0. Indeed all of them are special cases of 14. For the latter, it easy to see that $\Sigma^+ = \psi(u_l) - \psi(u_r)$ and $\Sigma^- = \big(\psi(u_l) - \psi(u_m)\big) - \big(\psi(u_m) - \psi(u_r)\big)$, hence $\Delta\mathbf{V} = 0$.

It remains to check only Case 6.

CASE 6. This is the only case which requires condition (4.9). Indeed, $u_l(u_l - u_m) > 0$ and it follows that
$$\begin{aligned}\Delta\mathbf{V} &= \psi(u_l) - \psi\big(\varphi(u_l)\big) + \psi\big(\varphi(u_m)\big) - \psi\big(\varphi(u_l)\big) + \\ &\quad - \big(\psi(u_m) - \psi(u_l)\big) - \big(\psi(u_m) - \psi\big(\varphi(u_m)\big)\big) \\ &= 2\Big[\big(\psi(u_l) - \psi\big(\varphi(u_l)\big)\big) - \big(\psi(u_m) - \psi\big(\varphi(u_m)\big)\big)\Big] \leq 0.\end{aligned}$$

This completes the proof. □

The existence of a function ψ satisfying the condition (4.9) will be established in Section 5. The equivalence between $\mathbf{TV}$ and $\mathbf{V}$ will be proved there, too. Now we are ready to conclude with the main result of the present paper.

Theorem 4.3. *Consider the conservation law* (1.1) *together with the nonclassical Riemann solver characterized by the function* $\varphi : \boldsymbol{R} \to \boldsymbol{R}$. *Suppose that* φ *satisfies the assumptions* [H1]–[H5] *listed in Section* 2.

Given an initial data u_0 *with bounded total variation, there exists a positive constant* $\tilde{C}$ *depending only on* φ *and the* $\mathbf{L}^\infty$*-norm of* u_0 *such that the approximate solutions* $u_\nu(t)$ *(constructed by wave-front tracking) satisfy*

$$\mathbf{TV}\big(u_\nu(t)\big) \le \tilde{C}\, \mathbf{TV}(u_0) \tag{4.11}$$

for all times $t \ge 0$.

A subsequence of u_ν *converges in the* $\mathbf{L}^1$ *norm toward a weak solution of the conservation law* (1.1)-(1.2) *which satisfies the entropy inequality* (1.3).

Proof. By (4.8)-(4.10), the approximate solutions constructed above have uniformly bounded $\mathbf{L}^\infty$-norm and total variation. We can apply Helly's theorem to find a (sub)sequence which converges in ${\mathbf{L}^1}_{loc}$ to a function u. Since the modified and the usual strengths of waves are equivalent (see (5.10)) u is a nonclassical weak solution of (1.1)-(1.2) satisfying also the entropy inequality (1.3). □

5. Construction of the Function ψ

In this section we prove the existence of a function ψ satisfying (4.9) needed in Propositions 4.1 and 4.2. This will be accomplished by a fixed-point argument in a suitable function space X defined below.

Denote by $\mathrm{Lip}_I(\psi)$ the Lipschitz constant of a function ψ defined on some interval I. Let $M > 0$ be a constant greater than the $\mathbf{L}^\infty$-norm of u_0 and define $J_M := [-M, M] \cup \big[\varphi(M), \varphi(-M)\big]$. Finally, let L_M be the Lipschitz constant of φ in the set J_M. Introduce the space

$$X := \Big\{\psi \in C\big(J_M; \boldsymbol{R}\big) \;:\; \psi(0) = 0,\; \|\psi\|_X < \infty\Big\},$$

endowed with the norm

$$\|\psi\|_X := \sup_{u \neq 0} \left| \frac{\psi(u)}{u - \varphi(u)} \right|.$$

Then define the subset $Y \subset X$ by

$$Y := \left\{ \psi \in X : \quad \psi \text{ is } \begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix} \text{ for } u \gtrless 0;\;\; \mathrm{Lip}_{I_0}(\psi) \le K \right\},$$

where $K \ge (1 + L_M)/(1 - \eta)$ is a fixed constant and η is the Lipschitz constant introduced in [H5].

The reason why we consider J_M instead of $[-M, M]$ is that we need a φ-invariant set and actually $\varphi(J_M) \subseteq J_M$ while in general this is not true for $[-M, M]$.

Lemma 5.1. $\big(X, \|\cdot\|_X\big)$ *is a Banach space and* Y *is a closed subset.*

Proof. It is clear that X is a normed space. Let us see that it is complete. Let $\psi_n \in X$, $n = 1, 2, \dots$ be a Cauchy sequence in the norm $\|\cdot\|_X$. By definition, for every $\varepsilon > 0$, there exists $\bar{n}$ such that for all $m, n \ge \bar{n}$ we have

$$\big|\psi_n(u) - \psi_m(u)\big| \le |u - \varphi(u)|\, \varepsilon \tag{5.1}$$

for all $u \neq 0$, but also for $u = 0$. Hence the sequence ψ_n is also Cauchy in the space $C\big(J_M; \boldsymbol{R}\big)$ with the sup-norm, and so it converges to a continuous function ψ. Moreover, by passing pointwise to the limit, we see that $\psi(0) = 0$. Finally by letting $m \to \infty$ in (5.1) we see that the convergence holds actually in the space X.

Finally let us see that Y is closed. Take $\psi_n \to \psi$ in X with $\psi_n \in Y$ for all n. First of all, by passing pointwise to the limit, it follows that ψ satisfies the monotonicity properties. By hypothesis we have

$$\left|\frac{\psi_n(u) - \psi_n(v)}{u - v}\right| \leq K \tag{5.2}$$

for all n and all $u, v \in I_0$ with $u \neq v$. Since ψ_n converges to ψ pointwise, by passing to the limit in (5.2) we get $\mathrm{Lip}_{I_0}(\psi) \leq K$, hence $\psi \in Y$ and Y is closed. □

Now define the map $T : X \mapsto X$ by the relation

$$\big(T\psi\big)(u) := \psi\big(\varphi(u)\big) + |u|, \qquad u \in \boldsymbol{R}. \tag{5.3}$$

Theorem 5.2. *T maps X into X and is a contraction.*

Proof. Let $\psi \in X$ be fixed. It is clear that $\big(T\psi\big)(0) = 0$. Let us see that T is a contraction. For all $\psi, \bar\psi \in X$ and $u \neq 0$ we have

$$\left|\frac{T\psi(u) - T\bar\psi(u)}{u - \varphi(u)}\right| = \left|\frac{\varphi(u) - \varphi\big(\varphi(u)\big)}{u - \varphi(u)}\right| \left|\frac{\psi\big(\varphi(u)\big) - \bar\psi\big(\varphi(u)\big)}{\varphi(u) - \varphi\big(\varphi(u)\big)}\right|,$$

hence

$$\big\|T\psi - T\bar\psi\big\|_X \leq \sup_{u \neq 0}\left|\frac{\varphi(u) - \varphi\big(\varphi(u)\big)}{u - \varphi(u)}\right| \cdot \big\|\psi - \bar\psi\big\|_X,$$

and by taking $\bar\psi \equiv 0$ in this last inequality, it follows that $\|T\psi\|_X < \infty$ and T maps X into itself. Now, it is easy to see (i.e. geometrically) that

$$\left|\frac{\varphi(u) - \varphi\big(\varphi(u)\big)}{u - \varphi(u)}\right| < 1, \quad u \neq 0, \tag{5.4}$$

or even more

$$\left|\frac{\varphi(u) - \varphi\big(\varphi(u)\big)}{u - \varphi(u)}\right| = \frac{\varphi\big(\varphi(u)\big) - \varphi(u)}{u - \varphi(u)} = 1 - \frac{1 - \varphi\big(\varphi(u)\big)/u}{1 - \varphi(u)/u}, \tag{5.5}$$

for all $u \neq 0$. By (2.6) and (5.5) it follows that

$$\sup_{u \neq 0}\left|\frac{\varphi(u) - \varphi\big(\varphi(u)\big)}{u - \varphi(u)}\right| < 1.$$

Hence T is a contraction. □

By the contraction principle the map T has a unique fixed point in X. Denote it by $\psi : J_M \to \boldsymbol{R}$. By construction the function $\psi(u) - \psi\big(\varphi(u)\big)$ is monotone increasing (resp. monotone decreasing) for u positive (resp. negative). More precisely in view of (5.3) and $T(\psi) = \psi$, we have

$$\begin{aligned} \psi(u) - \psi\big(\varphi(u)\big) &= u, \qquad \text{for } u > 0, \\ \psi(u) - \psi\big(\varphi(u)\big) &= -u, \qquad \text{for } u < 0. \end{aligned} \tag{5.6}$$

Therefore the assumption of Propositions 4.1 and 4.2 holds together with the uniform $\mathbf{L}^\infty$ bound (4.8) and the bound for the new functional $\mathbf{V}\big(u_\nu(t)\big)$, i.e. (4.10).

At this point it seemed we could not say anything about the regularity of ψ close to 0. And we will need ψ to be Lipschitz continuous on I_0 to prove equivalence between **TV** and **V**.

Let us consider the second iterate of $T : X \mapsto X$.

Lemma 5.3. *$T^{[2]} : X \mapsto X$ and is a contraction. Moreover $T^{[2]}$ maps Y into itself.*

Proof. The first assertion is trivial. Take $\psi_0 \in Y$. By our definition and [H2] it follows that $T^{[2]}\psi$ is increasing (resp. decreasing) for u positive (resp. negative). Iterating (5.3), we get that $T^{[2]}$ is defined by

$$T^{[2]}\psi(u) = \psi\big(\varphi^{[2]}(u)\big) + \operatorname{sgn}(u)\big(u - \varphi(u)\big), \quad u \in \boldsymbol{R}. \tag{5.7}$$

The relation (5.7) together with $\varphi^{[2]}\big(I_0\big) \subset I_0$, imply

$$\mathrm{Lip}_{I_0}\big(T^{[2]}\psi_0\big) \leq 1 + \mathrm{Lip}_{I_0}(\varphi) + \mathrm{Lip}_{\varphi^{[2]}(I_0)}(\psi_0) \cdot \mathrm{Lip}_{I_0}\big(\varphi^{[2]}\big) \leq 1 + L_M + K\eta \leq K, \tag{5.8}$$

by the choice of K. Hence $T^{[2]}\psi_0 \in Y$. □

Now, $T^{[2]}$ is a contraction on X, hence it admits a unique fixed point. Since $T^{[2]}$ maps Y into Y and Y is closed, it follows that this fixed point belongs to Y. Every fixed point of T is also a fixed point of $T^{[2]}$, hence $T^{[2]}$ and T have the *same* fixed point. Thus the fixed point of T belongs to Y and so it is Lipschitz continuous on a neighborhood of 0 and satisfies the monotonicity properties.

Remark 5.4. The operator T does not map Y into Y. Nevertheless, since $T^{[2]}$ maps Y into Y and φ is Lipschitz, it follows that, for every $\psi \in X$, also the Lipschitz constant of $T^{[2n+1]}\psi$ cannot grow too much as $n \to \infty$.

We point out that if ψ_0 were a fixed point of $T^{[2]}$ only, then we could not recover the relations (5.6). So we need ψ to be a fixed point of *both* T and $T^{[2]}$.

Finally we prove that the functional $\mathbf{V}$ is equivalent to the usual total variation.

Lemma 5.5. *Given $M > 0$, there exist positive constants C_1, C_2 such that*

$$C_1\, \mathbf{V}(u) \leq \mathbf{TV}(u) \leq C_2\, \mathbf{V}(u) \tag{5.9}$$

for any piecewise constant function u with $\|u\|_{\mathbf{L}^\infty} \leq M$.

Proof. It is sufficient to prove that

$$C_1\, \sigma(u_l, u_r) \leq |u_r - u_l| \leq C_2\, \sigma(u_l, u_r), \tag{5.10}$$

for all u_l, u_r with $|u_l|, |u_r| \leq M$. Without loss of generality we can assume $u_l > 0$. For all $u_r > 0$, by the monotonicity of ψ and φ we have

$$\big|\psi(u_r) - \psi(u_l)\big| = \big|\psi\big(\varphi(u_r)\big) - \psi\big(\varphi(u_l)\big)\big| + |u_r - u_l| \geq |u_r - u_l|,$$

hence

$$\left|\frac{u_r - u_l}{\sigma(u_l, u_r)}\right| = \left|\frac{u_r - u_l}{\psi(u_r) - \psi(u_l)}\right| \leq 1. \tag{5.11}$$

If, instead, $u_r < 0$ we have

$$\left|\frac{u_r - u_l}{\sigma(u_l, u_r)}\right| \leq \left|\frac{u_l - \varphi(u_l)}{\psi(u_l) - \psi\big(\varphi(u_l)\big)}\right| = \left|\frac{u_l - \varphi(u_l)}{u_l}\right| \leq (1 + L_M) =: C_2, \tag{5.12}$$

since $|\varphi(u)| \leq L_M|u|$ for all $|u| \leq M$.

Next we prove that ψ is Lipschitz continuous on $I := [-M, M]$ (hence also on J_M). First of all, we can assume $M > \varepsilon_0$. Since ψ is a fixed point of $T^{[2]}$ it follows that

$$\psi(u) = \psi\big(\varphi^{[2]}(u)\big) + \operatorname{sgn}(u)\big(u - \varphi(u)\big),$$

which implies

$$\mathrm{Lip}_I(\psi) \leq \mathrm{Lip}_I\big(\varphi^{[2]}\big) \cdot \mathrm{Lip}_{\varphi^{[2]}(I)}(\psi) + \mathrm{Lip}_I(\varphi) + 1. \tag{5.13}$$

Note that by (5.13) the Lipschitz constant of ψ on the interval $[-M, M]$ can be controlled by that on the (strictly) smaller interval $\big[\varphi^{[2]}(-M), \varphi^{[2]}(M)\big]$.

More precisely, even though $\mathrm{Lip}_I(\varphi^{[2]})$ may be greater than 1, it happens that the function $\varphi^{[2]}$ has only one fixed point on $(-\infty, +\infty)$, namely $u = 0$. Hence, having fixed $M > \varepsilon_0$, there exists an integer p such that the iterates $\varphi^{[2p]}(u) \in [-\varepsilon_0, \varepsilon_0]$ for all $|u| \in [\varepsilon_0, M]$, where p depends only on ε_0 and M. By iterating (5.13), this implies that

$$\mathrm{Lip}_I(\psi) \leq K_1 \cdot \mathrm{Lip}_{I_0}(\psi) + K_2, \tag{5.14}$$

where K_1, K_2 are constants depending only on M, ε_0 and the Lipschitz constant of φ. Since $\mathrm{Lip}_{I_0}(\psi) \leq K$, (5.14) says that ψ is Lipschitzian.

Then the conclusion holds with

$$C_1 := \Big(K_1 \cdot \mathrm{Lip}_{I_0}(\psi) + K_2\Big)^{-1}.$$

□

6. Remarks on the Construction

The present result is stronger than the one presented in [**3**]. On one hand we consider a more general flux-function; moreover we drop both the assumption that the solution should coincide with the classical one in a small neighborhood of 0 (see (H2) in [**3**]), and the assumption that α should be decreasing. Concerning this last hypothesis, notice that in the cubic-flux case with the choice $\psi(u) = |u|$ (as we considered in [**3**]) we have

$$\operatorname{sgn}(u)\big(\psi(u) - \psi\big(\varphi(u)\big)\big) = -\alpha(u).$$

So, α is decreasing iff (4.9) holds. This means that the monotonicity request on α comes out by the *particular* choice $\psi(u) = |u|$. The assumption can be drop just by carefully choosing the function ψ.

The choice (4.2) appears to be a sort of nonlinear generalization of the definition of $\sigma(u_l, u_r)$ given in [**3**], the latter corresponding to the case $\psi(u) = |u|$. Unfortunately this last choice does not work in the general case mainly because the flux-function f is not symmetric.

The case $\varphi \equiv \tau$ corresponds to the classical case in which the Oleinik-Liu solutions [**26,27**] are selected. Notice that in view of Lemma 2.1 hypotheses [H1]-[H5] are automatically satisfied. So, we expects that a sufficient condition for the nonclassical solution to be in **BV** is that φ and τ have the same behavior near $u = 0$, roughly speaking $\varphi'(0) = \tau'(0)$. In fact, we could prove existence in **BV** under the weaker hypothesis [H5].

The function $|u|$ on the right-hand side of (5.3) can be replaced by a more general Lipschitz continuous function $G(u)$, i.e. we can look for a function ψ satisfying the equation

$$\psi(u) = \psi\big(\varphi(u)\big) + G(u),$$

with G increasing (resp. decreasing) for u positive (resp. negative), and behaving like $|u|$ for u close to 0. The corresponding function ψ obtained by a fixed-point argument similar to the one presented in the previous section, depends on G and, in general, is *nonlinear*. Indeed, if one tries to use a *piecewise linear* ψ of the form

$$\psi(u) := \begin{cases} \lambda^+ u, & \text{for } u > 0, \\ \lambda^- u, & \text{for } u < 0, \end{cases} \tag{6.1}$$

for some positive λ^+ and negative λ^-, then the condition $\sigma \geq 0$ (more precisely $\operatorname{sgn}(u)\big(\psi(u) - \psi\big(\varphi(u)\big)\big) \geq 0$) implies that $m := \lambda^-/\lambda^+$ must satisfy

$$\sup_{v<0} \left| \frac{\varphi(v)}{v} \right| =: A^- \leq |m| \leq A^+ := \inf_{u>0} \left| \frac{u}{\varphi(u)} \right|.$$

So a necessary condition is $A^- \leq A^+$. If $\varphi(u) = -\alpha u + o(u)$, then the previous condition is violated as long as there exists a state w such that $|\varphi(w)| > \alpha^{-1}|w|$, and this could be the case when the flux is not symmetric. Nevertheless the choice (6.1) works for (1.1) with a symmetric flux function, and in this case one can take $\lambda^+ = -\lambda^- = 1$, provided that $\varphi'(u) > -1$ for all u.

If we are interested only in *small* data it is possible to choose $\psi(u) = |u|$ even for general fluxes and regular φ. Indeed, if $\varphi \in C^1$ and $\varphi'(0) > -1$, then (5.6) reduces to

$$\big[\operatorname{sgn}(u)(\psi - \psi(\varphi))\big]'(u) = \big(1 + \varphi'(u)\big),$$

which is positive for u close to zero.

Finally, our hypothesis [H5] seems to be unavoidable, as there are counterexamples (see Section 7) in which $\varphi'(0) = -1$ and the total variation of the solution blows up in finite time.

7. Examples of Blow-Up of the Total Variation

In this section we present two examples in which hypothesis [H5] does not hold and the total variation of the *exact* nonclassical solution blows up in finite time. For a recent important result about blow-up for systems of conservation laws, see Jenssen [**19**].

Example 7.1. Consider the equation (1.1) with the following flux-function

$$f(u) := \begin{cases} u^h, & u \geq 0, \\ u^k, & u < 0, \end{cases}$$

with $k > h$ odd and greater than 1. It should be stressed that this function does not satisfy our regularity conditions since it is only Lipschitz continuous at the origin. Nevertheless, the example presented now is of interest since it shows new features not encountered in the classical case. We recall that, when the classical Oleinik entropy condition is enforced, the solution of the Cauchy problem (1.1)-(1.2) with Lipschitz continuous flux has bounded variation and in fact is total variation diminishing.

In the context of nonclassical solutions, we will produce an example where the initial data is in **BV** but the total variation of the solution blows up *instantaneously* at $t = 0$. This actually happens for a particular choice of the kinetic function φ for which $\varphi'(0+) = -\infty$. It should be noticed that also $\tau'(0+) = -\infty$, nevertheless the classical solution exists and is in **BV**. This means that in the case of a Lipschitz continuous flux-function, whether the total variation of the solution blows up or not, is not determined by the value of $\varphi'(0)$ but, as we shall see, can be related to the behavior of the function $\alpha - \varphi$ near $u = 0$.

It is not difficult to see that for u positive $\varphi^*(u) = -u^\gamma$, where $\gamma = \frac{h-1}{k-1} < 1$, and that $\tau(u) < \tau_k(u)$ where $\tau_k(u)$ satisfies (2.1) with $f(u) = u^k$ for all u. Hence $\tau(u) < -C_k u$ for a positive constant C_k depending only on k, and so $\tau(u) < -2u^2$ if for example $0 < 2u < C_k$. Choose now $\alpha(u) = -u^2 > \tau(u)$ for all $u \geq 0$. It follows that $\varphi(u) = -u^\gamma\big(1 + O(u)\big)$, hence

$$\alpha(u) - \varphi(u) \geq -u^2 + \frac{1}{2}u^\gamma,$$

if u is sufficiently small.

Choose now an integer n_0 such that $1/n_0^\beta < C_k$ where $\beta = 1/\gamma > 1$. Take the initial data of the form

$$u_0(x) := \begin{cases} 1/n_0^\beta, & \text{if } x \in (-\infty, 2n_0], \\ 1/n^\beta, & \text{if } x \in (2n, 2n+1),\ n \geq n_0, \\ -2/n^{2\beta}, & \text{if } x \in \big(2n+1, 2(n+1)\big),\ n \geq n_0. \end{cases}$$

An easy estimate implies that

$$\mathbf{TV}(u_0) \leq 4 \sum_{n=n_0}^{\infty} \frac{1}{n^\beta} < \infty.$$

For small positive t, the solution is obtained just by solving the Riemann problems at each discontinuity point in u_0. Notice that $-2/n^{2\beta} > \tau(1/n^\beta)$, hence the Riemann problem with data $(1/n^\beta, -2/n^{2\beta})$ is solved by a nonclassical shock from $1/n^\beta$ to $\varphi\big(1/n^\beta\big)$ followed by a classical shock from $\varphi\big(1/n^\beta\big)$ to $-2/n^{2\beta}$. In particular it follows that

$$\begin{aligned} \Delta\mathbf{TV}(0) &\geq \sum_{n=n_0}^{\infty} \big(\alpha(1/n^\beta) - \varphi(1/n^\beta)\big) \\ &\geq \frac{1}{2}\sum_{n=n_0}^{\infty} \frac{1}{n^{\gamma\beta}} - \sum_{n=n_0}^{\infty} \frac{1}{n^{2\beta}} \geq \frac{1}{2}\sum_{n=n_0}^{\infty} \frac{1}{n}. \end{aligned}$$

This implies that $\Delta\mathbf{TV}(0) = +\infty$ hence $\mathbf{TV}(0+) = +\infty$.

Finally, notice that $u(t,\cdot) \notin \mathbf{BV}$ but $u(t,\cdot) \in \mathbf{BV}_{loc}$.

Example 7.2. Now we will take $f(u) = u^3$, so our hypotheses on the flux-function are satisfied. Since the total variation of the solution of the Riemann problem (u_l, u_r) depends in a Lipschitz continuous way on $|u_l - u_r|$, it appears that in this case the total variation can not blow up instantaneously. In fact, we shall prove that for suitable initial data u_0 and choice of the kinetic function φ, there exists a time $\bar{t}$ such that

$$\mathbf{TV}\big(u(t,\cdot)\big) = +\infty,$$

for all $t \geq \bar{t}$, where u is the solution of (1.1)-(1.2).

We shall consider the case $\varphi(u) = \varphi^*(u) = -u$ for all u, hence $\varphi(u)$ does not satisfy [H5]. In this situation *every* Riemann problem with $u_l u_r < 0$ generates a nonclassical shock; more precisely the solution is given by a nonclassical shock connecting u_l to $-u_l$ followed by a classical shock connecting $-u_l$ to u_r, no matter how small u_r is. This means that arbitrarily small oscillations near 0 can produce nonclassical shocks of arbitrarily large strength. For related results connected with the study of radially symmetric systems, see the works of Freistühler, for instance in [**10, 11**].

Now let us construct initial data for which the total variation of the solution blows up. We define $u_0(x)$ to be equal to 1 for $x < 0$ and equal to 0 for $0 < x < x_0 := 1$. In x_0 a rarefaction will originate. First of all, the Riemann problem in $x = 0$ is solved by a single classical shock traveling with speed $\lambda_0 := \lambda(1, 0) = 1$. We want to define inductively points x_n, y_n and states u_n such that $x_{n-1} < y_n < x_n$ for all n and u_0 is given by

$$u_0(x) := \begin{cases} u_n, & \text{if } x_{n-1} < x < y_n, \\ 0, & \text{if } y_n < x < x_n, \\ 0, & \text{if } x > \sup_n x_n. \end{cases} \tag{7.1}$$

The idea is the following: start at x_0 and take u_1 small and negative to be defined later. The Riemann problem at x_0 is solved by a rarefaction wave which will interact with the original shock outgoing from the origin, at the point $P_0 := (1, 1)$ in the (x, t)-plane. This interaction will produce a slower nonclassical shock connecting 1 to -1 and a faster classical shock which will interact with the rarefaction until the point P_1 (see Figure 7.1). Let x_1 be the x-coordinate of point P_1. Now, draw back the line with slope $\lambda(u_1, 0)$ passing through P_1. Let y_1 be the x-coordinate of the intersection point between this line and the x-axis. Notice that $0 < \lambda(u_1, 0) < \lambda(u_1)$ hence we have $x_0 < y_1 < x_1$. Moreover the Riemann problem at y_1 is actually solved by the shock traveling with speed $\lambda(u_1, 0)$. Since P_1 depends only on the speed at the right of the rarefaction (that is $\lambda(u_1) = 3u_1^2$), then it is clear that once u_1 is known, so x_1, y_1 are.

Let us now proceed inductively: assume points x_n, y_n and value u_{n+1} are given and assume that the rarefaction originating at x_n interacts with the shock originating at y_n at the point P_n, producing a nonclassical shock connecting $(-1)^n$ to $(-1)^{n+1}$ traveling with speed 1 and a classical shock interacting with the previous rarefaction until point P_{n+1}. As before let x_{n+1}, y_{n+1} be the x-coordinates of the point P_{n+1} and the intersection-point between the x-axis and the line with slope $\lambda(u_{n+1})$ passing through P_{n+1}, respectively. Again $x_n < y_{n+1} < x_{n+1}$.

We notice that each interaction at P_n generates a nonclassical shock between the states 1 and -1, traveling with speed 1. Hence these fronts will never interact

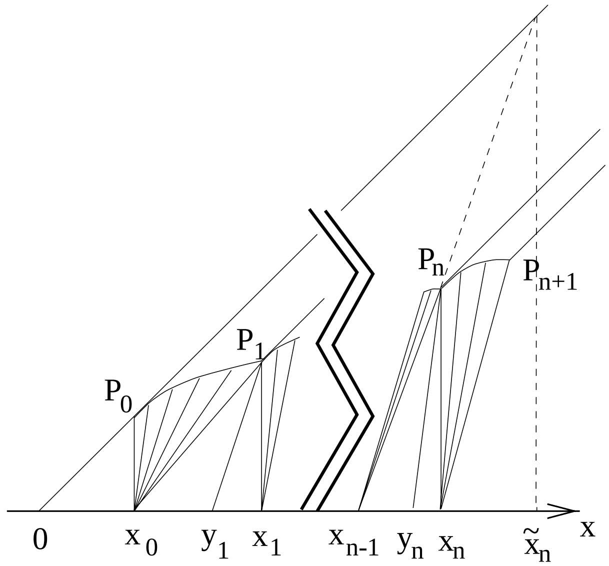

FIGURE 7.1

in the future. If we can generate infinitely many nonclassical shocks in finite time, then the total variation of the solution will blow up.

This is achieved by suitably choosing the states u_n in such a way that the sequence x_n converges to some finite $\bar{x}$. We define u_n inductively by letting

$$u_{n+1} := (-1)^{n+1} \cdot \left| \lambda^{-1}\left(\frac{1}{1+2^n x_n}\right)\right| .$$

Indeed, let $\tilde{x}_n$ be the x-coordinate of the intersection point between the nonclassical shock originating at P_0 and the line with slope $\lambda(u_n)$ and originating at $(x_{n-1}, 0)$ (see Figure 7.1). Then an easy computation gives

$$x_{n+1} - x_n \le \tilde{x}_{n+1} - x_n = \frac{\lambda(u_{n+1})}{1-\lambda(u_{n+1})} x_n = \frac{1}{2^n}.$$

This implies that $x_n \to \bar{x} \le \sum_{n=0}^{\infty} 1/2^n = 2$. It is easy to see that the points P_n will converge to a point $\bar{P} = (\bar{t}, \bar{x})$ with $\bar{t} \le \bar{x}$. By construction, at time $\bar{t}$ the solution will have infinitely many nonclassical shocks connecting the states 1 and -1, hence $\mathbf{TV}\big(u(\bar{t}, \cdot)\big) = \infty$, and since they will never interact in the future, this is true even for all $t > \bar{t}$. On the other hand we have

$$\mathbf{TV}\big(u(0,\cdot)\big) = 1 + 2\sum_{n=1}^{\infty} \left|\lambda^{-1}\left(\frac{1}{1+2^n x_n}\right)\right| \le 1 + \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n < \infty.$$

REMARK 7.3. It is possible to construct an example similar to Example 7.2, when we request only the existence of a single point $\bar{u} > 0$ such that $\varphi^{[2]}(\bar{u}) = \bar{u}$, and even if [H4] does not hold.

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