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Formation of Singularities in Relativistic Fluid Dynamics and in Spherically Symmetric Plasma Dynamics

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1. Introduction

Quasilinear hyperbolic systems have a special place in the theory of partial differential equations since most of the PDEs arising in continuum physics are of this form. Well-known examples are the Euler equations for a perfect compressible fluid, the equations of elastodynamics for a perfect elastic solid, and equations describing a variety of field-matter interactions, such as magnetohydrodynamics etc. It is well-known that for all these systems the Cauchy problem is well-posed, i.e., it has a unique classical solution in a small neighborhood (in space-time) of the hypersurface on which the initial data are given.

On the other hand, it is not expected that these systems will have a global-in-time regular solution, because shock discontinuities are expected to form at some point, at least as long as the initial data are not very small. In more than one space dimension, there are no general theorems to that effect however, mainly because in higher dimensions, the method of characteristics, which is a powerful tool in one dimension for the study of hyperbolic systems, becomes intractable.

In 1985 T. C. Sideris published a remarkable paper on the formation of singularities in three-dimensional compressible fluids [**13**], proving that the classical solution to Euler equations has to break down in finite time. His proof was based on studying certain averaged quantities formed out of the solution, showing that they satisfy differential inequalities whose solutions have finite life-span. Such a technique was already employed by Glassey [**3**] in the case of a nonlinear Schrödinger equation. The idea is that by using averaged quantities one is able to avoid local analysis of solutions. The same technique was subsequently used to prove other formation of singularity theorems: for a compressible fluid body surrounded by vacuum in the nonrelativistic [**7**] and relativistic [**10**] cases, for the spherically symmetric Euler-Poisson equations in the attractive [**6**] and repulsive [**8**] cases, for

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magnetohydrodynamics [**9**], and for elastodynamics [**14**]. In this paper we present two more such "Siderian" blowup theorems: one in relativistic fluid mechanics, and the other in plasma dynamics.

2. Relativistic Fluid Dynamics

Let (M, g) be the Minkowski spacetime, with (x^μ), $\mu = 0, \ldots, 3$ the global coordinate system on M in which $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. We will use the standard convention that Greek indices run from 0 to 3, while Latin ones run from 1 to 3. Indices are raised and lowered using the metric tensor g, and all up-and-down repeated indices are summed over the range. We also denote $t = x^0$ and $\mathbf{x} = (x^1, x^2, x^3)$. In the following, we adopt the notation and terminology of [**1**] and quote from it some of the basic facts regarding relativistic dynamics:

The energy tensor for a relativistic perfect fluid is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}. \tag{2.1}$$

In this formula,

1. $u = (u^0, \mathbf{u})$ is the four-velocity field of the fluid, a unit future-directed time-like vectorfield on M, so that $g(u, u) = -1$ and hence
$$u^0 = \sqrt{1 + |\mathbf{u}|^2}.$$
We note that here, unlike the nonrelativistic case considered by Sideris, *all* components of the energy tensor are quadratic in the velocity.
2. $\rho \geq 0$ is the *proper energy density* of the fluid, the eigenvalue of T corresponding to the eigenvector u. It is a function of the (nonnegative) thermodynamic variables n, the *number density* and s, the *entropy per particle.* The particular dependence of ρ on these variables is given by the *equation of state*
$$\rho = \rho(n, s). \tag{2.2}$$
3. $p \geq 0$ is the fluid pressure, defined by
$$p = n\frac{\partial \rho}{\partial n} - \rho. \tag{2.3}$$
Basic assumptions on the equation of state of a perfect fluid are
$$\frac{\partial \rho}{\partial n} > 0, \quad \frac{\partial p}{\partial n} > 0, \quad \frac{\partial \rho}{\partial s} \geq 0 \text{ and } = 0 \text{ iff } s = 0. \tag{2.4}$$
In particular, these insure that η, the speed of sound in the fluid, is always real:
$$\eta^2 := \left(\frac{dp}{d\rho}\right)_s.$$
In addition, the energy tensor (2.1) must satisfy the positivity condition, which implies that we must have
$$p \leq \rho. \tag{2.5}$$

A typical example of an equation of state is that of a polytropic gas. A perfect fluid is called *polytropic* if the equation of state is of the form

$$\rho = n + \frac{A(s)}{\gamma - 1} n^\gamma, \tag{2.6}$$

where $1 < \gamma < 2$ and A is a positive increasing function of s (The speed of light is equal to one). This implies that $p = An^\gamma$ and thus the sound speed $\eta(n, s)$ is determined as follows:

$$\eta^2 = \left(\frac{dp}{d\rho}\right)_s = \frac{\partial p/\partial n}{\partial \rho/\partial n} = \frac{\gamma(\gamma-1)An^{\gamma-1}}{\gamma - 1 + \gamma A n^{\gamma-1}}.$$

In particular, the sound speed is increasing with density and is bounded above by $\sqrt{\gamma - 1}$.

The equations of motion for a relativistic perfect fluid are:

$$\partial_\nu T^{\mu\nu} = 0. \tag{2.7}$$

Moreover, $n = n(x)$ satisfies the *continuity equation*

$$\partial_\nu (n u^\nu) = 0. \tag{2.8}$$

Given an equation of state (2.2), the system of equations (2.7-2.8) provides 5 equations for the 5 unknowns $n(x)$, $s(x)$ and $\mathbf{u}(x)$. The component of (2.7) in the direction of u is

$$u^\nu \partial_\nu \rho + (\rho + p)\partial_\nu u^\nu = 0. \tag{2.9}$$

As long as the solution is C^1, this is equivalent to the *adiabatic condition*

$$u^\nu \partial_\nu s = 0. \tag{2.10}$$

The component of (2.7) in the direction orthogonal to u is

$$(\rho + p) u^\nu \partial_\nu u^\mu + h^{\mu\nu}\partial_\nu p = 0, \tag{2.11}$$

where

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$$

is the projection tensor onto the orthogonal complement of $u(x)$ in $T_x M$.

Thus the system of equations for a relativistic fluid can be written as follows:

$$\left\{ \begin{array}{rcl} \partial_\nu (n u^\nu) &=& 0, \\ (\rho + p) u^\nu \partial_\nu u^\mu + h^{\mu\nu}\partial_\nu p &=& 0, \\ u^\nu \partial_\nu s &=& 0. \end{array} \right. \tag{2.12}$$

The *Cauchy problem* for a relativistic fluid consists of specifying the values of n, s and $\mathbf{u}$ on a spacelike hypersurface Σ_0 of M,

$$n_{|\Sigma_0} = n_0, \quad s_{|\Sigma_0} = s_0, \quad \mathbf{u}_{|\Sigma_0} = \mathbf{u}_0, \tag{2.13}$$

and finding a solution $(n, \mathbf{u}, s)$ to (2.12,2.13) in a neighborhood of Σ_0 in M. In particular, let $\Sigma_0 = \mathbb{R}^3 \times \{0\}$ be the hyperplane $t = 0$ in M and suppose the initial data (2.13) correspond to a smooth compactly supported perturbation of a quiet fluid filling the space, i.e., assume

(2.14) n_0, s_0 and $\mathbf{u}_0$ are smooth functions on $\mathbb{R}^3$ and there are positive constants R_0, $\bar{n}$ and $\bar{s}$ such that outside the ball $B_{R_0}(0)$ we have $n_0 = \bar{n}$, $s_0 = \bar{s}$, and $\mathbf{u}_0 = 0$.

Let $\bar{\eta} = \eta(\bar{n}, \bar{s})$ be the sound speed in the background quiet state. We then have

Proposition 2.1. *Any C^1 solution of* (2.12,2.13,2.14) *will satisfy*

$$n = \bar{n}, \quad s = \bar{s}, \quad \mathbf{u} = 0,$$

outside the ball $B_{R(t)}(0)$ where $R(t) = R_0 + \bar{\eta} t$.

Proof. It is enough to check that the system (2.12) can be written in symmetric hyperbolic form:

$$A^{\mu}_{ij}(U)\partial_\mu U^j = 0 \text{ where } A^{\mu}_{ij} = A^{\mu}_{ji} \text{ and } A^0_{ij} \text{ is positive definite.} \tag{2.15}$$

This can be accomplished for example by using p instead of n as an unknown. By (2.4), we can think of n as a function of p and and s and thus of ρ as a function of p and s. By (2.3) it is then easy to see that (2.12) is equivalent to the following system for the unknowns $U = (p, u, s)$:

$$\left\{ \begin{array}{rcl} \frac{1}{(\rho+p)\eta} u^\nu \partial_\nu p + \eta \partial_\nu u^\nu & = & 0 \\ \eta h^{\mu\nu}\partial_\nu p + (\rho+p)\eta u^\nu \partial_\nu u^\mu & = & 0 \\ u^\nu \partial_\nu s & = & 0. \end{array} \right. \tag{2.16}$$

Let $\bar{U} = (\bar{p}, 1, 0, 0, 0, \bar{s})$ denote the constant background solution to (2.16). Let $\bar{\zeta} := (\bar{\rho} + \bar{p})\bar{\eta} > 0$. We then have that the differential operator $P = \bar{A}^\mu \partial_\mu$ corresponding to the linearization of (2.15) at $\bar{U}$ is symmetric hyperbolic, with

$$\bar{A}^0 = A^0(\bar{U}) = \mathrm{diag}(\frac{1}{\bar{\zeta}}, \bar{\zeta}, \bar{\zeta}, \bar{\zeta}, \bar{\zeta}, 1), \qquad \bar{A}^i = A^i(\bar{U}) = \begin{pmatrix} 0 & 0 & \bar{\eta}\mathbf{e}_i^T & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\eta}\mathbf{e}_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Once we have this, we can use energy estimates, as in [**12**], to conclude the desired domain of dependence statement. □

We now prove that for large enough initial data, the solution to (2.12,2.13,2.14) cannot remain C^1 for all $t > 0$. Such a result was announced in [**10**], but the unpublished proof contained an error which invalidated the argument [**11**].

First of all, a scaling analysis shows that without loss of generality we can set $R_0 = 1$. Let

$$B_t = \{x \in M \mid x^0 = t,\ |\mathbf{x}| \le R(t) = 1 + \bar{\eta}t\}$$

denote the time t slice of the range of influence of the data, and let

$$Q(t) := \int_{\mathbb{R}^3} g_{ij} x^i T^{0j} = \int \mathbf{x}\cdot \mathbf{u} u^0 (\rho + p) \tag{2.17}$$

be the total radial momentum of the fluid at time t. We then have

$$\begin{aligned} Q'(t) &= \int g_{ij} x^i \partial_0 T^{0j} = -\int g_{ij} x^i \partial_k T^{kj} \\ &= \int g_{ij}(T^{ij} - \bar{T}^{ij}) = \int (\rho+p)|\mathbf{u}|^2 + 3(p - \bar{p}). \end{aligned} \tag{2.18}$$

Let

$$E = \int_{\mathbb{R}^3} T^{00} - \bar{T}^{00} = \int (\rho + p)|\mathbf{u}|^2 + \rho - \bar{\rho} \tag{2.19}$$

be the total energy of the perturbation. By (2.7) it is a conserved quantity, $E = E_0$. Our goal is to use E to obtain a differential inequality for Q that would lead to blowup.

We are going to make two assumptions on the equation of state of the fluid, which are quite natural from a physical point of view. First we note that, as mentioned before, we can use the pressure p as a thermodynamic variable in place of n. The equation of state of the fluid then has the form $\rho = \rho(p,s)$. The two assumptions are:

(A1) $\rho(p,s)$ is a non-increasing function of s, for each p.
(A2) $\eta(p,s)$ is a non-decreasing function of p, for each s.

These two assumptions are in particular satisfied for a polytropic equation of state (2.6). In order to see that, we observe that $n = (p/A(s))^{1/\gamma}$, and from there we get

$$\rho(p,s) = \frac{1}{\gamma-1}p + \frac{1}{A^{1/\gamma}(s)}p^{1/\gamma}.$$

It is then clear that **(A1)** holds. Moreover

$$\eta^2(p,s) = \frac{\gamma(\gamma-1)A^{1/\gamma}(s)p^{(\gamma-1)/\gamma}}{\gamma-1+\gamma A^{1/\gamma}(s)p^{(\gamma-1)/\gamma}}$$

shows that **(A2)** is satisfied.

We also make the following assumptions on the initial data:

(D1) $\bar{\eta} < \frac{1}{3}$.
(D2) $E > 0$.
(D3) $s_0(\mathbf{x}) \geq \bar{s}$ for all $\mathbf{x} \in B_0$.

By (2.10), the entropy s is constant along the flow lines, and thus **(D3)** implies that $s(x) \geq \bar{s}$ for $x \in B_t$. By **(A1)** and **(A2)** we then have

$$\begin{aligned}\rho - \bar{\rho} &= \rho(p,s) - \rho(\bar{p},\bar{s}) = \rho(p,s) - \rho(p,\bar{s}) + \rho(p,\bar{s}) - \rho(\bar{p},\bar{s}) \leq \rho(p,\bar{s}) - \rho(\bar{p},\bar{s}) \\ &= \int_{\bar{p}}^{p} \frac{\partial\rho}{\partial p}(p',\bar{s})\,dp' = \int_{\bar{p}}^{p} \frac{1}{\eta^2(p',\bar{s})}\,dp' \leq \frac{1}{\bar{\eta}^2}(p-\bar{p}).\end{aligned}$$

By (2.18) and (2.19) we then obtain

$$Q'(t) \geq 3\bar{\eta}^2 E + (1-3\bar{\eta}^2)\int(\rho+p)|\mathbf{u}|^2,$$

which implies, by virtue of **(D1)** and **(D2)** that

$$Q'(t) \geq (1-3\bar{\eta}^2)\int(\rho+p)|\mathbf{u}|^2 > 0.$$

In particular $Q(t) > 0$ if $Q(0) > 0$.

On the other hand, we can always estimate $Q(t)$ from above, using (2.5):

$$\begin{aligned}Q^2(t) &\leq \left(\int(\rho+p)|\mathbf{u}|^2\right)R^2(t)\left(\int_{B_t}(\rho+p)(|\mathbf{u}|^2+1)\right) \\ &\leq 2\left(\int(\rho+p)|\mathbf{u}|^2\right)R^2(t)\left(\int_{B_t}(\rho+p)|\mathbf{u}|^2+\rho-\bar{\rho}+\bar{\rho}\right) \\ &\leq \frac{2}{1-3\bar{\eta}^2}Q'(t)R^2(t)[E+\frac{4\pi}{3}\bar{\rho}R^3(t)].\end{aligned}$$

Integrating this differential inequality and changing the integration variable to $r = R(t)$, we obtain

$$\frac{1}{Q(t)} \leq \frac{1}{Q(0)} - \frac{1-3\bar{\eta}^2}{2\bar{\eta}}\int_1^{R(t)}\frac{dr}{Er^2+\frac{4\pi}{3}\bar{\rho}r^5},$$

which contradicts the positivity of Q for all time provided the initial data satisfies the following final assumption:

(D4) $Q(0) > \dfrac{2\bar\eta}{1-3\bar\eta^2}\left(\int_1^\infty \dfrac{dr}{Er^2+\frac{4\pi}{3}\bar\rho r^5}\right)^{-1}.$

The contradiction implies that there exists a certain $T^* < \infty$ by which time a C^1 solution has to have broken down. In particular, the domain of dependence may break down at an earlier time, perhaps because a shock discontinuity forms. We have thus proved

THEOREM 2.2. *Suppose that the equation of state of a fluid satisfies* **(A1)** *and* **(A2)**. *Then the Cauchy problem* (2.12,2.13,2.14) *with initial data satisfying* **(D1–D4)** *cannot have a global-in-time* C^1 *solution.*

REMARK 2.3. It is easy to obtain a simpler, sufficient condition for blowup: Let

$$f(y) := \left(\int_1^\infty \frac{dr}{r^2(r^3+y)}\right)^{-1}.$$

By **(D4)** we thus need

$$Q(0) > \frac{2\bar\eta}{1-3\bar\eta^2}\frac{4\pi}{3}\bar\rho\, f\Big(\frac{E}{\frac{4\pi}{3}\bar\rho}\Big). \tag{2.20}$$

It is easy to see that $f(0) = 4$, $f'(0) = 16/7$ and that f is a concave function of y, so that $f(y) < \frac{16}{7}y + 4$. It is therefore enough to have

$$Q(0) > \frac{32\bar\eta}{7(1-3\bar\eta^2)}(E + \frac{7\pi}{3}\bar\rho). \tag{2.21}$$

We note that unlike the nonrelativistic case, the lower bound for the initial radial momentum in **(D4)** or (2.21) depends on the initial energy, and thus on the initial velocity. Since Q is of the same order of magnitude as E, it is worthwhile to show that there *exist* data sets satisfying these largeness conditions. In fact, (2.21) can be satisfied for $\bar n$ small enough. All that is needed is $\partial\rho/\partial n > 0$ at $n = 0$. We illustrate this in the following by considering the polytropic case.

Let us consider a fluid with a polytropic equation of state (2.6), and consider initial data of the following form

$$n_0(\mathbf{x}) = \bar n\psi(r), \qquad \mathbf{u}_0(\mathbf{x}) = \frac{\mathbf{x}}{r}\phi(r), \qquad s_0(\mathbf{x}) = \bar s + \phi(r), \tag{2.22}$$

where $r = |\mathbf{x}|$. ϕ and ψ are smooth, positive functions on $[0,\infty)$ such that

$$\phi(r) \equiv 0 \text{ for } r \geq 1, \qquad \phi(0) = 0, \tag{2.23}$$

and

$$\psi(r) \equiv 1 \text{ for } r \geq 1, \qquad \int_0^1 (\psi(r)-1)r^2 dr = 0. \tag{2.24}$$

We then compute

$$\begin{aligned} E &= \int_{B_0} (\rho_0 + p_0)|\mathbf{u}_0|^2 + \rho_0 - \bar\rho \\ &= 4\pi\bar n\int_0^1 \left\{\psi\phi^2 + \frac{1}{\gamma-1}\bar n^{\gamma-1}[A(s)\psi^\gamma(\gamma\phi^2+1) - A(\bar s)]\right\} r^2 dr, \end{aligned}$$

and thus $E > 0$ by **(D3)** and (2.24). Now,

$$Q(0) = 4\pi\bar n\int_0^1 \phi\sqrt{1+\phi^2}(\psi + A\frac{\gamma}{\gamma-1}\bar n^{\gamma-1}\psi^\gamma)r^3 dr.$$

Dividing (2.21) by $4\pi\bar{n}$, all we need is that the following inequality be satisfied for $\bar{n}$ small enough:

(2.25)
$$\int_0^1 \psi\phi\sqrt{1+\phi^2}r^3dr + O(\bar{n}^{\gamma-1}) > \frac{32}{7(1-3\bar{\eta}^2)}\bar{\eta}\left\{\int_0^1 \psi\phi^2r^2dr + \frac{7}{12} + O(\bar{n}^{\gamma-1})\right\}.$$

This is clearly true since $\bar{\eta} \to 0$ as $\bar{n} \to 0$. We have thus shown

PROPOSITION 2.4. *Let ϕ and ψ be two smooth, positive functions on* $[0,\infty)$ *satisfying* (2.23,2.24). *Then there exists $\bar{n} > 0$ small enough (depending on ϕ, ψ and γ) such that the initial data set $(n_0, \mathbf{u}_0, s_0)$ of the form* (2.22) *satisfy the conditions* **(D1–D4)**, *and thus lead to a blowup for* (2.12,2.13,2.14).

3. Euler-Maxwell with Constant Background Charge

A simple two-fluid model to describe plasma dynamics is the so called Euler-Maxwell system, where a compressible electron fluid interacts with a constant ion background. Let $n(t,\mathbf{x})$, $s(t,\mathbf{x})$ and $\mathbf{v}(t,\mathbf{x})$ be the average electron density, entropy, and velocity, let $\bar{n}$ be the constant ion density, and let $\mathbf{E}(t,\mathbf{x})$ and $\mathbf{B}(t,\mathbf{x})$ be the electric and magnetic fields. Let c = speed of light in vacuum, e = the charge of an electron, and m = the mass of an electron. The Euler-Maxwell system (see [**5**, pp. 490–491]) then takes the form:

$$\text{(3.1)}\qquad \left\{\begin{array}{rcl} \partial_t n + \partial_i(nv^i) &=& 0 \\ \partial_t \Pi^i + \partial_j T^{ij} &=& \dfrac{e\bar{n}}{m}E^i \\ \partial_t s + v^i\partial_i s &=& 0 \end{array}\right. \qquad \begin{array}{rcl} \partial_t B^i + c(\nabla\times\mathbf{E})^i &=& 0 \\ \partial_t E^i - c(\nabla\times\mathbf{B})^i &=& -4\pi env^i, \end{array}$$

together with the constraint equations

$$\text{(3.2)}\qquad \partial_i E^i = 4\pi e(n-\bar{n}), \qquad \partial_i B^i = 0.$$

In the above, Π is the momentum vector,

$$\Pi = n\mathbf{v} + \frac{1}{4\pi mc}(\mathbf{E}\times\mathbf{B}),$$

and T is the stress tensor, which can be decomposed into material and electromagnetic parts: $T = T_M + T_E$, with

$$\begin{aligned} T_M^{ij} &= nv^iv^j + \frac{1}{m}p\delta^{ij}, \\ T_E^{ij} &= \frac{1}{4\pi m}[\frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2)\delta^{ij} - E^iE^j - B^iB^j]. \end{aligned}$$

p is the electron pressure, which is modeled by a polytropic law $p(n,s) = A(s)n^\gamma$, where $\gamma > 1$ and A is a positive increasing function.

The system (3.1) being hyperbolic, we once again have the domain of dependence property. However, this time the largest characteristic speed in the background will be c, the speed of light. We recall that in Sideris's original argument [**13**], the largeness condition on the initial data implied that the initial velocity had to be supersonic at some point, relative to the sound speed in the background. An analogous result in the Euler-Maxwell case would thus require that the initial velocity be superluminar at some point, which is absurd. However, we note that if the data is spherically symmetric, so will be the solution, and thus there will be no electromagnetic waves, and the largest characteristic speed will once again be the sound speed, so a Siderian blowup theorem is possible in the spherically symmetric case. Moreover, since spherical symmetry implies that the flow is irrotational, such a blowup result is complementary to the recent construction [**4**] of global smooth irrotational solutions with small amplitude for the above system. We note that a blowup result in the spherically symmetric, isentropic case with no background charge has been obtained [**2**] using Riemann invariants.

Remark 3.1. Under the assumption of spherical symmetry, the Euler-Maxwell system reduces to what is often referred to as the spherically symmetric Euler-Poisson system (with repulsive force). We note the important distinction between this, and the general Euler-Poisson system obtained by taking the Newtonian limit $c \to \infty$ in (3.1). The latter is not a hyperbolic system, and does not have finite propagation speeds.

We have the following theorem:

Theorem 3.2. *Let ν_0, σ_0 and u_0 be smooth functions on $\mathbb{R}^+$ satisfying*

$$u_0(r) \equiv \sigma_0(r) \equiv \nu_0(r) \equiv 0 \text{ for } r \geq 1, \qquad u_0(0) = 0, \qquad \sigma_0(r) \geq 0,$$

and the neutrality condition

$$\int_0^1 \nu_0(r) r^2 dr = 0. \tag{3.3}$$

Let $\bar{s} \geq 0$ be fixed. Then

(a) *There exists $T > 0$ and functions $\nu, \sigma, u, E \in C^1([0,T) \times \mathbb{R}^+)$ such that*

$$\begin{array}{l} \nu(0,r) = \nu_0(r) \\ \sigma(0,r) = \sigma_0(r) \\ u(0,r) = u_0(r) \end{array} \qquad E(0,r) = \frac{4\pi e}{r^2}\int_0^r \nu_0(r') {r'}^2 dr'.$$

and such that the Euler-Maxwell system (3.1) *has a unique solution of the form:*

$$\begin{array}{rcl} n(t,\mathbf{x}) & = & \bar{n} + \nu(t,r) \\ s(t,\mathbf{x}) & = & \bar{s} + \sigma(t,r) \\ \mathbf{v}(t,\mathbf{x}) & = & u(t,r)\dfrac{\mathbf{x}}{r} \end{array} \qquad \begin{array}{rcl} \mathbf{E}(t,\mathbf{x}) & = & E(t,r)\dfrac{\mathbf{x}}{r} \\ \mathbf{B}(t,\mathbf{x}) & \equiv & 0, \end{array} \tag{3.4}$$

where $r = |\mathbf{x}|$.

(b) *For $t \in [0,T)$, $(n, s, \mathbf{v}, \mathbf{E})$ satisfy the reduced Euler-Maxwell system:*

$$\left\{ \begin{array}{rcl} \partial_t n + \partial_i(n v^i) & = & 0 \\ \partial_t s + v^i \partial_i s & = & 0 \\ \partial_t(n v^i) + \partial_j T^{ij} & = & \dfrac{e\bar{n}}{m} E^i \\ \partial_t E^i + 4\pi e n v^i & = & 0, \end{array} \right. \tag{3.5}$$

where

$$T^{ij} = n v^i v^j + \frac{1}{m} p \delta^{ij} + \frac{1}{4\pi m}(\frac{1}{2}|\mathbf{E}|^2 \delta^{ij} - E^i E^j),$$

together with the constraint Poisson equation:

$$\partial_i E^i = 4\pi e(n - \bar{n}).$$

(c) *Let* $\bar{\eta} = \sqrt{\gamma A(\bar{s})\bar{n}^{\gamma-1}}$ *be the sound speed in the background,* $R(t) := 1 + \bar{\eta}t$, *and let*

$$D_T := \{(t, \mathbf{x}) \mid 0 \le t < T,\ |\mathbf{x}| \ge R(t)\}.$$

Then we have $(n, s, \mathbf{v}, \mathbf{E}) \equiv (\bar{n}, \bar{s}, 0, 0)$ *on* D_T.

(d) *For any fixed* $\nu_0(r)$ *which satisfies* (3.3), *there exists* $u_0(r)$ *sufficiently large, such that the life-span of the* C^1 *solution* (3.4) *is finite.*

Proof. (a) The Euler-Maxwell system (3.1) can be written as a positive, symmetric hyperbolic system, and therefore has a unique, local C^1 solution with $n > 0$ provided its initial data are sufficiently smooth. Notice that the initial data are spherically symmetric. Because of the rotational covariance properties of the Euler-Maxwell system and the uniqueness of the local solution, the solution remains spherically symmetric and (a) follows.

(b) follows since $B^i \equiv 0$.

(c) Notice that from the Poisson equation at $t = 0$,

$$E(0, r) = \frac{4\pi e}{r^2}\int_0^r \nu_0(r)r^2 dr \equiv 0 \text{ for } r \ge 1$$

by the neutrality assumption (3.3). Now the reduced Euler-Maxwell system (3.5) is still a hyperbolic system, and we can deduce (c) via the Proposition in [**12**].

(d) Let

$$Q(t) := \frac{1}{4\pi}\int_{\mathbb{R}^3} x \cdot \Pi = \int_0^\infty rnu\, r^2 dr.$$

A direct computation yields:

$$Q'(t) = \int_0^\infty \left\{ nu^2 + \frac{3}{m}(p - \bar{p}) + \frac{1}{8\pi m}E^2 \right\} r^2 dr + \frac{e\bar{n}}{m}\int_0^\infty rE\, r^2 dr,$$

where $\bar{p} = p(\bar{n}, \bar{s})$. Meanwhile, by the first and fourth equations in (3.5),

$$\int_0^\infty rE(t, r)r^2 dr = \int_0^\infty rE(0, r)r^2 dr - 4\pi e\int_0^t Q(t')dt'.$$

Integrating by parts, we notice that

$$\int_0^\infty rE(0, r)r^2 dr = -4\pi e\int_0^\infty \nu_0(r)r^4 dr.$$

We now define $y(t) := \int_0^t Q(t')dt'$ and obtain

$$y''(t) + \omega^2 y(t) = G(t), \tag{3.6}$$

where $\omega^2 = \dfrac{4\pi e^2\bar{n}}{m}$ is the *plasma frequency*, and

$$G(t) := -\omega^2\int_0^\infty \nu_0(r)r^4 dr + \int_0^\infty \left\{ nu^2 + \frac{3}{m}(p - \bar{p}) + \frac{1}{8\pi m}E^2 \right\} r^2 dr.$$

Therefore, from solving the ODE (3.6) for $y(t)$, we have

$$y''(t) = -\omega y'(0)\sin\omega t + G(t) - \omega\int_0^t \sin\omega(t - \tau)G(\tau)d\tau. \tag{3.7}$$

We recall the conserved quantities energy:

$$\mathcal{E} = \int_0^\infty \left\{ \frac{1}{2} n u^2 + \frac{1}{m(\gamma-1)}(A(s)n^\gamma - A(\bar{s})\bar{n}^\gamma) + \frac{1}{8\pi m} E^2 \right\} r^2 dr,$$

and mass

$$M = \frac{1}{4\pi} \int_{\mathbb{R}^3} (n - \bar{n}) = \int_0^\infty \nu(t,r) r^2 dr.$$

From the neutrality condition (3.3) we have $M \equiv 0$. Also, $s(0,\mathbf{x}) \geq \bar{s}$ since $\sigma_0 \geq 0$ by assumption. By the adiabatic condition (the second equation in (3.5)) entropy is constant along flow lines, and thus $s(t,\mathbf{x}) \geq \bar{s}$ for $t < T$. Since A is an increasing function,

$$\int A(s)n^\gamma - A(\bar{s})\bar{n}^\gamma \geq A(\bar{s}) \int n^\gamma - \bar{n}^\gamma \geq \bar{\eta}^2 \int n - \bar{n} = 0.$$

Hence we have

$$\alpha \mathcal{E} \leq G(t) + \omega^2 \int_0^\infty \nu_0(r) r^4 dr \leq \beta \mathcal{E},$$

with $\alpha = \min\{1, 3(\gamma-1)\}$, $\beta = \max\{2, 3(\gamma-1)\}$. But for large enough $u_0(r)$, $\int_0^\infty \nu_0(r) r^4 dr$ is dominated by $\mathcal{E}(0)$. Hence, we have

$$\frac{\alpha}{2}\mathcal{E} \leq G(t) \leq 2\beta\mathcal{E}$$

for sufficiently large $u_0(r)$. Moreover, we have

$$Q^2(t) \leq R^2(t) \int_0^\infty n u^2 \int_0^{R(t)} n \leq C R^5(t) \bar{n} \mathcal{E}. \tag{3.8}$$

C will henceforth denote a generic numerical constant. By choosing $u_0(r)$ large such that $\mathcal{E}(t) = \mathcal{E}(0) \geq 1$, we have

$$|y'(0)| = |Q(0)| \leq C\sqrt{\bar{n}}\mathcal{E}.$$

Thus from (3.7), there exists $T_0 = T_0(\gamma, \bar{n}, \omega) > 0$ such that for $0 \leq t \leq T_0$,

$$Q'(t) \geq C\mathcal{E}.$$

Together with (3.8), we deduce for $0 \leq t \leq T_0$,

$$Q'(t) \geq \frac{C}{R^5(t)\bar{n}} Q^2(t).$$

Integrating over $[0, T_0]$ we obtain

$$\frac{1}{Q(0)} - \frac{1}{Q(T_0)} \geq \frac{C}{\bar{n}\bar{\eta}}[1 - \frac{1}{(1+\bar{\eta}T_0)^4}]. \tag{3.9}$$

We can then choose $u_0(r)$ sufficiently large, so that $Q(0)$ is so large to contradict (3.9).

□

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