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Genuinely Nonlinear Hyperbolic Systems of Two Conservation Laws

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Abstract. This is an expository paper discussing the regularity and large time behavior of admissible BV solutions of genuinely nonlinear, strictly hyperbolic systems of two conservation laws. The approach will be via the theory of generalized characteristics.

1. Introduction

As is well-known, the theory of the Cauchy problem for nonlinear hyperbolic systems of conservation laws has to surmount numerous obstacles. The multi-space dimensional case is still terra incognita. Considerable progress has been made in one-space dimension, but the theory has yet to achieve definitive status. The source of the difficulty lies in that first derivatives of solutions starting out from even smooth and "small" initial values eventually blow up, triggering the development of jump discontinuities which propagate as shock waves. Thus, at best, only weak solutions may exist in the large. On the other hand, uniqueness generally fails within the class of weak solutions, so that extraneous "entropy" conditions have to be imposed in order to single out the admissible solution.

For strictly hyperbolic systems and initial data of small total variation, the random scheme [11;16] as well as front tracking algorithms [1;18] have successfully been employed for constructing admissible weak solutions in the class BV of functions of bounded variation. Furthermore, it has been established [3] that these solutions are L^1-stable, at least when the system is genuinely nonlinear. When the total variation of the initial data is large, even the L^∞ norm may blow up in finite time [13], so that the existence of even weak solutions is problematic.

Conditions are more favorable for genuinely nonlinear systems endowed with a coordinate system of Riemann invariants, in particular, systems of two conservation laws. In that case, the coupling between distinct characteristic families is weaker and, as a result of the spreading of rarefaction waves, even solutions starting out from initial values with unbounded total variation instantaneously acquire bounded variation. This remarkable property was first derived in the pioneering memoir [12], by appealing to the notion of approximate characteristics, within the framework of

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the random choice scheme. The same technique was subsequently employed by several authors in order to study the local structure and the large time behavior of solutions.

The aim of this expository paper is to provide an outline of a comprehensive theory of genuinely nonlinear, strictly hyperbolic systems of two conservation laws, developed from a different standpoint: One is to consider, at the outset, an admissible solution in an appropriate function class and derive its properties, without regard to any particular method of construction. The principal tool in the investigation will be the theory of generalized characteristics [5]. The results will include bounds on the total variation of the trace of the solution along any space-like curve, as well as a description of local structure and large time asymptotics of solutions under initial data in L^1, of compact support, or periodic. The above shall be reported here without demonstration; detailed proofs will be presented in Chapter XII of the forthcoming book [7] by the author. Proofs obtained under stronger a priori restrictions on the function class of solutions, have appeared in [5;6;19].

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2. BV Solutions and Generalized Characteristics

Consider a genuinely nonlinear, strictly hyperbolic system of two conservation laws:

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0. \tag{2.1}$$

Thus U takes values in $\mathbb{R}^2$ and F is a given, smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that, for any $U \in \mathbb{R}^2$, the Jacobian matrix $DF(U)$ has real distinct eigenvalues $\lambda(U) < \mu(U)$ associated with linearly independent eigenvectors $R(U), S(U)$, which satisfy

$$D\lambda(U)R(U) < 0 \ , \qquad D\mu(U)S(U) > 0 \, . \tag{2.2}$$

The system is endowed with a coordinate system of Riemann invariants (z, w), normalized by

$$DzR = 1 \ , \ \ DzS = 0 \, , \ \ DwR = 0 \, , \ \ DwS = 1. \tag{2.3}$$

By taking composition with the inverse of the local diffeomorphism $U \mapsto (z, w)$, one may realize functions of U as functions of (z, w); for economy in notation, the same symbol shall be employed to denote both representations.

It is assumed, further, that the system has the Glimm-Lax interaction property, namely the collision of any two shocks of the same family produces a shock of that family together with a rarefaction wave of the opposite family. This condition is here expressed by

$$S^T D^2 z S > 0 \ , \qquad R^T D^2 w R > 0 \, . \tag{2.4}$$

The normalization (2.3) in conjunction with the direction of the inequalities (2.2) and (2.4) imply that z increases across admissible weak 1-shocks and 2-shocks while w decreases across admissible weak 1-shocks and 2-shocks.

We now assume that $U(x,t)$ is a weak solution of (2.1) on $(-\infty, \infty) \times [0, \infty)$, which is a bounded measurable function of class BV_{loc}. In particular, $(-\infty, \infty) \times [0, \infty) = \mathcal{C} \bigcup \mathcal{J} \bigcup \mathcal{I}$, where $\mathcal{C}$ is the set of points of approximate continuity of U, $\mathcal{J}$ is the shock set of U and $\mathcal{I}$ is the set of irregular points of U. The one-dimensional

Hausdorff measure of $\mathcal{I}$ is zero. $\mathcal{J}$ is essentially covered by the countable union of C^1 arcs. With any $(\overline{x},\overline{t}) \in \mathcal{J}$ are associated one-sided approximate limits $U^\pm$ and a tangent line of slope (shock speed) s, which are related through the Rankine-Hugoniot jump condition

$$F(U^+) - F(U^-) = s[U^+ - U^-]\,. \tag{2.5}$$

We will be assuming that the solution U has sufficiently small oscillation, in which case (2.5) implies that s must be close to either λ or μ. This allows us to classify each shock point as belonging to the first or the second characteristic family. We then assume that the solution satisfies the Lax E-condition, namely

$$\lambda(U^+) < s < \lambda(U^-)\ , \tag{2.6$_1$}$$

$$\mu(U^+) < s < \mu(U^-)\ , \tag{2.6$_2$}$$

for 1-shocks or 2-shocks, respectively.

For convenience, we normalize U by requiring that it assumes at any point $(\overline{x},\overline{t}) \in \mathcal{C}$ the approximate value U^0 at that point, $U(\overline{x},\overline{t}) = U^0$. Furthermore, we extend $U^\pm$ from $\mathcal{J}$ to $\mathcal{J} \bigcup \mathcal{C}$ by setting $U^+ = U^- = U^0$ at any $(\overline{x},\overline{t}) \in \mathcal{C}$.

Characteristics of the first or second family, associated with a classical, Lipschitz continuous, solution U of (2.1) are integral curves of the ordinary differential equations

$$\frac{dx}{dt} = \lambda(U(x,t))\ , \tag{2.7$_1$}$$

or

$$\frac{dx}{dt} = \mu(U(x,t)). \tag{2.7$_2$}$$

Extending this notion, we associate characteristics with weak solutions in the function class discussed above by adopting the same definition, except that now (2.7) have to be interpreted as generalized ordinary differential equations, in the sense of Filippov [10]:

Definition 2.1. A *generalized characteristic* of the first or second family on the time interval $[\sigma,\tau] \subset [0,\infty)$, associated with the weak solution U, is a Lipschitz curve $\xi : [\sigma,\tau] \to (-\infty,\infty)$ which satisfies the differential inclusion

$$\dot{\xi} \in [\lambda(U^+)\,,\ \lambda(U^-)]\ , \tag{2.8$_1$}$$

or

$$\dot{\xi} \in [\mu(U^+)\,,\ \mu(U^-)]\ , \tag{2.8$_2$}$$

almost everywhere on $[\sigma,\tau]$.

In particular, shocks of either family are generalized characteristics of that family.

By standard theory of differential inclusions, through any fixed point $(\overline{x},\overline{t}) \in (-\infty,\infty) \times [0,\infty)$ pass two (not necessarily distinct) generalized characteristics of each family, associated with U and defined on $[0,\infty)$, namely the *minimal* $\xi_-(\cdot)$ and the *maximal* $\xi_+(\cdot)$, with $\xi_-(t) \le \xi_+(t)$ for $t \in [0,\infty)$. The funnel-shaped region confined between the graphs of ξ_- and ξ_+ comprises the set of points (x,t) that may be connected to $(\overline{x},\overline{t})$ by a generalized characteristic of that family. The extremal backward characteristics will be playing a pivotal role throughout the paper. Their first important property is that they propagate with classical characteristic speed:

Theorem 2.1. *Let $\xi(\cdot)$ denote any of the four extremal backward characteristics emanating from some point $(\overline{x},\overline{t})$ of the upper half-plane. Then $(\xi(t),t) \in \mathcal{C}$ for almost all $t \in [0,\overline{t}]$. In particular, for almost all $t \in [0,\overline{t}]$,*

$$\dot{\xi} = \lambda(U^{\pm}) \ , \tag{2.9$_1$}$$

if ξ is a 1*-characteristic, or*

$$\dot{\xi} = \mu(U^{\pm}) \ , \tag{2.9$_2$}$$

if ξ is a 2*-characteristic.*

The extremal backward characteristics mark the paths of signals travelling with extremal speed and may thus be employed in order to characterize space-like curves:

Definition 2.2. A Lipschitz curve, with graph $\mathcal{A}$ embedded in the upper half-plane, is called *space-like* relative to U when every point $(\overline{x},\overline{t}) \in \mathcal{A}$ has the following property: The set $\{(x,t) : 0 \le t < \overline{t},\ \zeta(t) < x < \xi(t)\}$ of points confined between the maximal backward 2-characteristic ζ and the minimal backward 1-characteristic ξ, emanating from $(\overline{x},\overline{t})$, has empty intersection with $\mathcal{A}$.

Clearly, any generalized characteristic, of either family, associated with U, is space-like relative to U. Similarly, all time-lines, t=constant, are space-like.

We now impose the following *structural condition* on our solution U: The traces of the Riemann invariants (z,w) along any space-like curve are functions of (locally) bounded variation.

The justification for the above assumption shall be provided, a posteriori, in Section 3, where such bounds on the variation will indeed be established. In fact, only part of the assumption is necessary in the analysis: The condition need only be tested for special space-like curves, namely generalized characteristics and time-lines. It should also be noted that, as shown in [4], any solution satisfying the structural condition must necessarily coincide with the solution with the same initial data constructed by either the random choice method or the front tracking algorithm.

In consequence of the structural condition, one-sided limits $U(x\pm,t)$ exist for all $-\infty < x < \infty,\ t > 0$, and $(x,t) \in \mathcal{C}$ implies $U(x-,t) = U(x+,t) = U(x,t)$ while $(x,t) \in \mathcal{J}$ implies $U(x-,t) = U^-, U(x+,t) = U^+$.

If U were a classical, Lipschitz continuous, solution of (2.1), then the trace of z along any 1-characteristic and the trace of w along any 2-characteristic would be constant. On the other hand, if U were a piecewise smooth admissible solution, then the trace of z along classical 1-characteristics and the trace of w along classical 2-characteristics would be step functions, with jumps at the points where the characteristic crosses shocks of the opposite family. Moreover, by classical theory, the sign of the jump would be fixed and the strength of the jump would be of cubic order in the strength of the crossed shock. It is interesting that the above essentially hold even in the context of weak solutions, for the extremal backward characteristics:

Theorem 2.2. *Let ξ be the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from any fixed point $(\overline{x}, \overline{t})$ of the upper half-plane. Set*

$$\overline{z}(t) = z(\xi(t)-, t)\ ,\quad \overline{w}(t) = w(\xi(t)+, t)\ ,\quad 0 \le t \le \overline{t}. \tag{2.10}$$

Then $\overline{z}(\cdot)$ (or $\overline{w}(\cdot)$) is a nonincreasing saltus function whose variation is concentrated in the set of jump points of $\overline{w}(\cdot)$ (or $\overline{z}(\cdot)$). Furthermore, if $\tau \in (0, \overline{t})$ is any point of jump discontinuity of $\overline{z}(\cdot)$ (or $\overline{w}(\cdot)$), then

$$\overline{z}(\tau-) - \overline{z}(\tau+) \le a[\overline{w}(\tau+) - \overline{w}(\tau)]^3\ , \tag{2.11_1}$$

or

$$\overline{w}(\tau-) - \overline{w}(\tau+) \le a[\overline{z}(\tau+) - \overline{z}(\tau)]^3\ , \tag{2.11_2}$$

where a is a constant depending solely on F.

The proof of the above theorem is based on estimates induced by entropy inequalities and is quite lengthy. It is given in [7, Ch. XII]. For earlier proofs, requiring more restrictive structural conditions on U, see [5;6].

Since we are operating in the realm of solutions with small oscillation, (2.11) imply that z and w are nearly constant along the extremal backward characteristics of the corresponding family. From the perspective of the present approach, it is this property that induces the decoupling of the two characteristic families and thereby all the distinctive properties of solutions of our system that will be presented in following sections.

3. Bounds on the Variation

A priori bounds are reported here on admissible weak solutions U of (2.1) of class BV_{loc}, with small oscillation, which satisfy the structural condition laid down in Section 2. They are similar to the estimates derived in [12], in the context of the random choice scheme. The proofs are found in [7, Ch. XII] or, under somewhat stronger assumptions on U, in [19].

The solution is conveniently monitored through its Riemann invariants (z, w). The oscillation is controlled by a small positive constant δ:

$$|z(x,t)| + |w(x,t)| < 2\delta\ ,\quad -\infty < x < \infty\ ,\quad 0 < t < \infty\ . \tag{3.1}$$

The first set of estimates depends on the initial data. We assume

$$\sup_{(-\infty,\infty)} |z(\cdot,0)| + \sup_{(-\infty,\infty)} |w(\cdot,0)| \le \delta\ , \tag{3.2}$$

$$TV_{(-\infty,\infty)} z(\cdot,0) + TV_{(-\infty,\infty)} w(\cdot,0) < b\delta^{-1}\ , \tag{3.3}$$

where b is a fixed, small constant, independent of δ. Thus, there is a tradeoff, allowing for arbitrarily large total variation at the expense of keeping the oscillation sufficiently small.

Theorem 3.1. *Consider any space-like curve $t = t^*(x)$, $x_\ell \le x \le x_r$, in the upper half-plane, along which the trace of (z, w) is denoted by (z^*, w^*). Then*

$$TV_{[x_\ell,x_r]} z^*(\cdot) \le TV_{[\xi_\ell(0),\xi_r(0)]} z(\cdot,0) + c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]} z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]} w(\cdot,0)\}\ , \tag{3.4_1}$$

$$TV_{[x_\ell,x_r]} w^*(\cdot) \le TV_{[\zeta_\ell(0),\zeta_r(0)]} w(\cdot,0) \tag{3.4_2}$$

$$+c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0)+TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)\},$$

where $\xi_\ell(\cdot),\xi_r(\cdot)$ are the minimal backward 1-characteristics and $\zeta_\ell(\cdot),\zeta_r(\cdot)$ are the maximal backward 2-characteristics emanating from the end-points (x_ℓ,t_ℓ) and (x_r,t_r) of the graph of $t^*(\cdot)$.

The estimates (3.4) reflect the fact that z and w are nearly constant along minimal backward 1-characteristics and maximal backward 2-characteristics, respectively. Indeed, we notice that the effect of the coupling of the two characteristic families is $O(\delta^2)$.

Since generalized characteristics are space-like curves, one may combine Theorems 2.2 and 3.1 to deduce the following corollary:

Theorem 3.2. *For any point (x,t) of the upper half-plane:*

$$\sup_{(-\infty,\infty)} z(\cdot,0)\geq z(x,t)\geq \inf_{(-\infty,\infty)} z(\cdot,0)-cb\delta\ , \tag{3.5$_1$}$$

$$\sup_{(-\infty,\infty)} w(\cdot,0)\geq w(x,t)\geq \inf_{(-\infty,\infty)} w(\cdot,0)-cb\delta\ . \tag{3.5$_2$}$$

Thus, on account of (3.2) and by selecting b sufficiently small, we secure a posteriori that the solution will satisfy (3.1).

Due to the spreading of rarefaction waves, solutions acquire instantaneously bounded variation, independently of the initial data. This is reflected in the following proposition, which applies to any solution with small oscillation (3.1), without any assumptions on the initial data:

Theorem 3.3. *For any $-\infty<x<y<\infty$ and $t>0$,*

$$TV_{[x,y]}z(\cdot,t)+TV_{[x,y]}w(\cdot,t)\leq\beta\frac{y-x}{t}+\gamma\delta\ , \tag{3.6}$$

where β and γ are constants that may depend on F but are independent of the initial data.

The oscillation of the solution is also controlled by just the oscillation, and not the variation, of the initial data:

Theorem 3.4. *There is a positive constant κ, depending solely on F, such that solutions generated by initial data with small oscillation*

$$|z(x,0)|+|w(x,0)|<\kappa\delta^2\ ,\qquad -\infty<x<\infty\ , \tag{3.7}$$

but unrestricted, possibly infinite, total variation, satisfy (3.1).

4. Regularity of Solutions

The invariance of the system (2.1) under uniform stretching of the space-time variables suggests that, in the vicinity of any fixed point $(\overline{x},\overline{t})$ of the upper half-plane, the solution U should behave like a self-similar solution relative to that point: In the most general situation, shocks and/or centered compression waves converge and collide at $(\overline{x},\overline{t})$ to produce a jump discontinuity which is then resolved into an outgoing fan of shocks and/or rarefaction waves, corresponding to the solution of a Riemann problem. Indeed, such behavior has been established in [9], for solutions constructed by the random choice scheme. See also [2]. Similar results will be reported here for our solution U, which satisfies the conditions laid down in Section 2. The proofs, found in [7, Ch. XII], rely heavily on Theorem 2.2.

With any fixed point $(\overline{x}, \overline{t})$ of the upper half-plane, we associate eight generalized characteristics emanating from it, namely, four backward, $\xi_-, \xi_+, \zeta_-, \zeta_+$, and four forward, $\phi_-, \phi_+, \psi_-, \psi_+$, determined as follows: ξ_- is the minimal backward 1-characteristic, ξ_+ is the maximal backward 1-characteristic, ζ_- is the minimal backward 2-characteristic, ζ_+ is the maximal backward 2-characteristic, ϕ_+ is the maximal forward 1-characteristic, and ψ_- is the minimal forward 2-characteristic. For $t > \overline{t}$, $\phi_-(t)$ is identified by the property that the minimal backward 1-characteristic ξ emanating from any point (x, t) is intercepted by the $\overline{t}$-time line at $\xi(\overline{t})$, with $\xi(\overline{t}) < \overline{x}$ if $x < \phi_-(t)$ and $\xi(\overline{t}) \geq \overline{x}$ if $x \geq \phi_-(t)$. Similarly, $\phi_+(t)$ is identified by the property that the maximal backward 2-characteristic ζ emanating from any point (x, t) is intercepted by the $\overline{t}$-time line at $\zeta(\overline{t})$, with $\zeta(\overline{t}) > \overline{x}$ if $x > \psi_+(t)$ and $\zeta(\overline{t}) \leq \overline{x}$ if $x \leq \psi_+(t)$. Of course, the above eight characteristics are not necessarily distinct: we may have coincidence of ξ_- with ξ_+, ζ_- with ζ_+, ϕ_- with ϕ_+, and/or ψ_- with ψ_+.

The characteristics $\xi_-, \xi_+, \zeta_-, \zeta_+, \phi_-, \phi_+, \psi_-$ and ψ_+ border regions

$$\mathcal{S}_W = \{(x,t) : x < \overline{x},\ \zeta_-^{-1} = (x) < t < \phi_-^{-1}(x)\}\ , \tag{4.1}$$

$$\mathcal{S}_E = \{(x,t) : x > \overline{x},\ = \xi_+^{-1}(x) < t < \psi_+^{-1}(x)\}\ , \tag{4.2}$$

$$\mathcal{S}_N = \{(x,t) : t > \overline{t},\ \phi_+(t) < x < \psi_-(t)\}\ , \tag{4.3}$$

$$\mathcal{S}_S = \{(x,t) : t < \overline{t},\ \zeta_+(t) < x < \xi_-(t)\}. \tag{4.4}$$

Theorem 4.1. *The solution U, with Riemann invariants (z, w), has the following properties, at any fixed point $(\overline{x}, \overline{t})$ of the upper half-plane:*

(a) *As (x,t) tends to $(\overline{x}, \overline{t})$ through any one of the four regions $\mathcal{S}_W, \mathcal{S}_E, \mathcal{S}_N$* or *$\mathcal{S}_S$ $(z(x,t), w(x,t))$ tend to respective limits $(z_W, w_W), (z_E, w_E), (z_N, w_N)$ or (z_S, w_S). In particular $z_W = z(\overline{x}-, \overline{t}), w_W = w(\overline{x}-, \overline{t})$, $z_E = z(\overline{x}+, \overline{t}), w_E = w(\overline{x}+, \overline{t})$.*

$(\mathrm{b})_1$ *If $p_\ell(\cdot)$ and $p_r(\cdot)$ are any two backward* 1-*characteristics emanating from $(\overline{x}, \overline{t})$, with $\xi_-(t) \leq p_\ell(t) < p_r(t) \leq \xi_+(t)$, for $t < \overline{t}$, then*

$$\begin{aligned} z_S = \lim_{t\uparrow\overline{t}} z(\xi_-(t)\pm, t) &\leq \lim_{t\uparrow\overline{t}} z(p_\ell(t)-, t) \leq \lim_{t\uparrow\overline{t}} z(p_\ell(t)+, t) \\ &\leq \lim_{t\uparrow\overline{t}} z(p_r(t)-, t) \leq \lim_{t\uparrow\overline{t}} z(p_r(t)+, t) \leq \lim_{t\uparrow\overline{t}} z(\xi_+(t)\pm, t) = z_E\ , \end{aligned} \tag{4.5$_1$}$$

$$\begin{aligned} w_S = \lim_{t\uparrow\overline{t}} w(\xi_-(t)\pm, t) &\geq \lim_{t\uparrow\overline{t}} w(p_\ell(t)-, t) \geq \lim_{t\uparrow\overline{t}} w(p_\ell(t)+, t) \\ &\geq \lim_{t\uparrow\overline{t}} w(p_r(t)-, t) \geq \lim_{t\uparrow\overline{t}} w(p_r(t)+, t) \geq \lim_{t\uparrow\overline{t}} w(\xi_+(t)\pm, t) = w_E. \end{aligned} \tag{4.6$_1$}$$

$(\mathrm{b})_2$ *If $q_\ell(\cdot)$ and $q_r(\cdot)$ are any two backward* 2-*characteristics emanating from $(\overline{x}, \overline{t})$, with $\zeta_-(t) \leq q_\ell(t) < q_r(t) \leq \zeta_+(t)$, for $t < \overline{t}$, then*

$$\begin{aligned} w_W = \lim_{t\uparrow\overline{t}} w(\zeta_-(t)\pm, t) &\geq \lim_{t\uparrow\overline{t}} w(q_\ell(t)-, t) \geq \lim_{t\uparrow\overline{t}} w(q_\ell(t)+, t) \\ &\geq \lim_{t\uparrow\overline{t}} w(q_r(t)-, t) \geq \lim_{t\uparrow\overline{t}} w(q_r(t)+, t) \geq \lim_{t\uparrow\overline{t}} w(\zeta_+(t)\pm, t) = w_S\ , \end{aligned} \tag{4.5$_2$}$$

$$\begin{aligned} z_W = \lim_{t\uparrow\overline{t}} z(\zeta_-(t)\pm, t) &\leq \lim_{t\uparrow\overline{t}} z(q_\ell(t)-, t) \leq \lim_{t\uparrow\overline{t}} z(q_\ell(t)+, t) \\ &\leq \lim_{t\uparrow\overline{t}} z(q_r(t)-, t) \leq \lim_{t\uparrow\overline{t}} z(q_r(t)+, t) \leq \lim_{t\uparrow\overline{t}} z(\zeta_+(t)\pm, t) = z_S. \end{aligned} \tag{4.6$_2$}$$

(c)$_1$ *If* $\phi_-(t) = \phi_+(t)$, *for* $\bar{t} < t < \bar{t} + s$, *then* $z_W \leq z_N, w_W \geq w_N$. *On the other hand, if* $\phi_-(t) < \phi_+(t)$, *for* $\bar{t} < t < \bar{t} + s$, *then, as* (x,t) *tends to* $(\bar{x},\bar{t})$ *through the region* $\{(x,t) : t > \bar{t},\ \phi_-(t) < x < \phi_+(t)\}, w(x,t)$ *tends to* w_W. *Furthermore, if* $p_\ell(\cdot)$ *and* $p_r(\cdot)$ *are any two forward* 1*-characteristics issuing from* $(\bar{x},\bar{t})$, *with* $\phi_-(t) \leq p_\ell(t) \leq p_r(t) \leq \phi_+(t)$, *for* $\bar{t} < t < \bar{t} + s$, *then*

$$(4.7)_1 \qquad z_W = \lim_{t\downarrow\bar{t}} z(\phi_-(t)\pm, t) \geq \lim_{t\downarrow\bar{t}} z(p_\ell(t)-, t) = \lim_{t\downarrow\bar{t}} z(p_\ell(t)+, t)$$

$$\geq \lim_{t\downarrow\bar{t}} z(p_r(t)-, t) = \lim_{t\downarrow\bar{t}} z(p_r(t)+, t) \geq \lim_{t\downarrow\bar{t}} z(\phi_+(t)\pm, t) = z_N.$$

(c)$_2$ *If* $\psi_-(t) = \psi_+(t)$, *for* $\bar{t} < t < \bar{t} + s$, *then* $w_N \geq w_E,\ \ z_N \leq z_E$. *On the other hand, if* $\psi_-(t) < \psi_+(t)$, *for* $\bar{t} < t < \bar{t} + s$, *then, as* (x,t) *tends to* $(\bar{x},\bar{t})$ *through the region* $\{(x,t) : t > \bar{t},\ \psi_-(t) < x < \psi_+(t)\}, z(x,t)$ *tends to* z_E. *Furthermore, if* $q_\ell(\cdot)$ *and* $q_r(\cdot)$ *are any two forward* 2*-characteristics issuing from* $(\bar{x},\bar{t})$, *with* $\psi_-(t) \leq q_\ell(t) \leq q_r(t) \leq \psi_+(t)$, *for* $\bar{t} < t < \bar{t} + s$, *then*

$$(4.7)_2 \qquad w_N = \lim_{t\downarrow\bar{t}} w(\psi_-(t)\pm, t) \leq \lim_{t\downarrow\bar{t}} w(q_\ell(t)-, t) = \lim_{t\downarrow\bar{t}} w(q_\ell(t)+, t)$$

$$\leq \lim_{t\downarrow\bar{t}} w(q_r(t)-, t) = \lim_{t\downarrow\bar{t}} w(q_r(t)+, t) \leq \lim_{t\downarrow\bar{t}} w(\psi_+(t)\pm, t) = w_E.$$

Statements (b)$_1$ and (b)$_2$ regulate the incoming waves, allowing for any combination of admissible shocks and focussing compression waves. Statements (c)$_1$ and (c)$_2$ characterize the outgoing wave fan. In particular, (c)$_1$ implies that the state (z_W, w_W), on the left, may be joined with the state (z_N, w_N), on the right, by a 1-rarefaction wave or admissible 1-shock; while (c)$_2$ imples that the state (z_N, w_N), on the left, may be joined with the state (z_E, w_E), on the right, by a 2-rarefaction wave or admissible 2-shock. Thus, the outgoing wave fan is locally approximated by the solution of the Riemann problem with end-states $(z(\bar{x}-,\bar{t}), w(\bar{x}-,\bar{t}))$ and $(z(\bar{x}+,\bar{t}), w(\bar{x}+,\bar{t}))$.

In Section 2 we noted that membership in $BV_{\rm loc}$ endows the solution U with certain regularity. This is now improved, in consequence of Theorem 4.1: Any point $(\bar{x},\bar{t}) \in \mathcal{C}$ of approximate continuity is actually a point of continuity, characterized by the property that the four states $(z_W, w_W), (z_E, w_E), (z_N, w_N)$ and (z_S, w_S) coincide. Similarly, any point $(\bar{x},\bar{t}) \in \mathcal{J}$ of the shock set is a point of jump discontinuity, characterized by either $(z_W, w_W) = (z_S, w_S) \neq (z_E, w_E) = (z_N, w_N)$, for 1-shocks, or $(z_W, w_W) = (z_N, w_N) \neq (z_E, w_E) = (z_S, w_S)$, for 2-shocks. Finally, $\mathcal{I}$ comprises all points $(\bar{x},\bar{t})$ for which at least three of the four states $(z_W, w_W), (z_E, w_E), (z_N, w_N)$ and (z_S, w_S) are distinct. It can be shown that $\mathcal{I}$ is at most countable.

The focussing of characteristics, induced by genuine nonlinearity, is responsible for the demise of Lipschitz continuity and the generation of shocks. However, this same pattern, viewed in reverse time, has the opposite effect of lowering the Lipschitz constant of the solution. This "schizophrenic" role of genuine nonlinearity is reflected in the following

Theorem 4.2. *Assume the set* $\mathcal{C}$ *of points of continuity of the solution* U *has nonempty interior* $\mathcal{C}^0$. *Then* U *is locally Lipschitz continuous on* $\mathcal{C}^0$.

5. Initial Data in L^1

Genuine nonlinearity gives rise to a host of dissipative mechanisms that affect the large time behavior of solutions. The following proposition reports $O(t^{-\frac{1}{2}})$ decay when the initial data are summable. The proof is given in [7, Ch. XII].

Theorem 5.1. *When* $(z(\cdot,0),w(\cdot,0)) \in L^1(-\infty,\infty)$, *then, as* $t \to \infty$,

$$(z(\cdot,t),w(\cdot,t)) = O(t^{-\frac{1}{2}}) \ , \tag{5.1}$$

uniformly in x *on* $(-\infty,\infty)$.

6. Initial Data with Compact Support

Here we discuss the large time behavior of solutions with initial data $(z(\cdot,0),w(\cdot,0))$ that vanish outside a bounded interval $[-\ell,\ell]$. In the first place, by virtue of Theorem 5.1, the Riemann invariants decay at the rate $O(t^{-\frac{1}{2}})$. As a result, the two characteristic families decouple asymptotically, and each one develops a N-wave profile of width $O(t^{\frac{1}{2}})$ and strength $O(t^{-\frac{1}{2}})$, which propagates into the rest state at characteristic speed. This asymptotic portrait was established in [8;15;17], for solutions constructed by the random choice scheme. For BV solutions satisfying the structural condition, the study of the spreading of generalized characteristics leads to the following, sharp result, whose proof is given in [7, Ch. XII]:

Theorem 6.1. *Employing the notation introduced in Section 4, consider the special forward characteristics* $\phi_-(\cdot),\psi_-(\cdot)$ *issuing from* $(-\ell,0)$ *and* $\phi_+(\cdot),\psi_+(\cdot)$ *issuing from* $(\ell,0)$. *Then*

(a) *For* t *large,* ϕ_-,ψ_-,ϕ_+ *and* ψ_+ *propagate according to*

$$\phi_-(t) = \lambda(0,0)t - (p_-t)^{\frac{1}{2}} + O(1) \ , \tag{6.1$_1$}$$

$$\psi_+(t) = \mu(0,0)t + (q_+t)^{\frac{1}{2}} + O(1) \ , \tag{6.1$_2$}$$

$$\phi_+(t) = \lambda(0,0)t + (p_+t)^{\frac{1}{2}} + O(t^{\frac{1}{4}}) \ , \tag{6.2$_1$}$$

$$\psi_-(t) = \mu(0,0)t - (q_-t)^{\frac{1}{2}} + O(t^{\frac{1}{4}}) \ , \tag{6.2$_2$}$$

where p_-,p_+,q_- *and* q_+ *are nonnegative constants.*

(b) *For* $t>0$ *and either* $x<\phi_-(t)$ *or* $x>\psi_+(t)$,

$$z(x,t)=0 \ , \qquad w(x,t)=0. \tag{6.3}$$

(c) *For* t *large,*

$$TV_{[\phi_-(t),\psi_+(t)]}z(\cdot,t) + TV_{[\phi_-(t),\psi_+(t)]}w(\cdot,t) = O(t^{-\frac{1}{2}}). \tag{6.4}$$

(d) *For* t *large and* $\phi_-(t)<x<\phi_+(t)$,

$$\lambda(z(x,t),0) = \frac{x}{t} + O(\frac{1}{t}) \ , \tag{6.5$_1$}$$

while for $\psi_-(t)<x<\psi_+(t)$,

$$\mu(0,w(x,t)) = \frac{x}{t} + O(\frac{1}{t}). \tag{6.5$_2$}$$

(e) *For t large and $x > \phi_+(t)$, if $p_+ > 0$ then*

$$0 \le -z(x,t) \le c[x - \lambda(0,0)t]^{-\frac{3}{2}}, \tag{6.6$_1$}$$

while for $x < \psi_-(t)$, if $q_- > 0$ then

$$0 \le -w(x,t) \le c[\mu(0,0)t - x]^{-\frac{3}{2}}. \tag{6.6$_2$}$$

Thus, in the wake of nondegenerate N-waves the Riemann invariants decay at the rate $O(t^{-\frac{3}{4}})$. In cones properly contained in the wake, the decay is even faster, $O(t^{-\frac{3}{2}})$.

The pointwise decay estimates of Theorem 6.1 induce the following asymptotic behavior of solutions in $L^1(-\infty,\infty)$:

Theorem 6.2. *Assume $p_+ > 0$ and $q_- > 0$. Then, as $t \to \infty$,*

$$\|U(x,t) - M(x,t;p_-,p_+)R(0,0) - N(x,t;q_-,q_+)S(0,0)\|_{L^1(-\infty,\infty)} = O(t^{-\frac{1}{4}}), \tag{6.7}$$

where M and N denote the N-wave profiles:

$$M(x,t;p_-,p_+) = \begin{cases} \dfrac{x - \lambda(0,0)t}{\lambda_z(0,0)t}\ , & \text{for } -(p_-t)^{\frac{1}{2}} \le x - \lambda(0,0)t \le (p_+t)^{\frac{1}{2}} \\ 0 & \text{otherwise}\ , \end{cases} \tag{6.8$_1$}$$

$$N(x,t;q_-,q_+) = \begin{cases} \dfrac{x - \mu(0,0)t}{\mu_w(0,0)t}\ , & \text{for } -(q_-t)^{\frac{1}{2}} \le x - \mu(0,0)t \le (q_+t)^{\frac{1}{2}} \\ 0 & \text{otherwise.} \end{cases} \tag{6.8$_2$}$$

7. Periodic Solutions

The study of genuinely nonlinear systems of two conservation laws will be completed with a discussion of the large time behavior of solutions that are periodic,

$$U(x+\ell,t) = U(x,t)\ , \quad -\infty < x < \infty\ , \quad t > 0\ , \tag{7.1}$$

and have zero mean:

$$\int_y^{y+\ell} U(x,t)dx = 0\ , \qquad -\infty < y < \infty\ , \quad t > 0. \tag{7.2}$$

The confinement of waves resulting from periodicity induces active interactions and cancellation. As a result, the total variation per period decays at the rate $O(t^{-1})$:

Theorem 7.1. *For any $x \in (-\infty,\infty)$,*

$$TV_{[x,x+\ell]}z(\cdot,t) + TV_{[x,x+\ell]}w(\cdot,t) \le \frac{\beta\ell}{t}. \tag{7.3}$$

The proof of (7.3), originally given in [12], is an immediate corollary of (3.6) and periodicity.

An important feature of periodic solutions is the existence of divides. A *divide* of the first (or second) characteristic family associated with U, is a curve $\chi : [0,\infty) \to (-\infty,\infty)$ with the property that for any $t \in [0,\infty)$ the restriction of χ to the interval $[0,t]$ coincides with the minimal (or maximal) backward characteristic of the first (or second) family emanating from the point $(\chi(t),t)$. It can be shown

[6] that in the case of our periodic solution at least one (and probably generically just one) divide of each family originates from any interval of the x-axis of period length ℓ. Of course, the ℓ-translate of any divide is necessarily a divide.

As shown in [6;7, Ch. XII], an interesting mechanism is at work here which decouples the two characteristic families, as $t \to \infty$, and induces the formation of saw-toothed shaped profiles, familiar in the case of scalar conservation laws [14], of strength $O(t^{-1})$:

Theorem 7.2. *The upper half-plane is partitioned by divides of the first (or second) family along which z (or w) decays at the rate $O(t^{-2})$. Let $\chi_-(\cdot)$ and $\chi_+(\cdot)$ be any two adjacent divides of the first (or second) family, with $\chi_-(t) < \chi_+(t)$. Then $\chi_+(t) - \chi_-(t)$ approaches a constant at the rate $O(t^{-1})$, as $t \to \infty$. Furthermore, between χ_- and χ_+ lies a characteristic ψ, of the first (or second) family, such that, as $t \to \infty$,*

$$\psi(t) = \frac{1}{2}[\chi_-(t) + \chi_+(t)] + o(1) \ , \tag{7.4}$$

$$\lambda_z(0,0)z(x,t) = \begin{cases} \dfrac{x-\chi_-(t)}{t} + o(\dfrac{1}{t}) \ , & \chi_-(t) < x < \psi(t) \ , \\[2ex] \dfrac{x-\chi_+(t)}{t} + o(\dfrac{1}{t}) \ , & \psi(t) < x < \chi_+(t) \ , \end{cases} \tag{7.5_1}$$

or

$$\mu_w(0,0)w(x,t) = \begin{cases} \dfrac{x-\chi_-(t)}{t} + o(\dfrac{1}{t}) \ , & \chi_-(t) < x < \psi(t) \ , \\[2ex] \dfrac{x-\chi_+(t)}{t} + o(\dfrac{1}{t}) \ , & \psi(t) < x < \chi_+(t). \end{cases} \tag{7.5_2}$$

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