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# Vanishing Viscosity Limit for Initial-Boundary Value Problems for Conservation Laws

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Abstract. The convergence of the vanishing viscosity method for initial-boundary value problems is analyzed for nonlinear hyperbolic conservation laws through several representative systems. Some techniques are developed to construct the global viscous solutions and establish the $H^{-1}$ compactness of entropy dissipation measures for the convergence of the viscous solutions with general initial-boundary conditions. The representative examples considered include the systems of isentropic gas dynamics, nonlinear elasticity, and chromatography.

## 1. Introduction

We are concerned with the convergence of the vanishing viscosity method for initial-boundary value problems for nonlinear hyperbolic systems of conservation laws. Physical motivation is the vanishing viscosity limit from viscous compressible fluids to the inviscid ones with initial-boundary conditions, which is a natural way to determine entropy solutions for the inviscid equations in the interior of fluid domains under consideration. Our analysis focuses on several representative examples of nonlinear systems including the models for isentropic gas dynamics, nonlinear elasticity, and chromatography. The main objective in addressing the particular systems is to expose a general procedure of analysis for such a problem with general boundary conditions, especially *nonhomogeneous* ones. In order to make the main steps clearer and to avoid superfluous technicalities, we restrict our analysis here to the standard domains of form $Q_T = (0,T) \times (0,1)$ or $Q = (0,\infty) \times (0,1)$. Many other examples may be treated by following the same procedure.

We start with a general system of conservation laws in one space variable:

$$\partial_t u + \partial_x f(u) = 0, \qquad (t,x) \in Q, \tag{1.1}$$

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with $u \in U \subset \mathbf{R}^m$, $f \in C^1(U;\mathbf{R}^m)$, for some domain $U \subset \mathbf{R}^m$. We consider the following initial-boundary conditions for (1.1):

$$u|_{t=0} = u_0(x), \tag{1.2}$$

$$u|_{x=0} = a_0(t), \qquad u|_{x=1} = a_1(t). \tag{1.3}$$

We assume that the initial-boundary data satisfy

$$u_0 \in L^\infty((0,1);\mathbf{R}^m), \qquad a_0, a_1 \in L^\infty((0,\infty);\mathbf{R}^m). \tag{1.4}$$

Since the boundary layers generally exist for arbitrarily given $a_i, i = 1, 2$, our focus here is to expose a procedure to construct weak entropy solutions in the interior of $Q$ such that the solutions obtained are natural for the case that there is no boundary layer (see [**5**]). With this in mind, a definition of entropy solutions for the initial-boundary value problem was given in [**5**] for general multidimensional systems of conservation laws in more general (not necessarily cylindrical) domains, motivated from [**1**] for the scalar case. Some discussions about the initial-boundary problem in different contexts, related to [**1**], have been made for hyperbolic systems of conservation laws (*e.g.* [**2, 13, 19, 20, 30, 34, 36**]).

We say that $\eta \in C^1(\mathbf{R}^m)$ is an *entropy* for (1.1), with associated *entropy flux* $q \in C^1(\mathbf{R}^m)$, if

$$\nabla q(u) = \nabla \eta(u) \nabla f(u). \tag{1.5}$$

We call $F(u) = (\eta(u), q(u))$ an *entropy pair*. If $\eta(u)$ is convex, we say $F(u)$ is a convex entropy pair. An entropy pair $\mathcal{F}(u,v) = (\alpha(u,v), \beta(u,v))$ is called a *boundary entropy pair* if, for each fixed $v \in \mathbf{R}^m$, $\alpha(u,v)$ is convex with respect to $u$, and

$$\alpha(v,v) = \beta(v,v) = \partial_u \alpha(v,v) = 0. \tag{1.6}$$

We say that $\mathcal{F}(u,v) = (\alpha(u,v), \beta(u,v))$ is a *generalized boundary entropy pair* if it is the uniform limit of a sequence of boundary entropy pairs over compact sets.

Definition 1.1. We say that $u \in L^\infty(Q_T;\mathbf{R}^m)$ is a *weak entropy solution* of (1.1)-(1.3) in $Q_T$ if it satisfies

- Conservation Laws (1.1): For all $\phi \in C_0^\infty(Q_T)$, $\phi \geq 0$, and any convex entropy pair $(\eta, q)$,

$$\iint_{Q_T} (\eta(u)\partial_t \phi + q(u)\partial_x \phi)\, dx\, dt \geq 0; \tag{1.7}$$

- Initial Condition (1.2):

$$\operatorname*{ess\,lim}_{t\to 0+} \int_0^1 |u(t,x) - u_0(x)|\, dx = 0; \tag{1.8}$$

- Boundary Condition (1.3): For any $\gamma(t) \in L^1(0,T)$, $\gamma(t) \geq 0$, a.e., and any boundary entropy pair $\mathcal{F} = (\alpha, \beta)$,

(1.9)
$$\operatorname*{ess\,lim}_{x\to 0+} \int_0^T \beta(u(t,x), a_0(t))\gamma(t)\, dt \leq 0, \quad \operatorname*{ess\,lim}_{x\to 1-} \int_0^T \beta(u(t,x), a_1(t))\gamma(t)\, dt \geq 0.$$

To illustrate some features of the above definition, we consider a simple example. Let (1.1) be strictly hyperbolic, which means that the Jacobian $\nabla f$ is diagonalizable and all eigenvalues are real and distinct. In (1.2)-(1.3), let $a_0$, $u_0$, and $a_1$ be constant states $u_l$, $u_m$, and $u_r$, respectively, such that $u_l$ and $u_m$ are connected by a shock with a negative speed, while $u_m$ and $u_r$ are connected by a shock with a positive speed. To simplify, we assume that both shocks belong to genuinely nonlinear families, that is, the corresponding eigenvalues of $\nabla f$ are monotone along the integral curves of the associated right-eigenvectors (*cf.* [**21**]). Then it is easy to prove that $u(t,x) \equiv u_m$, $(t,x) \in Q$, is an entropy solution for the corresponding initial-boundary value problem, according to the above definition, provided that $u_l$, $u_m$, and $u_r$ are sufficiently close to each other.

Indeed, (1.7) and (1.8) are trivially satisfied. For (1.9), it suffices to check the inequality for the left boundary, $x = 0$, since the other is similar. Then this reduces to showing that $\beta(u_m, u_l) \le 0$. Now, from the Lax shock condition, it follows that $\beta(u_m, u_l) \le s\alpha(u_m, u_l)$, for $|u_m - u_l|$ sufficiently small (see [**22**]). Now, from the properties of boundary entropies, one has $\alpha(u, v) \ge 0$, and hence the desired inequality follows since $s < 0$.

This solution is actually consistent with the natural physical solution in the interior of $Q$, with boundary layers on the boundaries, via the characteristic analysis. It would be interesting to analyze systematically the uniqueness of entropy solutions in the sense of (1.7)-(1.9) for such problems.

We next recall an important fact about the solutions of (1.1)-(1.3) according to (1.7)-(1.9), established in [**5**], which holds even for the general multidimensional case in general (not necessarily cylindrical) domains.

Theorem 1.1. *Assume that* (1.1) *is endowed with a strictly convex entropy. A function* $u(t,x) \in L^\infty(Q_T; \mathbf{R}^m)$ *satisfies* (1.7)-(1.9) *if and only if the following conditions hold:*

(a). $u(t,x)$ *satisfies* (1.1) *in the sense of distributions in* $Q_T$*;*

(b). *Given any boundary entropy pair* $(\alpha(u,v), \beta(u,v))$*, there exists a constant* $M > 0$ *such that, for any nonnegative* $\phi(t,x) \in C_0^\infty\big((-\infty, T) \times \mathbf{R}\big)$ *and any* $v \in \mathbf{R}^m$*,*

$$\begin{aligned} &\int_0^T \int_0^1 (\alpha(u(t,x),v)\partial_t\phi + \beta(u(t,x),v)\partial_x\phi)\,dx\,dt \\ &\qquad + \int_0^1 \alpha(u_0(x),v)\phi(0,x)\,dx + M\int_\Gamma \alpha(u^b,v)\phi\,dt \ge 0, \end{aligned} \tag{1.10}$$

*where* $\Gamma = \cup_{j=0}^1 \{x = j, t > 0\}$*, and* $u^b(t) = a_i(t), i = 0, 1$.

In the subsequent sections, we solve problem (1.1)–(1.3), in the sense of Definition 1.1, for the representative systems of nonlinear elasticity, chromatography, and isentropic gas dynamics, according to the following scheme. In Section 2, we establish some general results for the parabolic systems obtained from (1.1) with an additional viscosity term in its right-hand side. In particular, we obtain a useful uniform estimate (2.22) for the derivative of the viscous solutions, which is essential in order to establish the $H^{-1}$-compactness of entropy dissipation measures for *nonhomogeneous* boundary conditions addressed here (see §3.1). In Section 3, we apply the results of Sections 1-2 and the compensated compactness methods to obtain the existence of entropy solutions when (1.1) is either the $2 \times 2$ system of nonlinear elasticity, or the $m \times m$ system of chromatography in Langmuir coordinates, or some other systems mentioned therein. Finally, in Section 4, we analyze the convergence of the viscous approximate solutions of the initial-boundary value problem for the system of isentropic Euler equations for gas dynamics with the aid of the results in Sections 1-2, especially estimate (2.22).

## 2. Parabolic Systems

In this section we consider the initial-boundary value problem for the parabolic system obtained from (1.1) with an additional viscosity term. Namely, we are concerned with the following initial-boundary value problem:

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_{xx} u, \qquad (t,x) \in Q, \tag{2.1}$$

$$u|_{x=0} = a_{0,\varepsilon}(t), \qquad u|_{x=1} = a_{1,\varepsilon}(t), \quad t > 0, \tag{2.2}$$

$$u|_{t=0} = u_{0,\varepsilon}(x), \qquad x \in (0,1). \tag{2.3}$$

To simplify the statements of the results, we assume that $a_{0,\varepsilon}, a_{1,\varepsilon}$ are smooth and

$$\sup_{\varepsilon>0} \|(a_{0,\varepsilon}, \varepsilon a_{0,\varepsilon}', a_{1,\varepsilon}, \varepsilon a_{1,\varepsilon}')\|_{L^\infty(0,\infty)} < \infty, \tag{2.4}$$

and $u_{0,\varepsilon} \in C_0^\infty(0,1)$ with

$$\sup_{\epsilon>0} \|u_{0,\varepsilon}\|_{L^\infty(0,1)} < \infty, \tag{2.5}$$

and the compatibility conditions:

$$a_{0,\varepsilon}^{(k)}(0) = a_{1,\varepsilon}^{(k)}(0) = 0, \qquad \text{for all } k \in \mathbf{N}. \tag{2.6}$$

For the purposes of the applications given in this paper, it suffices to consider the situation in which $f$ satisfies:

$$f \in C^3(\mathbf{R}^m; \mathbf{R}^m) \text{ is globally Lipschitz, and } f(-u) = f(u). \tag{2.7}$$

Denote by $K^\varepsilon(t,x)$ the fundamental solution of the heat equation $\partial_t v = \varepsilon \partial_{xx} v$, that is,

$$K^\varepsilon(t,x) = \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{x^2}{4\varepsilon t}}.$$

We will make use of the fact that

$$\|K^\varepsilon(t)\|_1 = 1, \qquad \|K_x^\varepsilon(t)\|_1 = \frac{1}{\sqrt{\varepsilon \pi t}}, \tag{2.8}$$

where $\| \; \|_1$ denotes the norm of $L^1(\mathbf{R})$.

For a function $\zeta(x)$ defined in $(0,1)$, denote by $\tilde{\zeta}$ the function defined in $\mathbf{R}$ such that

$$\begin{cases} \tilde{\zeta}(x) = \zeta(x), & 0 < x < 1, \\ \tilde{\zeta}(-x) = -\tilde{\zeta}(x), & x \in \mathbf{R}, \\ \tilde{\zeta}(x+2n) = \tilde{\zeta}(x), & x \in \mathbf{R}, \; n \in \mathbf{Z}. \end{cases} \tag{2.9}$$

Set $h_\varepsilon(t,x) = (1-x)a_{0,\varepsilon}(t) + x a_{1,\varepsilon}(t)$, $x \in (0,1)$, $t \geq 0$. If $u(t,x)$ is a smooth solution of (2.1)–(2.3) in $[0,T] \times (0,1)$, then $w(t,x) = u(t,x) - h_\varepsilon(t,x)$ is a smooth solution of the initial-boundary value problem:

$$\partial_t w - \varepsilon\partial_{xx} w = -\partial_x f(u) - \partial_t h_\varepsilon, \qquad (t,x)\in Q, \tag{2.10}$$

$$w|_{x=0} = 0, \qquad w|_{x=1} = 0, \qquad t>0, \tag{2.11}$$

$$w|_{t=0} = u_{0,\varepsilon}(x), \qquad x\in(0,1). \tag{2.12}$$

Hence, we expect that $u(t,x)$ satisfies

$$u(t) = K^\varepsilon(t) * \tilde u_{0,\varepsilon} - \int_0^t K^\varepsilon(t-s) * (\widetilde{\partial_x f(u)}(s) + \widetilde{\partial_t h_\varepsilon}(s))\,ds + h_\varepsilon(t), \tag{2.13}$$

for $(t,x)\in Q$.

To see this fact, we denote the right-hand side of (2.10) by $r(t,x)$ and approximate it in $L^1_{loc}(Q)$ by a sequence $r^n(t,x)$ in $C_0^\infty(Q)$. If $w^n(t,x)$ is the corresponding solution for the modified (2.10) with conditions (2.11)-(2.12), then we clearly have

$$w^n(t) = K^\varepsilon(t) * \tilde u_{0,\varepsilon} - \int_0^t K^\varepsilon(t-s) * \widetilde{r^n}(s)\,ds.$$

Now, letting $n\to\infty$, we see that $w^n$ converges in $L^1_{loc}(Q)$ to a certain $w$ satisfying

$$w(t) = K^\varepsilon(t) * \tilde u_{0,\varepsilon} - \int_0^t K^\varepsilon(t-s) * \tilde r(s)\,ds.$$

From the properties of the heat kernel and the function $\tilde r(t,x)$, we deduce that $w$ satisfies (2.10) in the sense of distributions in $Q$. Then $w-(u-h_\varepsilon)$ satisfies the homogeneous heat equation. By the standard regularity theory, $w$ is then smooth. Since (2.11) and (2.12) are easily verified from (2.13), the uniqueness of the solution of (2.10)–(2.12) implies that $w = u - h_\varepsilon$. On the other hand, if $u$ satisfies (2.13) and has continuous derivatives of first order in $t$ and up to second order in $x$, throughout $Q$, then applying the heat operator $\partial_t - \varepsilon\partial^2_{xx}$ to both sides of (2.13), for $(t,x)\in Q$, yields that $u$ satisfies (2.1) in $Q$ in the sense of distributions and, hence, in the classical sense. Conditions (2.2)-(2.3) are also immediately deduced from (2.13).

Then, for smooth solutions in $[0,T]\times(0,1)$, (2.13) is equivalent to

$$\begin{aligned} u(t) = K^\varepsilon(t) * \tilde u_{0,\varepsilon} &- \int_0^t \partial_x K^\varepsilon(t-s) * f(\tilde u)(s)\,ds \\ &- \int_0^t K^\varepsilon(t-s) * \widetilde{\partial_t h_\varepsilon}(s)\,ds + h_\varepsilon(t), \end{aligned} \tag{2.14}$$

where we have used the definition of the extension $\widetilde{\partial_x f(u)}(s)$ of $\partial_x f(u)(s)$, from $(0,1)$ to $\mathbf{R}$ according to (2.9). Indeed, to pass the derivative from $\widetilde{\partial_x f(u)}(s)$ to the heat kernel to obtain (2.14) from (2.13), we write the convolution as a sum of integrals in $(0,1)$, using (2.9), then apply integration by parts, and observe that the sums like

$$\sum_{n\in\mathbf{Z}} \int_0^t \{K^\varepsilon(t-s,x-2n-1) - K^\varepsilon(t-s,x-2n+1)\} f(a_{1,\varepsilon}(s))\,ds,$$

resulting also from this process, vanish identically. Hence a possible strategy for solving (2.1)–(2.3) is to obtain first a solution of (2.14), and then to prove that it possesses the required regularity.

Let $\mathcal{G}_T = L^\infty((0,T); L^\infty(0,1))$ and define the operator $\mathcal{L} : \mathcal{G}_T \to \mathcal{G}_T$ by

$$\mathcal{L}(v)(t) = K^\varepsilon(t) * \tilde{u}_{0,\varepsilon} - \int_0^t \partial_x K^\varepsilon(t-s) * f(\tilde{v})(s)\, ds - \int_0^t K^\varepsilon(t-s) * \widetilde{\partial_t h_\varepsilon}(s)\, ds + h_\varepsilon(t). \tag{2.15}$$

Let $\|\nabla f(u)\|_{L^\infty} = C_0$. We have

$$\|\mathcal{L}(v_1)(t) - \mathcal{L}(v_2)(t)\|_{L^\infty} \le C_0 \sqrt{\frac{T}{\varepsilon\pi}} \|v_1 - v_2\|_{L^\infty},$$

and so $\mathcal{L}$ is a contraction in $\mathcal{G}_T$ as long as

$$C_0 \sqrt{\frac{T}{\varepsilon\pi}} < 1. \tag{2.16}$$

By the Banach Fixed Point Theorem, there exists a unique $u \in \mathcal{G}_T$, which satisfies $\mathcal{L}(u) = u$ when $0 \le t \le T$.

LEMMA 2.1. *Let $u$ be the unique fixed point of $\mathcal{L}$ in $\mathcal{G}_T$ with $T = \alpha_0\varepsilon$ for $\alpha_0 << 1$ independent of $\varepsilon$, such that* (2.16) *holds. Then there exists $C_1 > 0$ such that,*

$$\|\partial_x u(t)\|_{L^\infty} \le \frac{C_1}{\sqrt{\varepsilon t}}, \qquad 0 < t \le T, \tag{2.17}$$

*with $C_1$ independent of $\varepsilon$. Furthermore, there exists a constant $C_2$, depending on $\varepsilon$, such that*

$$\|\partial_x u(t)\|_{L^\infty} \le C_2, \qquad 0 < t \le T. \tag{2.18}$$

PROOF. The proof of (2.17) reduces to proving the following assertion: there exists a constant $C_1 > 0$ such that, if $v \in \mathcal{G}_T \cap C((0,T] \times (0,1))$, $v|_{x=0} = a_{0,\varepsilon}$, $v|_{x=1} = a_{1,\varepsilon}$, and

$$\|\partial_x v(t)\|_{L^\infty} \le \frac{C_1}{\sqrt{\varepsilon t}}, \tag{2.19}$$

then $\mathcal{L}(v)$ also satisfies these properties.

Indeed, since $u$ is the unique fixed point of $\mathcal{L}$ in $\mathcal{G}_T$, we have $u = \lim_{k\to\infty} u^k$ in $\mathcal{G}_T$, where $u^1 = h$, $u^{k+1} = \mathcal{L}(u^k)$. Hence, from the assertion, we have that (2.19) is satisfied for $v = u^k$, $u^k|_{x=0} = a_{0,\varepsilon}(t)$, and $u^k|_{x=1} = a_{1,\varepsilon}(t)$, for all $k \in \mathbf{N}$. Therefore, the standard arguments yield that $u$ must also satisfy these properties.

We now pass to the proof of the assertion. From the hypothesis on $v$, one has

$$\partial_x \mathcal{L}(v)(t) = \partial_x K^\varepsilon(t) * \tilde{u}_{0,\varepsilon} - \int_0^t \partial_x K^\varepsilon(t-s) * \widetilde{\partial_x f(v)}(s)\, ds - \int_0^t \partial_x K^\varepsilon(t-s) * \partial_t \widetilde{h_\varepsilon}(s)\, ds + \partial_x h_\varepsilon(t),$$

and so

$$\begin{aligned}\|\partial_x \mathcal{L}(v)(t)\|_{L^\infty} \leq & \frac{\|\tilde{u}_{0,\varepsilon}\|_{L^\infty}}{\sqrt{\pi\varepsilon t}} + \frac{CC_1}{\varepsilon} + 2\sqrt{\frac{T}{\pi\varepsilon}}\|\partial_t h_\varepsilon\|_{L^\infty} + \|\partial_x h_\varepsilon\|_{L^\infty} \\ \leq & \frac{1}{\sqrt{\pi\varepsilon t}}\big(\|u_{0,\varepsilon}\|_{L^\infty} + CC_1\sqrt{\frac{\pi T}{\varepsilon}} + 2T\|\partial_t h_\varepsilon\|_{L^\infty} + \sqrt{\pi\varepsilon T}\|\partial_x h_\varepsilon\|_{L^\infty}\big) \\ \leq & \frac{C_1}{\sqrt{\varepsilon t}},\end{aligned}$$

provided that

$$C_1 \geq \frac{\|u_{0,\varepsilon}\|_{L^\infty} + 2T\|\partial_t h_\varepsilon\|_{L^\infty} + \sqrt{\varepsilon\pi T}\|\partial_x h_\varepsilon\|_{L^\infty}}{\sqrt{\pi}(1 - C\sqrt{\frac{T}{\varepsilon}})}, \tag{2.20}$$

where $C > 0$ depends only on $C_0$, independent of $\varepsilon$ and $v$. Since $T = \alpha_0\varepsilon$ for $\alpha_0 << 1$ independent of $\varepsilon$, the fact that $C_1$ can be taken independent of $\varepsilon$ is clearly seen from (2.20), because of (2.4) and (2.5).

The second part of the statement follows similarly. We only need to prove that there exists a constant $C_2$ such that, if $\|\partial_x v\|_{L^\infty} \leq C_2$, $v|_{x=0} = a_{0,\varepsilon}$ and $v|_{x=1} = a_{1,\varepsilon}$, then $\mathcal{L}(v)$ has also these properties. To this end, we observe that we may write

$$\begin{aligned}\partial_x \mathcal{L}(v)(t) = K^\varepsilon(t) * \widetilde{\partial_x u_{0,\varepsilon}} - \int_0^t \partial_x K^\varepsilon(t-s) * \widetilde{\partial_x f(v)}(s)\, ds \\ - \int_0^t \partial_x K^\varepsilon(t-s) * \partial_t \widetilde{h_\varepsilon}(s)\, ds + \partial_x h_\varepsilon(t).\end{aligned}$$

Hence

(2.21)

$$\|\partial_x \mathcal{L}(v)\|_{L^\infty} \leq \|\widetilde{\partial_x u_{0,\varepsilon}}\|_{L^\infty} + CC_2\sqrt{\frac{T}{\varepsilon}} + 2\sqrt{\frac{T}{\pi\varepsilon}}\|\partial_t h_\varepsilon\|_{L^\infty} + \|\partial_x h_\varepsilon\|_{L^\infty} \leq C_2,$$

provided that

$$C_2 \geq \frac{\|\partial_x u_{0,\varepsilon}\|_{L^\infty} + 2\sqrt{T/(\pi\varepsilon)}\|\partial_t h_\varepsilon\|_{L^\infty} + \|\partial_x h_\varepsilon\|_\infty}{1 - C\sqrt{T/\varepsilon}},$$

where $C$ depends only on $C_0$. This concludes the proof. □

Let $u$ be the unique fixed point of $\mathcal{L}$ in $\mathcal{G}_T$. By Lemma 2.1, $\partial_x u$ is bounded in $[0,T] \times (0,1)$. Clearly, $u$ is a weak solution of (2.1)–(2.3) in the sense that $u$ belongs to the space

$$W(T) = \{v \in L^2([0,T]; W^{1,2}(0,1)) \mid \partial_t v \in L^2([0,T]; W^{-1,2}(0,1))\},$$

satisfying

$$\langle \partial_t u(t), \phi\rangle + \varepsilon\int_0^1 \partial_x u \partial_x \phi\, dx = \int_0^1 f(u)(t)\partial_x \phi\, dx,$$

for almost all $t \in [0,T]$ and all $\phi \in W_0^{1,2}(\Omega)$, $u(0) = u_{0,\varepsilon}$, and $u(t) - h_\varepsilon(t) \in W_0^{1,2}(0,1)$, for almost all $t \in [0,T]$. The fact that $\partial_t u \in L^2([0,T]; W^{-1,2}(0,1))$ follows the observation that $u$ is the limit in $L^\infty$ of a sequence $u^n$ satisfying

$$\partial_t u^{n+1} - \varepsilon\partial_{xx} u^{n+1} = -\partial_x f(u^n),$$

with $\|\partial_x u^n\|_{L^\infty}$ uniformly bounded in $n$. Therefore, $\partial_t u^n(t,x)$ is uniformly bounded in $L^2([0,T];W^{-1,2}(0,1))$ and must converge weakly to $\partial_t u(t,x)$.

Applying the regularity theory for parabolic equations (see [**17, 23, 28**]), one deduces that $u$ is a smooth solution of (2.1)–(2.3) in $[0,T]\times(0,1)$. It is easy to see that the hypothesis that $f$ is globally Lipschitz allows one to repeat the above procedures step by step in time, indefinitely, to obtain a global smooth solution to (2.1)–(2.3) satisfying (2.17)-(2.18).

If system (2.1) is endowed with a bounded invariant region (see [**8**]), then the global smooth solution $u$ is uniformly bounded. We now prove that this allows us to obtain a useful estimate for $\partial_x u$.

THEOREM 2.1. *Let $u$ be the unique smooth solution of* (2.1)-(2.3). *Assume that $u$ is uniformly bounded in* $[0,\infty)\times(0,1)$*, independently of $\varepsilon$, and that* (2.4)-(2.6) *hold. Then, for all $\delta>0$ sufficiently small, there exists a positive constant $M$, independent of $\varepsilon$, such that*

$$\begin{cases} \|\varepsilon\partial_x u(t)\|_{L^\infty}\le M, & \text{for } t>\varepsilon\delta,\\ \|\varepsilon\partial_x u(t)\|_{L^\infty}\le M\sqrt{\frac{\varepsilon}{t}}, & \text{for } 0<t\le\varepsilon\delta. \end{cases} \tag{2.22}$$

*The same result holds for the smooth solution of the Cauchy problem.*

PROOF. For any $t_0>0$, consider the operator $\mathcal{L}$ in $\mathcal{G}_T=L^\infty([t_0,t_0+T];L^\infty(\Omega))$, given by

$$\begin{aligned} \mathcal{L}(v)(t) =& K^\varepsilon(t-t_0)*\tilde{u}(t_0)-\int_{t_0}^t \partial_x K^\varepsilon(t-s)*f(\tilde{v})(s)\,ds\\ &-\int_{t_0}^t K^\varepsilon(t-s)*\widetilde{\partial_t h_\varepsilon}(s)\,ds+h_\varepsilon(t). \end{aligned} \tag{2.23}$$

Exactly as above, we easily see that $\mathcal{L}$ is a contraction mapping in $\mathcal{G}_T$ if (2.16) is satisfied. Also, identically as in the proof of Lemma 2.1, we prove the assertion that, for $C_1$ satisfying (2.20) with $\|u_{0,\varepsilon}\|_{L^\infty}$ replaced by $\|u(t_0)\|_{L^\infty}$, if $v\in\mathcal{G}_T\cap C((t_0,t_0+T]\times\Omega)$, $v|_{x=0}=a_{0,\varepsilon}$, $v|_{x=1}=a_{1,\varepsilon}$, and

$$\|\partial_x v(t)\|_{L^\infty}\le\frac{C_1}{\sqrt{\varepsilon(t-t_0)}}, \tag{2.24}$$

then $\mathcal{L}(v)$ also satisfies these properties. Thus, using the fact that $u$ is the unique fixed point of $\mathcal{L}$, we deduce from the standard arguments that $u$ must satisfy (2.24). Take $T=2\varepsilon\delta$, for any $\delta>0$ such that (2.16) holds. Then, for any $t>\varepsilon\delta$, we take some $t_0=t-T/2$ in (2.24) to obtain

$$\|\varepsilon\partial_x u(t)\|_{L^\infty}\le\frac{C_1}{\sqrt{\delta}}. \tag{2.25}$$

On the other hand, for $0<t\le\varepsilon\delta$, we have that (2.17) holds. From Lemma 2.1, $C_1$ can be taken independent of $\varepsilon$. Therefore, taking $M=C_1/\sqrt{\delta}$, we conclude the proof of (2.22).

The proof of estimate (2.22) for the smooth solution of the Cauchy problem is completely similar. □

In order to prove (1.10) for hyperbolic systems, we need to get a corresponding inequality for the associated parabolic systems. To this end, we will make use of a construction as in [**28, 30**]. For $\delta > 0$ sufficiently small, define

$$d(x) = \begin{cases} x, & 0 < x < \delta, \\ \delta, & \delta < x < 1 - \delta, \\ 1 - x, & 1 - \delta < x < 1, \end{cases}$$

and, for some $M > 0$, set

$$\xi_\varepsilon(x) \equiv 1 - e^{-\frac{M}{\varepsilon} d(x)}.$$

For any $\varphi \in C_0(\mathbf{R})$, $\varphi \geq 0$, the function $\xi_\varepsilon(x)$ satisfies

$$M \int_0^1 |\xi_\varepsilon'(x)| \varphi(x)\, dx \leq \varepsilon \int_0^1 \xi_\varepsilon'(x) \varphi'(x)\, dx + M(\varphi(0) + \varphi(1)). \tag{2.26}$$

Indeed,

$$\begin{aligned} \int_0^1 \xi_\varepsilon'(x)\varphi'(x)\, dx &= \int_{\{0<d<\delta\}} \xi_\varepsilon'(x)\varphi'(x)\, dx \\ &= - \int_{\{0<d<\delta\}} \xi_\varepsilon''(x)\varphi(x)\, dx - \frac{M}{\varepsilon}(\varphi(1) + \varphi(0)) \\ &\quad + \frac{M}{\varepsilon} e^{-\frac{M\delta}{\varepsilon}} (\varphi(1-\delta) + \varphi(\delta)) \\ &\geq \frac{M^2}{\varepsilon^2} \int_{\{0<d<\delta\}} e^{-\frac{M d(x)}{\varepsilon}} \varphi(x)\, dx - \frac{M}{\varepsilon}(\varphi(1) + \varphi(0)) \\ &= \frac{M}{\varepsilon} \int_O^1 |\xi_\varepsilon'(x)|\varphi(x)\, dx - \frac{M}{\varepsilon}(\varphi(1) + \varphi(0)), \end{aligned}$$

which immediately give (2.26).

THEOREM 2.2. *Let $u$ be the smooth solution of* (2.1)–(2.3)*, and let* $(\alpha(u,v), \beta(u,v))$ *be a boundary entropy pair for* (1.1)*. Then there exists a constant $M > 0$ such that, for all $\phi \in C_0^\infty((-\infty, T) \times \mathbf{R})$, $\phi \geq 0$, and $v \in \mathbf{R}^m$,*

$$\begin{aligned} &- \textstyle\int_0^T \int_0^1 \{\alpha(u,v)\partial_t \phi + \beta(u,v)\partial_x \phi + \varepsilon \alpha(u,v)\partial_{xx}\phi\} \xi_\varepsilon \, dx\, dt \\ &\leq \textstyle\int_0^1 \alpha(u_{0,\varepsilon}, v)\phi(x,0)\xi_\varepsilon\, dx + M \int_\Gamma \alpha(u_\varepsilon^b, v)\phi\, dt \\ &\quad +2\varepsilon \textstyle\int_0^T \int_0^1 \alpha(u,v)\xi_\varepsilon' \partial_x \phi\, dx\, dt, \end{aligned} \tag{2.27}$$

*where $\Gamma = \cup_{j=0}^1 \{x = j, t > 0\}$, and $u_\varepsilon^b = a_{i,\varepsilon}, i = 0, 1$.*

PROOF. Denote $\eta(u) = \alpha(u,v)$, $q(u) = \beta(u,v)$. By the convexity of $\alpha$ with respect to $u$ and (1.6), we easily see that there must exist a constant $M > 0$, independent of $v$, such that $|q(u)| \leq M\eta(u)$. Now, multiplying (2.1) by $\nabla\eta(u)$, one obtains

$$\partial_t \eta(u) + \partial_x q(u) \leq \varepsilon \partial_{xx} \eta(u). \tag{2.28}$$

Then, multiplying (2.28) by $\xi_\varepsilon\phi$, integrating in $Q_T = (0,T)\times(0,1)$, and using integration by parts, we obtain

$$
\begin{aligned}
& -\int_0^T\int_0^1 \{\eta(u)\partial_t\phi + q(u)\partial_x\phi + \varepsilon\eta(u)\partial_{xx}\phi\}\,\xi_\varepsilon\,dx\,dt \\
\leq\ & \int_0^1 \eta(u_{0,\varepsilon})\phi(x,0)\xi_\varepsilon\,dx + M\int_0^T\int_0^1 \eta(u)\phi|\xi_\varepsilon'|\,dx\,dt \\
& -\varepsilon\int_0^T\int_0^1 \partial_x(\eta(u)\phi)\xi_\varepsilon'\,dx\,dt + 2\varepsilon\int_0^T\int_0^1 \eta(u)\xi_\varepsilon'\partial_x\phi\,dx\,dt,
\end{aligned}
$$

where we have used $|q(u)| \leq M\eta(u)$. Applying (2.26) with $\eta(u)\phi$ replacing $\varphi$ in the inequality displayed above, we then obtain (2.27). ☐

## 3. Nonlinear Elasticity, Chromatography, and Other Systems

In this section we apply the results in Sections 1-2 to solving the initial-boundary value problem for two specific systems: the one arising in one-dimensional nonlinear elasticity and the other appearing in chromatography with Langmuir coordinates. We also discuss other applications which follow in a similar fashion.

**3.1. Nonlinear Elasticity.** Consider the one-dimensional nonlinear elasticity system:

$$
\begin{cases}
\partial_t u_1 - \partial_x\sigma(u_2) = 0,\\
\partial_t u_2 - \partial_x u_1 = 0,
\end{cases}
\tag{3.1}
$$

where $\sigma$ is a smooth function satisfying $\sigma'(\tau) > 0$, and $\tau\sigma''(\tau) > 0$ if $\tau \neq 0$. Then, in this case, $f(u) = (-\sigma(u_2), -u_1)^\top$. System (3.1) is endowed with the following strictly convex entropy:

$$\eta_*(u) = u_1^2 + \int_0^{u_2} \sigma(\tau)\,d\tau,$$

with entropy-flux:

$$q_*(u) = u_1\sigma(u_2).$$

Given a convex entropy $\eta(u)$, a boundary entropy pair $(\alpha(u,v), \beta(u,v))$ can be defined by taking the quadratic part of $\eta$ and its associated flux. That is,

$$
\begin{aligned}
\alpha(u,v) &= \eta(u) - \eta(v) - \nabla\eta(v)(u-v),\\
\beta(u,v) &= q(u) - q(v) - \nabla\eta(v)(f(u) - f(v)).
\end{aligned}
$$

Also, system (3.1) is endowed with a pair of independent Riemann invariants (*i.e.* the functions whose gradient are left-eigenvectors of $\nabla f$) given by

$$w_1 = u_1 + \int_0^{u_2}\sqrt{\sigma'(\tau)}\,d\tau, \qquad w_2 = u_1 - \int_0^{u_2}\sqrt{\sigma'(\tau)}\,d\tau.$$

The regions given by

$$R = \{u \in \mathbf{R}^2 \,|\, |w_1| < M,\ |w_2| < M\},$$

for any $M > 0$, are invariant under the flow of the parabolic system (2.1) corresponding to (3.1) (cf. [**8, 11, 14**]). Given uniformly bounded initial-boundary data, we take a region $R$ like the above with $M$ large enough so that the initial-boundary data assume values in $R$. In order to have the flux function $f$ of (3.1) satisfying condition (2.7), we first change from the coordinates $u$ to

$$\bar{u} = u - u_*, \tag{3.2}$$

where $u_* = (0, -A), A > 0$, is any point of the axis $u_1 = 0$ which does not belong to $R$. Then we replace $f$ by

$$\bar{f}(\bar{u}) = \varphi(\bar{u})f(\bar{u} + u_*), \text{ if } \bar{u} > 0; \qquad \bar{f}(-\bar{u}) = \bar{f}(\bar{u}), \tag{3.3}$$

with $\varphi \in C_0^\infty(\mathbf{R}^2)$ and $\varphi(\bar{u}) = 1$, if $\bar{u} \in R - u_*$, and such that $(0, -A)$ does not belong to the support of $\varphi$. The function $\bar{f}$ satisfies (2.7) and coincides with $f$ in the invariant region $R$. Now, by the invariant region arguments (cf. [**8**]), any smooth solution of system (2.1) associated with (3.1), with $\bar{f}$ replacing $f$, takes its values in $R$, as long as the initial-boundary data take values in $R$. Hence, replacing $f$ by $\bar{f}$ has no real effect, and the solution of the modified system is also a solution of the original one.

For given initial-boundary data (1.2)-(1.4), we can find smooth approximate functions $a_{0,\varepsilon}$, $a_{1,\varepsilon}$, and $u_{0,\varepsilon}$, which converge to $a_0$, $a_1$, and $u_0$, respectively, in $L^1_{loc}(0, \infty)$ and $L^1(0, 1)$ and satisfy (2.4)-(2.6), using the standard techniques of cutoff and mollification. We now consider the compactness of the smooth solution sequence $u^\varepsilon$ of the viscous systems (2.1) corresponding to (3.1). First, this sequence is uniformly bounded in $L^\infty(Q; \mathbf{R}^2)$, because all of the functions $u^\varepsilon$ assume values in $R$, which is a bounded region of $\mathbf{R}^2$. That is,

$$\|u^\varepsilon\|_{L^\infty(Q)} \le B_1, \tag{3.4}$$

for some $B_1 > 0$ independent of $\varepsilon$. To apply DiPerna's compactness result in [**11**], it suffices to verify the following:

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \quad \text{lies in a compact subset of } H^{-1}_{loc}(Q), \tag{3.5}$$

for any smooth entropy-entropy flux pair $(\eta, q)$.

With the aid of our estimate (2.22), property (3.5) can be seen as follows. We first multiply (2.1) by $\nabla\eta(u)$ to obtain

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) = \varepsilon \partial_{xx}\eta(u^\varepsilon) - \varepsilon(\partial_x u^\varepsilon)^\top \nabla^2\eta(u^\varepsilon)\partial_x u^\varepsilon. \tag{3.6}$$

If $\eta$ is strictly convex (*e.g.* $\eta = \eta_*$ given above), integrating (3.6) in $Q_T$ with any $T > 0$, we have

$$\begin{aligned} c_0 \iint_{Q_T} \varepsilon(\partial_x u^\varepsilon)^2 dxdt \le& \varepsilon \int_0^T (\eta'(u^\varepsilon)\partial_x u^\varepsilon)|_0^1 dt - \int_0^1 \eta(u^\varepsilon)|_0^T dx - \int_0^T q(u^\varepsilon)|_0^1 dt \\ \le& A_1 \int_0^\varepsilon \sqrt{\frac{\varepsilon}{t}} dt + A_2 \int_\varepsilon^T M dt + A_3 \le B_2, \end{aligned}$$

using estimate (2.22), where $A_i, i = 1, 2, 3$, and $B_2$ are independent of $\varepsilon$. Thus, we have

$$\sqrt{\varepsilon}\|\partial_x u^\varepsilon\|_{L^2(Q_T)} \le B_3, \tag{3.7}$$

for some constant $B_3 > 0$, depending on $T$, but independent of $\varepsilon$.

Now, from (3.7), we obtain as usual that, for any smooth entropy $\eta$,

$$\varepsilon(\partial_x u^\varepsilon)^\top \nabla^2\eta(u^\varepsilon)\partial_x u^\varepsilon$$

is uniformly bounded in $\mathcal{M}(Q_T)$, the space of signed Radon measures in $Q_T$. Therefore, by Sobolev's embeddings, it is compact in $W^{-1,p}(Q_T)$, for $1 < p < 2$. Also, from (3.7), we obtain that, for any smooth entropy $\eta(u)$,

$$\varepsilon\partial_{xx}\eta(u^\varepsilon)$$

is compact in $W^{-1,2}(Q_T)$ (in fact it converges to 0). Thus the right-hand side of (3.6) is compact in $W^{-1,p}(Q_T)$, using again Sobolev's embeddings. Now, because of (3.4), the left-hand side of (3.6) is uniformly bounded in $W^{-1,\infty}(Q_T)$. Then, an interpolation argument gives (3.5) (see [**10, 29**]).

Once we have proved (3.5), we can use DiPerna's compactness result in [**11**] to conclude the compactness of the sequence $u^\varepsilon$ in $L^1_{loc}(Q)$. Let $u$ be the limit of a subsequence $u^{\varepsilon_k}$ in $L^1_{loc}(Q)$ with $\varepsilon_k \to 0$ as $k \to \infty$. Hence, from (2.27) in Theorem 2.2, we obtain (1.10), using the fact that $\xi_\varepsilon$ and $\varepsilon\xi'_\varepsilon$ are uniformly bounded and converge pointwise to 1 and 0, respectively.

Thus, Theorem 1.1 can be applied to conclude that $u$ is an entropy solution of the initial-boundary value problem for (3.1).

THEOREM 3.1. *Let $a_0$, $a_1$, and $u_0$ satisfy* (1.4). *Then there exists a global entropy solution of the initial-boundary problem* (3.1) *and* (1.2)-(1.3) *in the sense of* (1.7)-(1.9).

**3.2. Chromatography:** The $m \times m$ chromatography system for Langmuir isotherms (cf. [**31**]) is given by

$$\partial_t u_i + \partial_x \left( \frac{k_i u_i}{1 + \sum_{j=1}^m u_j} \right) = 0, \quad x \in \mathbf{R}, \quad t \geq 0, \quad 1 \leq i \leq m, \tag{3.8}$$

where $0 < k_1 < k_2 < \cdots < k_m$ are given numbers. It is well known (cf. [**18**]) that (3.8) is endowed with $m$ linearly independent Riemann invariants $w_1, \ldots, w_m$, which have the property that the level surfaces $w_i = const.$ are affine hyperplanes in $\mathbf{R}^m$ (also see Temple [**38**]). For these systems, using the maximum principle (see [**33**]), it is easy to show that the regions

$$R = \{u \in \mathbf{R}^m \mid |w_i(u) - \bar{w}_i| \leq M_i, \ i = 1, \cdots, m\}$$

are invariant under the flow of the associated parabolic system (2.1), where $\bar{w} = (\bar{w}_1, \cdots, \bar{w}_n)$ is a constant state in $\mathbf{R}^m$ and $M_i > 0$ are arbitrary constants, as long as they are contained in the domain $\{u \in \mathbf{R}^m \,|\, u_i \geq 0, i = 1, \cdots, m\}$. Then, the same procedures as the one for the system of nonlinear elasticity can yield the existence of entropy solutions of the initial-boundary value problem for (3.8), where we apply the compactness theorem of James-Peng-Perthame [**18**], with the aid of Theorem 2.1 (i.e. (2.22)).

THEOREM 3.2. *Let $a_0, a_1$, and $u_0$ satisfy* (1.4). *Then there exists a global entropy solution of the initial-boundary problem* (3.8) *and* (1.2)-(1.3) *in the sense of* (1.7)-(1.9).

**3.3. Other Systems:** The same techniques can be used to prove the corresponding results for other systems such as the quadratic systems with umbilic degeneracy studied in [**6**], the class of conjugate type systems considered in [**15**], and the systems addressed in [**32**]. All of these systems have bounded invariant regions over which the flux functions are smooth, say, at least $C^3$ in the interior of the invariant regions and $C^2$ up to the boundaries.

## 4. System of Isentropic Euler Equations

The system of isentropic Euler equations reads

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x\big(\frac{m^2}{\rho} + p(\rho)\big) = 0, \end{cases} \tag{4.1}$$

where $\rho$ represents the density, $m$ is the momentum, and $p(\rho)$ is the pressure. The behavior of the pressure function $p(\rho)$ depends on the fluids under consideration. We assume at the onset that $p(\rho)$ satisfies

$$p'(\rho) > 0 \text{ (hyperbolicity)}, \quad \rho p''(\rho) + 2p'(\rho) > 0 \text{ (genuine nonlinearity)}, \tag{4.2}$$

away from the vacuum $\rho = 0$ and, when $\rho \to 0+$,

$$p(\rho) \approx \kappa \rho^\gamma (1 + P(\rho)), \qquad |P^{(n)}(\rho)| \le C\rho^{1-n},\ 0 \le n \le 4, \tag{4.3}$$

for some $\gamma > 1$. This means that, when $\rho \to 0$, the pressure law $p(\rho)$ has the same principal singularity as the $\gamma$-law, but allows additional singularities in the derivatives.

System (4.1) is endowed with a pair of independent Riemann invariants given by

$$w = \frac{m}{\rho} + \int_0^\rho \frac{1}{s}\sqrt{p'(s)}\,ds, \quad z = \frac{m}{\rho} - \int_0^\rho \frac{1}{s}\sqrt{p'(s)}\,ds. \tag{4.4}$$

Given positive constants $M_i$, $i = 1, 2$, consider the region $R$ of the plane $\rho$–$m$ given by $-z \le M_1$, $w \le M_2$, that is,

$$R = \big\{(\rho, m) \,|\, -M_1\rho + \rho\int_0^\rho \frac{1}{s}\sqrt{p'(s)}\,ds \le m \le M_2\rho - \rho\int_0^\rho \frac{1}{s}\sqrt{p'(s)}\,ds\big\}.$$

Then the region is invariant under smooth flows of the parabolic system (2.1) associated with (4.1) (cf. [**8**]), provided that we can show

$$\rho^\varepsilon(t, x) \ge \delta^\varepsilon(t), \quad 0 < t < \infty, \tag{4.5}$$

where $\delta^\varepsilon(t) > 0$ depends on $\varepsilon$ and $t$. Thus, we assume that the initial-boundary data (2.2)-(2.3), for the viscous systems (2.1) associated with (4.1), take values in $R$, for large $M_i, i = 1, 2$.

We notice that, in the region $R$, the flux function of (4.1) is only Lipschitz continuous because of the singularity in $\rho = 0$. In order to have the flux function $f(\rho, m) = (m, m^2/\rho + p(\rho))^\top$ partially satisfying (2.7), we artificially extend it to the half-plane $\{\rho < 0\}$ as an even function. The resultant function is smooth only away from the vacuum line $\{\rho = 0\}$. Hence local (in time) smooth solutions of the problem (2.1)–(2.3), corresponding to (4.1), can be extended only while they stay in the region $\rho > 0$. This is the main difference between the analyses for system (4.1) and for the systems in Section 3.

For given initial-boundary data $a_0, a_1$, and $u_0$ satisfying (1.4) and

$$\begin{cases} \rho_0(x) \ge 0, \quad |m_0(x)| \le C_0\rho_0(x), \quad C_0 > 0, \\ \rho(t, i) \ge 0, \quad |m(t, i)| \le C_0\rho(t, i), \quad i = 0, 1, \end{cases} \tag{4.6}$$

there exists sequences $a_{0,\varepsilon}$, $a_{1,\varepsilon}$, and $u_{0,\varepsilon}$ that converge to $a_0$, $a_1$, and $u_0$, respectively, in $L^1_{\rm loc}(0, \infty)$ and $L^1(0, 1)$ and that satisfy (2.4)-(2.6), and

$$\rho_{0,\varepsilon}(x), \rho^\varepsilon(t, 0), \rho^\varepsilon(t, 1) \ge \alpha^\varepsilon > 0, \tag{4.7}$$

where $\alpha^\varepsilon \to 0$ as $\varepsilon \to 0$. Under (4.7), there exists a unique global smooth solution of the Cauchy problem for the viscous system satisfying (4.5) (see [**12, 4**]). Actually, the key point to construct such a solution is to show that any local smooth solution, assuming values in $\{\rho > 0\}$, defined up to a certain time $T > 0$ must satisfy,

$$\rho^\varepsilon(t,x) \ge \delta^\varepsilon(T) > 0, \quad \text{for } 0 \le t < T, \tag{4.8}$$

for all $x \in (0,1)$ and some $\delta^\varepsilon(T) > 0$ depending on both $\varepsilon$ and $T$. The proof of (4.8) in [**12, 4**] can be easily adapted for the initial-boundary value problem with the help of an obvious version of Theorem 2.1 for local smooth solutions. Nevertheless, we will give an alternate proof here for (4.8) for our initial-boundary problem.

Consider the equation

$$\partial_t \rho + \partial_x(\rho u) = \varepsilon \partial_{xx}\rho. \tag{4.9}$$

Multiplying (4.9) by $v'(\rho)$ with $v(\rho) = 1/\rho$, we obtain

$$\partial_t v - \varepsilon\partial_{xx} v = \partial_x(uv) + v''(\rho u \partial_x \rho - \varepsilon(\partial_x\rho)^2) \le \partial_x(uv) + \frac{2vu^2}{2\varepsilon}. \tag{4.10}$$

Consider the equation

$$\partial_t g - \varepsilon\partial_{xx} g = \partial_x(ug) + \frac{2gu^2}{\varepsilon}, \quad x \in (0,1), \tag{4.11}$$

together with the conditions:

$$g|_{x=0} = v^\varepsilon(t,0), \quad g|_{x=1} = v^\varepsilon(t,1), \tag{4.12}$$

$$g|_{t=0} = v^\varepsilon(0,x), \quad x \in (0,1). \tag{4.13}$$

If $g$ is a smooth solution of (4.11)–(4.13) defined for $0 \le t < T$, the maximum principle, applied to the difference $v - g$, gives $v(t,x) \le g(t,x)$, for $(t,x) \in Q_T$. Thus, to get (4.8), all we have to do is to prove that

$$g(t,x) \le N^\varepsilon(T), \quad \text{for } (t,x) \in Q_T, \tag{4.14}$$

for some positive number $N^\varepsilon(T)$, depending on both $\varepsilon$ and $T$.

Now take any $t_0 \in (0,T)$ and consider the operator $\mathcal{L}$ in $\mathcal{G}_\tau = L^\infty((t_0, t_0+\tau) \times (0,1))$, with $t_0 + \tau \le T$, given by

$$\begin{aligned}\mathcal{L}(h) =& K^\varepsilon(t-t_0) * \tilde{g}(t_0) + \int_{t_0}^t K^\varepsilon(t-s) * \big(2\frac{\widetilde{hu^2}(s)}{\varepsilon} - \partial_t\tilde{\zeta}(s)\big)\,ds \\ &- \int_{t_0}^t \partial_x K^\varepsilon(t-s) * (\tilde{u}\tilde{h})\,ds + \zeta(t),\end{aligned} \tag{4.15}$$

where $\zeta(t,x) = (1-x)g(t,1) + xg(t,0)$, for $0 < x < 1$, $t > 0$, and the $\tilde{}$ has the same meaning as in Section 2. This operator is a contraction mapping in $\mathcal{G}_\tau$ if

$$2\max\{2M^2, MC_0, 1\}\sqrt{\frac{\tau}{\varepsilon}} < 1, \tag{4.16}$$

as one can easily verify, where $M$ is a constant such that $|u| \le M$ in $R$. In this case $g$ is its unique fixed point. Now take $t_0 = T - \tau_0$ with

$$\tau_0 = \varepsilon/[8(\max\{2M^2, MC_0, 1\})^2]$$

and $\tau = \tau_0$. Define

$$N(t_0) = \sup_{0\le t\le t_0} \|g(s)\|_{L^\infty(0,1)} + \|\zeta\|_{L^\infty(Q)}.$$

We now show that there exists a constant $N(T) > N(t_0)$ such that, if $h \in \mathcal{G}_\tau$ satisfies

$$\|h(t)\|_{L^\infty} \le N(T), \qquad 0 < t < T, \tag{4.17}$$

then $\mathcal{L}(h)$ also satisfies this inequality. We can seen this as follows.

$$\begin{aligned}\|\mathcal{L}(h)(t)\|_{L^\infty} \le & N(t_0) + 2M^2\frac{\tau_0}{\varepsilon}N(T) + \tau_0\|\partial_t\zeta\|_{L^\infty(Q)} + MN(T)\sqrt{\frac{\tau_0}{\varepsilon}} \\ \le & N(t_0) + \tau_0\|\partial_t\zeta\|_{L^\infty(Q)} + N(T)\bar{M}\sqrt{\frac{\tau_0}{\varepsilon}},\end{aligned}$$

where $\bar{M} = 2\max\{2M^2, MC_0, 1\}$. Therefore, one deduces that the assertion is true, provided

$$N(T) \ge \frac{N(t_0) + \tau_0\|\partial_t\zeta\|_{L^\infty(Q)}}{1 - \bar{M}\sqrt{\frac{\tau_0}{\varepsilon}}}. \tag{4.18}$$

Since $\mathcal{L}$ is a contraction mapping in $\mathcal{G}_\tau$, bound (4.17) must also hold for $g$, which then proves (4.8).

Once we have proven (4.8), we can easily show the existence of a unique global smooth solution of problem (2.1)–(2.3), corresponding to (4.1). The remaining of the proof of the existence of a solution to problem (1.1)–(1.3) follows the same procedure as the one for the systems in Section 3. In the polytropic case $p = \kappa\rho^\gamma$, after proving (3.5) with the help of Theorem 2.1, we may use the results in [**12**] ($\gamma = 1 + 2/(2k+1)$, $k > 1$), [**3**] ($1 < \gamma \le 5/3$), [**24**] ($\gamma \ge 3$), and [**25**] ($5/3 < \gamma < 3$) for the reduction of the Young measures to Dirac measures. The same can be done for more general pressure law $p(\rho)$ satisfying (4.2)-(4.3), by using the reduction procedure in the recent paper [**7**].

Theorem 4.1. *Let $a_0, a_1$, and $u_0$ satisfy* (1.4) *and* (4.6). *Then there exists a global entropy solution of the initial-boundary problem* (4.1)-(4.3) *and* (1.2)-(1.3) *in the sense of* (1.7)-(1.9) *and*

$$\rho(t,x) \ge 0, \quad |m(t,x)| \le C\rho(t,x), \qquad \textit{for some } C > 0 \textit{ independent of } t.$$

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