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The Harnack Inequality and Non-Divergence Equations

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The Harnack inequality is one of the central properties of solutions of linear (or appropriate nonlinear) second order elliptic equations in divergence or nondivergence form.

It is of particular interest when the coefficients are merely measurable, i.e., a solution u must satisfy

$$(*) \qquad D_i a_{ij}(x) D_j u = 0,$$

or

$$(**) \qquad a_{ij}(x) D_{ij} u = 0 \ ,$$

and

$$A(x) = a_{ij}(x)$$

is assumed only to be a measurable strictly elliptic matrix, i.e.,

$$\lambda \|y\|^2 \leq y^T A y \leq \Lambda \|y\|^2$$

for all values of x.

The importance of $A(x)$ being only measurable, is that constitutes in itself a class invariant under dilation. No matter how we try to "blow up" our solution, by dilations

$$u(x) = \mu u(\lambda x)$$

we remain in the same class, always far, from say, constant coefficients.

That is why such equations (and the Harnack inequality) played such an important rôle in the theory of non-linear (far from linear) equations, where there is no hope of "freezing the coefficients".

The Harnack inequality then states

Harnack inequality. Let u be an appropriate weak solution of $(*)$, resp. $(**)$.

If $u \geq 0$ in $B_r(x_0)$, then

$$\sup_{B_{r/2}(x_0)} u \leq C \inf_{B_{r/2}(x_0)} u \ ,$$

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where $B_r(x)$ is the ball of radius r and center x.

In case $(*)$, if u is a solution, $u - b$ is also a solution for any constant b.

In case $(**)$, if u is a solution $u - \ell$ is also a solution for any linear function ℓ.

In [**C**] (see also [**CC**]), following the Crandall-Lions theory of viscosity solutions to non divergence equations, we defined the class $\mathcal{S} = \mathcal{S}_\Lambda$ of "all viscosity solutions to some elliptic non divergence equations of ellipticity Λ".

More precisely,

Definition 1. The continuous function $u \in \underline{\mathcal{S}}_\Lambda$ (i.e., is a "subsolution") iff whenever a quadratic polynomial P touches u by above at some point x_0,

$$\|[D^2P]^-(x_0)\| \le \Lambda \|[D^2P]^+(x_0)\|$$

and u belongs to $\mathcal{S}$, if both $u, -u$ belong to $\underline{\mathcal{S}}$.

Here touches by above, means $P \ge u$ in a neighborhood of x_0, and $P(x_0) = u(x_0)$. Also $(D^2P)^\pm$ denotes the positive (negative) part of the symmetric matrix D^2P. Using the Krylov-Safanov technique ([**K-S**]) we proved

Theorem 1. *If $u \in \mathcal{S}$, it satisfies the Harnack inequality with a constant $C = C(\Lambda)$.*

Since, if $u \in \mathcal{S}_\Lambda$, $u - \ell \in \mathcal{S}_\Lambda$ for any linear function ℓ, actually, $u - \ell$ satisfies the Harnack inequality. The converse is elementary.

Theorem 2. *Assume that u is continuous and for any linear function ℓ, $u - \ell$ satisfies the Harnack inequality, with constant C. Then*

$$u \in \mathcal{S}_\Lambda$$

with $\Lambda = \Lambda(C)$.

Proof. Let P be a quadratic polynomial, tangent by below to u at the origin. By subtracting the linear part, ℓ, we may assume that

$$P = \sum_{\alpha_i > 0} \alpha_i x_i^2 + \sum_{\beta_i < 0} \beta_i x_i^2$$

and $P \le u$ in $B_h(0)$.

In particular

$$v = u - \min \beta_i h^2 \ge 0 \quad \text{in} \quad B_h(0)\ .$$

Thus, by Harnack inequality,

$$\sup_{B_{h/2}} v \le Cv(0) = C[-\min(\beta_i)h^2]\ .$$

But, always in B_h,

$$v = u - \min \beta_i h^2 \ge P - \min \beta_i h^2\ .$$

In particular

$$\alpha_{\max}\left(\frac{h}{2}\right)^2 \le (1 + C)[-\min \beta_i]h^2$$

□

A consequence of the argument above is the following remark:

Corollary 1. *Assume $u - \ell$ satisfies the Harnack inequality for every ℓ. If u has in $B_h(X_0)$ a tangent polynomial by below, i.e.,*
a) $u - P \geq 0$ *on* $B_h(X_0)$
b) $u - P = 0$ *at* X_0.
Then u has a tangent polynomial by above in $B_{h/2}(X_0)$,

$$Q = \ell + A|X - X_0|^2$$

with $A \leq C\|D^2 P\|$, and ℓ the linear part of P at X_0.

Proof. In the ball of radius ρ $\rho < h$), $v = u - \ell(X) + \|D^2 P\|\rho^2$, is a nonnegative function that satisfies the Harnack inequality. Since $v(0) = \|D^2 P\|\rho^2$, $\sup_{B_{\rho/2}} v \leq C\|D^2 P\|\rho^2$.

We would like to discuss now a more interesting case, that arises for instance in homogenization (see [**C1**]).

In that case, one has a function u_0, that, being a uniform limit of solutions u_ε to highly oscillatory equations, has inherited the following two properties
a) $(u_0 - \ell)$ satisfies the Harnack inequality for every ℓ,
b) $u_0(X) - u_0(X - X_0)$ satisfies the Harnack inequality for every X_0.
We want to show that in this case, u_0 satisfies a fully non linear uniformly elliptic equation

$$F(D^2 u, \nabla u) = 0 \ .$$

Although not necessary for this discussion, we start by pointing out the following Theorem, due to Cabre and Caffarelli.

Theorem 3. (see [**C-C**], Lemma 5.6) *Assume that, for any translation X_0,*

$$v(X) = u(X) - u(X - X_0)$$

satisfies the Harnack inequality. Then u is locally $C^{1,2}$.

We would like now to prove

Theorem 4. *Let u be a continuous function in the unit ball B_1, such that*
a) $|u| \leq 1$
b) $u \in \mathcal{S}_\Lambda$, *and for any X_0, for any constant C,*

$$v_{X_0} = u(X) - u(X - X_0) + C$$

satisfies the Harnack inequality.
Then, there exists a second order non linear operator $F(D^2\omega, D\omega)$, with $F(0, P) \equiv 0$, uniformly elliptic such that $u|_{B_1}$ is a viscosity solution of

$$F(D^2 u, Du) = 0.$$

Proof. The proof consists in building such an operator. Of course F is not unique: a linear function, ℓ, satisfies every possible non linear operator with $F(0, p) = 0$.

We start by recalling a basic Pucci extremal operator: For θ large, to be fixed, depending on Λ we define

$$\mathcal{P}(D^2 u) = \sum_{\lambda_j > 0} \lambda_j + \theta \sum_{\lambda_j < 0} \lambda_j,$$

where λ_j denote the eigenvalues of D^2u.

For any value of the vector $\vec{p}$ in R^n consider now the set of quadratic polynomials P, tangent by above to u at some point X_0, with $\nabla P(X_0) = \vec{p}$. That is:

$$T^+(\vec{p}) = \left\{ P/ \begin{array}{c} \text{i) } P \text{ quadratic,} \\ \text{ii) } P \text{ tangent by above to } u \text{ at a point } X_0 \in B_1, \\ \text{iii) } \nabla P(X_0) = p \ . \end{array} \right.$$

Similar definition for $T^-(\vec{p})$.

Define

$$\widetilde{F}(D^2u, Du) = \sup_{P \in T^+(\nabla u)} \mathcal{P}(D^2u - D^2p).$$

We first note that $\widetilde{F}$ is finite.

Indeed $\mathcal{P}$ is a Lipschitz function on D^2u, with $\|\mathcal{P}\|_{\mathrm{Lip}} = \theta$ so we just need to pin down $\widetilde{F}$ at zero. But we recall, from the proof of Theorem 2, that P being tangent by above to u, implies

$$|\lambda_{\min}(P)| \leq C|\lambda_{\max}(P)| \ .$$

Therefore

$$\mathcal{P}(-D^2P) \leq 0$$

if $\theta \geq nC$.

This makes, for every q,

$$\widetilde{F}(0, \vec{q}\,) \leq 0 \ .$$

To make $\widetilde{F}(0, \vec{q}\,) = 0$, we modify it to

$$F(D^2u, Du) = \max(\widetilde{F}(D^2u, Du), \mathcal{P}(D^2u)) \ .$$

This makes of F a uniformly elliptic a function, Lipschitz on D^2u, with

a) $\|F\|_{\mathrm{Lip}} \leq \theta$

b) $F(0), \ ) = 0$.

Let us check that u is a viscosity solution.

If P is tangent by above to u at x_0,

$$\mathcal{P}(D^2P - D^2P) = 0 \ .$$

Thus $F \geq 0$.

If Q is tangent to u be below at X_1 we make two observations

a) Always from Theorem 2,

$$|D^2P)^+| \leq C|D^2P^-| \ .$$

So, since we chose $\theta > nC$,

$$\mathcal{P}(D^2Q) \leq 0 \ .$$

b) If $\nabla Q(X_1) = \vec{q}$ and $P \in T^+(q)$, i.e., is tangent to u, by above, at some point X_0 and $\nabla P(X_0)$ is also $\vec{q}$, then

$$\widetilde{P} = P(X - X_0) - Q(X - X_1)$$

is tangent to

$$u(X - X_0) - u(X - X_1)$$

by above at the origin, and $\nabla\widetilde{P}(0) = 0$.

Again, from the proof of Theorem 2

$$|D^2(P-Q)^-| < C|D^2(P-Q)^+|.$$

Therefore

$$\mathcal{P}(Q-P) \le 0$$

for any Q in $T^-(\vec{q})$ and any P in $T^+(\vec{q})$.

Therefore

$$F(D^2Q, DQ) \le 0$$

for any Q in T.

This completes the proof that u is a viscosity solution of

$$F(D^2w, Dw) = 0\ .$$

Finally, we add the possibility of subtracting linear functions from $u_0(X) - u_0(X - X_0)$.

Theorem 5. *Assume that*

$$u_0(X) - u_0(X - X_0) - \ell(X)$$

satisfies the Harnack inequality for any linear function ℓ. Then $u_0(X)$ is a viscosity solution of a fully nonlinear, uniformly elliptic equation

$$F(D^2u) = 0.$$

Remark. The necessary condition in Theorem 5 is also sufficient (see [**C-C**] Theorem 5.3).

Further $u_0 \in C^{1,\alpha}$ (see Corollary 5.7 of [**C-C**]).

Proof of Theorem 5. The proof is the same as that of Theorem 4, but we now construct

$$\widetilde{F}(D^2u) = \sup_{P\in T^+} \mathcal{P}(D^2u - D^2P)$$

where the sup is now taken over all possible P in T^+, disregarding the value of ∇P at the contact point.

The last characterization we would like to discuss concerns solution to equations

$$F(D^2u) = 0$$

with F uniformly elliptic and concave.

It is shown in [**C-C**], that viscosity solutions, u of such equations have the property that any convex combination of its translations ($\lambda_i \ge 0$, $\sum \lambda_i = 1$)

$$v = \sum \lambda_i u(x - x_i)$$

is a viscosity subsolution of the same equation

$$F(D^2v) \ge 0$$

It is also shown that the difference of a sub and supersolution, in particular, $v - u$, belongs to the class $\underline{\mathcal{S}}$ of subsolutions to some elliptic operator. (Heuristically, pure second derivatives of u belong to $\underline{\mathcal{S}}$.)

We prove the inverse characterization.

Theorem 6. *Let u be a continuous function in B_1 such that*

a) $u_0 \in \mathcal{S}$.
b) $u_0(x + x_0) - u_0(x) \in \mathcal{S}$, *for any* x_0.
c) *For any convex combination*

$$v(x) = \Big[\sum \lambda_i u(x - x_i)\Big] - u(x)$$

belongs to $\underline{\mathcal{S}}$.

Then u is a ($C^{2,\alpha}$) solution of an equation

$$F(D^2 u) = 0$$

with F concave.

Proof. We will first construct the convex, uniformly elliptic level surface

$$F(M) = 0.$$

The full F can then be constructed by, for instance, linear extension in the direction of the identity.

More precisely, for every P in T^+ consider the cone

$$\Gamma_P = \{M : \mathcal{P}(M - D^2 P) \geq 0\}$$

with $\mathcal{P}$, as before defined by

$$\mathcal{P}(M) = \sum_{\lambda_i \geq 0} \lambda_i + \theta \sum_{\lambda_i \leq 0} \lambda_i$$

for some large $\theta(n)$ to be chosen later.

Next, let

$$\widetilde{\Gamma} = \Big(\bigcup_{P \in T^+} \Gamma_P \Big).$$

And finally, D, the convex envelope

$$D = \Big\{ M : M = \sum_1^{n \times n} p_i Q_i \text{ with } Q_i \in \widetilde{\Gamma} \text{ and } p_i \geq 0\ ,\ \sum p_i = 1 \Big\}\ .$$

Since all of the $\partial \Gamma_P$ define uniformly elliptic operators, so does $\partial \widetilde{\Gamma}$ and ∂D.

Let us show that

a) If $P \in T^+$, $D^2 P \in D$,
b) If $P \in T^-$, $D^2 P \notin D$.

That $D^2 P$, for any $P \in T^+$, belong to D happens by definition, since they belong to $\widetilde{\Gamma}$.

Assume that $Q \in T^-$ and $D^2 Q \in D$. Let us get a contradiction. Indeed, there are $Q_i \in \Gamma_{P_i}$, such that

$$Q = \sum_1^{n \times n} p_i Q_i$$

Let x_i be the points where P_i is tangent by above to u, and consider the function

$$v = \sum p_i u(x - x_i) - u(x - x_Q)$$

(with x_Q the point where Q is tangent to u). Then v has

$$\sum_{P_i} p_i P_i - Q$$

as a polynomial tangent by above at zero, that is

$$\sum_1^{n\times n} p_i(P_i - Q_i)$$

must satisfy, v belonging to the class $\bar{S}$, that the positive part of $D^2(\sum_1^{n\times n} p_i(P_i - Q_i)$ controls the negative part, i.e.

$$\|[D^2(\sum p_i(P_i - Q_i)]^-\| \leq C\|[D^2(\sum p_i(P_i - Q_i)]^+\|.$$

But according to the definition of $\mathcal{P}$,

$$\|[D^2(P_i - Q_i)]^-\| \geq \theta\|D^2(P_i - Q_i)\|^+$$

with θ large to be chosen. Since the number of matrices involved in the sum is a fixed one, $(n \times n)$, we have that

$$\|[\sum D^2 p_i(P_i - Q_i)]^-\| \geq \|[D^2(p_1(P_1 - Q_1))]^-\| - \sum \|D^2(p_i(P_i - Q_i)^+\|$$

If we choose $p_1(P_1 - Q_1)$ the one for which

$$\|[D^2(p_i(P_i - Q_i)]^-\|$$

is maximum, we get from the bounds above, that

$$\begin{aligned}\|[D^2(\sum p_i(P_i - Q_i))]^-\| &\geq \Big(1 - \frac{n\times n}{\theta}\Big)\|[D^2(p_1(P_1 - Q_1))]^-\| \\ &\geq \frac{\theta}{n\times n}\Big(1 - \frac{n\times n}{\theta}\Big)\|[D^2(\sum p_i(P_i - Q_i))]^+\|\end{aligned}$$

or

$$\|[D^2 \sum p_i(P_i - Q_i)]^-\| \geq \Big(\frac{\theta}{n\times n} - 1\Big)\|[D^2 \sum p_i(P_i - Q_i)]^+\|,$$

a contradiction to the fact that $v \in \underline{\mathcal{S}}$ if we choose θ large.

Finally, let us point out that if we choose P^+,

$$P^{\pm} = \pm\|u\|_{L^\infty(B_1)}|x|^2$$

an appropriate vertical translation of $P^{\pm}$ is tangent to u by above (resp. below) at some point.

Thus the Lipschitz graph ∂D is controlled by above and below at the origin. This completes the proof.

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