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Asymptotic Stability of Plane Diffusion Waves for the 2-D Quasilinear Wave Equation

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ABSTRACT. In this paper we consider the asymptotic stability of the solutions to the nonlinear damped wave equation in 2-D of space. In particular we deal with initial data which are small perturbation (in Sobolev norms) of a self-similar plane diffusive profile which solve a related parabolic equation. The results are achieved by using the classical energy method and in addition we provide polynomial rates of convergences.

1. Introduction

The present paper is part of a general program of understanding the connections between nonlinear nonhomogeneous hyperbolic systems and nonlinear parabolic equations. Concerning these problems, there are several points of view which can be pieced together in order to have a good comprehension of the underlying dynamics. Here we are concerned with the large time behavior of the solutions to the following nonlinear wave equation with a frictional damping term

$$\begin{aligned} &w_{tt} - \operatorname{div}\left[\vartheta\left(|\nabla w|\right)\nabla w\right] + \alpha w_t = 0, \\ &t \geq 0, \quad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{1.1}$$

where $\alpha > 0$, $w = w(x,y,t) \in \mathbb{R}$ and $\vartheta(\lambda) > 0$ is a smooth nonlinear function such that $\sigma(\lambda) = \vartheta(\lambda)\lambda$ satisfies $\sigma'(0) > 0$, $\sigma''(\lambda)\lambda > 0$ for any $\lambda \neq 0$. As usual, we denote

$$w_x(x,y,t) = v(x,y,t), \quad w_y(x,y,t) = m(x,y,t), \quad w_t(x,y,t) = u(x,y,t),$$

then the equation (1.1) can be reformulated as the following nonlinear hyperbolic system

$$\begin{cases} v_t - u_x = 0 \\ m_t - u_y = 0 \\ u_t - \operatorname{div}\left[\vartheta\left(|p|\right)p\right] = -\alpha u, \end{cases} \tag{1.2}$$

where $p = (v,m)$.

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We consider solutions of (1.2) which are small perturbations in H^s of a plane diffusion wave obtained from the caloric self-similar solution to the 1-D parabolic system related with (1.2)

$$\begin{cases} \overline{v}_t - \overline{u}_x = 0 \\ \alpha \overline{u} = [\vartheta(\overline{v})\,\overline{v}]_x\,, \end{cases} \tag{1.3}$$

with limiting conditions

$$\overline{v}(\pm\infty, t) = v_\pm, \quad \overline{u}(\pm\infty, t) = 0. \tag{1.4}$$

We wish to prove that these perturbed solutions to the full system (1.2) converge asymptotically, in higher order energy norms, to a related solution of the parabolic equation (1.3). Actually, this large-time dynamic is somehow decoupled into a typical 1-D phenomenon (see [**HL92, HL93, Nis96**]) and a more genuine 2-D convergence.

This type of analysis has been initiated by the previously mentioned papers of Hsiao and Liu [**HL92, HL93**] and later continued by Nishihara [**Nis96**] in one space dimension. All of these papers are based completely on the use of the classical energy techniques and they provide stability and polynomial decay rates. Recently, in [**MM**], it has been proved a related result concerning the initial-boundary value problem.

The asymptotic study for weak solutions of hyperbolic systems with damping has been carried out in [**MM90, MMS88**], by introducing an appropriate parabolic-type scaling and then by studying the related relaxation problem via the theory of compensated compactness. The general 2×2 case is treated in [**MR**], together with some multi-D results. Recently, the convergence obtained in [**MR**] in the general 2×2 case, which can be viewed as a convergence "in the mean", has been improved in [**LR97**] to an almost pointwise convergence, by following an idea of [**SX97**] for the p-system with linear damping. Related results for semiconductors hydrodynamic models have been obtained in [**MN95, Nat96, LM, Lat, JR**].

In the present paper, we prove that a 2-D perturbation of the plane wave $\overline{v}(x,t)$ converges as $t \uparrow +\infty$ with polynomial rates to the 1-D solution of [**HL92, Nis96**]. In particular, let us denote by $\widetilde{v}(x,t)$ this solution, our analysis is based on the splitting between the 1-D component and the 2-D component of the initial perturbation $\psi(x,y) = v(x,y,0) - \widetilde{v}(x,0)$, by using the condition

$$\int_{-\infty}^{+\infty} \psi(x,y)dx \equiv 0.$$

This zero-mean condition is necessary to avoid interactions between the one dimensional and the two dimensional dynamics, which could destabilize the convergence process.

In the next section, we will recall some properties of the self-similar solution of the parabolic equation [**HL92**] and of the solution of the 1-D hyperbolic problem [**HL92, Nis96**], which will be useful in the proof of the decay estimates.

The section 3 is devoted to prove the energy estimate which will show the convergence of the 2-D perturbation of $\widetilde{v}(x,t)$ as $t \uparrow +\infty$, thanks to the results of [**HL92, Nis96**], the convergence of the solutions of (1.2) toward the self-similar solutions of the parabolic system (1.3).

2. Statement of the Problem and Main Results

In this section we recall the main results regarding the 1-D problem [**HL92, PVD97, Nis96**]. Let us consider the nonlinear diffusion equation

$$f_t = -\frac{1}{\alpha}(\vartheta(f)f)_{xx}, \tag{2.1}$$

with the following conditions at $\pm\infty$

$$f(\pm\infty, t) = v_\pm, \quad v_+ > v_- > 0. \tag{2.2}$$

The problem (2.1)-(2.2) has a caloric self-similar solution

$$\overline{v}(x,t) = \varphi\left(\frac{x}{\sqrt{1+t}}\right), \quad \varphi(\pm\infty) = v_\pm, \;\; \varphi(\xi) > 0. \tag{2.3}$$

This solution verifies the inequalities [**HL92**]

$$\sum_{k=1}^{3}\left|\frac{d^k}{d\xi^k}\varphi(\xi)\right| + |\varphi(\xi) - v_+|_{\xi>0} + |\varphi(\xi) - v_-|_{\xi<0} \le C|v_+ - v_-|e^{-c\alpha\xi^2}, \tag{2.4}$$

and the pointwise decay estimates for all the derivatives of $\overline{v}$ can be easily obtained by differentiating (2.3) in x and t

$$\overline{v}_x = \frac{\varphi'(\xi)}{\sqrt{1+t}}, \qquad\qquad \overline{v}_t = -\frac{\xi\varphi'(\xi)}{2(1+t)}. \tag{2.5}$$

Then, let us consider a solution $(\widetilde{u}, \widetilde{v})$ of the 1-D system

$$\begin{cases} \widetilde{v}_t - \widetilde{u}_x = 0 \\ \widetilde{u}_t - \left[\vartheta\left(\widetilde{v}\right)\widetilde{v}\right]_x = -\alpha\widetilde{u}, \end{cases} \tag{2.6}$$

which verifies

$$\widetilde{v}(\pm\infty, 0) = v_\pm, \quad \widetilde{u}(\pm\infty, 0) = u_\pm. \tag{2.7}$$

Thus, it is known [**HL92, Nis96**] that the shift x_0 and the correctors $\widehat{u}$ and $\widehat{v}$ have the following expressions

$$\begin{aligned} x_0 &= \frac{u_+ - u_-}{\alpha(v_+ - v_-)} + \frac{1}{v_+ - v_-}\int_{-\infty}^{+\infty}(\widetilde{v}(x,0) - \overline{v}(x,0))\,dx, \\ \widehat{u}(x,t) &= e^{-\alpha t}\left[u_+ + (u_+ - u_-)\int_{-\infty}^{x} m_0(\xi)d\xi\right], \\ \widehat{v} &= \frac{u_+ - u_-}{-\alpha}e^{-\alpha t}m_0(x), \end{aligned}$$

where m_0 is a nonnegative test function such that $\displaystyle\int_{-\infty}^{+\infty} m_0(x)dx = 1$. Let us denote

$$\begin{aligned} \widetilde{V}(x,t) &= \int_{-\infty}^{x}(\widetilde{v}(\xi,t) - \overline{v}(\xi + x_0, t) - \widehat{v}(\xi,t))\,d\xi, \\ \widetilde{z}(x,t) &= \widetilde{u}(x,t) - \widehat{u}(x,t) - \overline{u}(x + x_0, t). \end{aligned}$$

With this notation, the problem (2.6)-(2.7) becomes

$$(2.8)\quad \begin{cases} \widetilde{V}_t - \widetilde{z} = 0 \\ \widetilde{z}_t - \left[\vartheta(\widetilde{V}_x + \overline{v} + \widehat{v})(\widetilde{V}_x + \overline{v} + \widehat{v}) - \vartheta(\overline{v})\overline{v}\right]_x + \alpha\widetilde{z} = -\overline{u}_t = -\frac{1}{\alpha}\left[\vartheta(\overline{v})\overline{v}\right]_x \\ \widetilde{V}(x,0) = \widetilde{V}_0(x) \\ \widetilde{z}(x,0) = \widetilde{z}_0(x) \\ \widetilde{V}_0(\pm\infty) = \widetilde{z}_0(\pm\infty) = 0. \end{cases}$$

Hence, the following theorem holds [**HL92, Nis96**].

THEOREM 2.1. *Suppose* $\delta = |v_+ - v_-| + |u_+ - u_-|$ *and* $\|\widetilde{V}_0\|_3 + \|\widetilde{z}_0\|_2$ *are sufficiently small. Then there exists a unique global solution* $(\widetilde{V}(x,t), \widetilde{z}(x,t))$ *to (2.8) which satisfies*

$$\widetilde{V} \in W^{i,\infty}([0,+\infty); H^i), \quad i = 0,\dots,3,$$

and moreover

$$\sum_{k=0}^{3}(1+t)^k\|\partial_x^k\widetilde{V}(\cdot,t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k\widetilde{z}(\cdot,t)\|^2 + \int_0^t\left[\sum_{j=1}^{3}(1+\tau)^{j-1}\|\partial_x^j\widetilde{V}(\cdot,\tau)\|^2 + \sum_{j=0}^{2}(1+\tau)^{j+1}\|\partial_x^j\widetilde{z}(\cdot,\tau)\|^2\right]d\tau$$
$$(2.9)\quad \le C(\|\widetilde{V}_0\|_3^2 + \|\widetilde{z}_0\|_2^2 + \delta).$$

REMARK 2.2. By using a recursive procedure, it is possible to improve the previous result when the initial data are more regular. In particular, the estimate (2.9) can be achieved for a larger k. For our purposes, we will assume $\|\widetilde{V}_0\|_8 + \|\widetilde{z}_0\|_7 \le \delta$ small enough to have

$$\sum_{k=0}^{8}(1+t)^k\|\partial_x^k\widetilde{V}(\cdot,t)\|^2 + \sum_{k=0}^{7}(1+t)^{k+2}\|\partial_x^k\widetilde{z}(\cdot,t)\|^2 + \int_0^t\left[\sum_{j=1}^{8}(1+\tau)^{j-1}\|\partial_x^j\widetilde{V}(\cdot,\tau)\|^2 + \sum_{j=0}^{7}(1+\tau)^{j+1}\|\partial_x^j\widetilde{z}(\cdot,\tau)\|^2\right]d\tau$$
$$(2.10)\quad \le C(\|\widetilde{V}_0\|_8^2 + \|\widetilde{z}_0\|_7^2 + \delta) \le C\delta.$$

REMARK 2.3. We know that the solution $\overline{v}$ of (2.1)-(2.2) is positive. Due to (2.10) and due to the expression of the corrector $\widehat{v}$, the difference $\widetilde{v} - \overline{v}$ is $O(\delta)$. Therefore, for δ sufficiently small, we have $\widetilde{v} > 0$.

Now we analyze the 2-D perturbation of this 1-D solution. Let us consider the following 2-D system given by the wave equation (1.1)

$$(2.11)\quad \begin{cases} v_t - u_x = 0 \\ m_t - u_y = 0 \\ u_t - \operatorname{div}\left[\vartheta\left(|p|\right)p\right] = -\alpha u, \end{cases}$$

where $p = (v, m)$. We choose the initial data $v(x, y, 0)$ and $\widetilde{v}(x, 0)$ so that

$$\int_{-\infty}^{+\infty} (v(x,y,0) - \widetilde{v}(x,0))\,dx = 0 \tag{2.12}$$

and we assume the following limiting conditions

$$\begin{array}{ll} v(\pm\infty, y, t) = v_\pm, & v(x, \pm\infty, t) = \widetilde{v}(x,t), \\ m(\pm\infty, y, t) = 0, & m(x, \pm\infty, t) = 0, \\ u(\pm\infty, y, t) = u_\pm e^{-\alpha t}, & u(x, \pm\infty, t) = \widetilde{u}(x,t). \end{array} \tag{2.13}$$

REMARK 2.4. The condition (2.12) implies in particular that the new perturbation due to the difference $v(x, y, 0) - \widetilde{v}(x, 0)$ does not affect the shift of the final plane wave. Therefore, the asymptotic profile of our 2-D solution is selected by the 1-D solution $\widetilde{v}$. This phenomenon provides a big advantage since it allows us to consider directly the convergence of the 2-D perturbation of $\widetilde{v}(x, t)$. Once we know this kind of convergence, we can simply make use of the estimate (2.10) to show the asymptotic behavior of the 2-D solution.

As in the 1-D case, we introduce a new set of variables

$$\begin{aligned} V(x,y,t) &= \int_{-\infty}^{x} (v(\xi,y,t) - \widetilde{v}(\xi,t))\,d\xi \\ M(x,y,t) &= \int_{-\infty}^{y} m(x,\eta,t)d\eta \\ z(x,y,t) &= u(x,y,t) - \widetilde{u}(x,t), \end{aligned}$$

and the problem (2.11)-(2.13) can be rewritten as follows

$$\begin{cases} V_t = z \\ M(x,y,t) = V(x,y,t) + \displaystyle\int_0^t \widetilde{u}(x,s)ds + M(x,y,0) - V(x,y,0) \\ z_t - \operatorname{div}\left[\vartheta\left(\left|\begin{array}{c} V_x + \widetilde{v} \\ V_y \end{array}\right|\right)\left(\begin{array}{c} V_x + \widetilde{v} \\ V_y \end{array}\right) + \vartheta(\widetilde{v})\left(\begin{array}{c} \widetilde{v} \\ 0 \end{array}\right)\right] + \alpha z = 0, \end{cases} \tag{2.14}$$

with the limiting conditions

(2.15)

$$V(\pm\infty, y, 0) = 0, \quad V(x, \pm\infty, 0) = 0, \quad z(\pm\infty, y, 0) = 0, \quad z(x, \pm\infty, 0) = 0.$$

Now we can state our main theorem. We recall that

$$\|f\| = \left(\int\int |f(x,y)|^2 dxdy\right)^{\frac{1}{2}}$$

denotes the classical $L^2(\mathbb{R}^2)$ norm and the Sobolev norm is given by

$$\|f\|_s = \left(\sum_{j=0}^{s} \int\int |D^j f(x,y)|^2 dxdy\right)^{\frac{1}{2}},$$

where D^j is any differential operator of the form $\dfrac{\partial^{j_1}}{\partial x^{j_1}}\dfrac{\partial^{j_2}}{\partial y^{j_2}}$ with $j_1 + j_2 = j$.

THEOREM 2.5. *Suppose δ and $\|V_0\|_7 + \|z_0\|_6$ are sufficiently small. Then there exists a unique global solution $(V(x,y,t), z(x,y,t))$ to (2.14)-(2.15) which satisfies*

$$V \in W^{i,\infty}([0, +\infty); H^{7-i}), \quad i = 0, \dots, 7,$$

and moreover

$$
\begin{aligned}
&\sum_{k=0}^{7}(1+t)^k\|D^kV(\cdot,t)\|^2+\sum_{k=0}^{6}(1+t)^{k+2}\|D^kz(\cdot,t)\|^2\\
&+\int_0^t\left[\sum_{j=1}^{7}(1+\tau)^{j-1}\|D^jV(\cdot,\tau)\|^2+\sum_{j=0}^{6}(1+\tau)^{j+1}\|D^jz(\cdot,\tau)\|^2\right]d\tau\\
&=O(1)\left(\|V_0\|_7^2+\|z_0\|_6^2+\delta\right). \qquad (2.16)
\end{aligned}
$$

REMARK 2.6. In view of the Sobolev embeddings, the estimate (2.16) of theorem 2.5 and the estimate (2.10) of remark 2.2 imply that the C^4 norm of $V(x,y,t)$ and $\widetilde{V}(x,t)$ decays in time with polynomial rates. Therefore, the same kind of C^4 convergence holds also for the quantity

$$\mathcal{V}(x,y,t)=\int_{-\infty}^{x}\left(v(\xi,y,t)-\overline{v}(\xi+x_0,t)-\widehat{v}(\xi,t)\right)d\xi.$$

Hence, the full 2-D solution converges toward the plane wave with the same rates established in [**HL92, Nis96**] for the 1-D problem.

3. The Proof of the Main Theorem

In this section we deal with the proof of the theorem 2.5, namely, the proof of the asymptotic behavior (2.16). We achieve this result by using energy methods, together with a continuation principle. As usual in this framework, we start with an a priori assumption

$$
\begin{aligned}
N(T)=\sup_{0<t<T}\Bigg\{&\sum_{k=0}^{7}(1+t)^k\left[\|\partial_x^kV(\cdot,t)\|^2+\|\partial_y^kV(\cdot,t)\|^2\right]\\
&+\sum_{k=0}^{6}(1+t)^{k+2}\left[\|\partial_x^kz(\cdot,t)\|^2+\|\partial_y^kz(\cdot,t)\|^2\right]\Bigg\}\le\varepsilon. \qquad (3.1)
\end{aligned}
$$

Let us use the following notations

$$\vartheta=\vartheta\left(\left|\begin{array}{c}V_x+\widetilde{v}\\ V_y\end{array}\right|\right),\quad \widetilde{\vartheta}=\vartheta(\widetilde{v}).$$

In the next lemma, we establish some useful properties of the nonlinear function ϑ.

LEMMA 3.1. *Let ϑ be a smooth function such that $\sigma(\lambda)=\vartheta(\lambda)\lambda$ satisfies $\sigma'(0)>0$ and $\sigma''(\lambda)\lambda>0$ for any $\lambda\neq0$. Then $\vartheta'(\lambda)\lambda>0$ for any $\lambda\neq0$.*

The following lemma concerns the bound of the H^1 norm of the solution V.

LEMMA 3.2. *Suppose ε, δ and $\|V_0\|_7^2+\|z_0\|_6^2$ are sufficiently small. Then*

$$
\begin{aligned}
\|V(t)\|_1^2+\|z(t)\|^2+&\int_0^t\left(\|V_x(\tau)\|^2+\|V_y(\tau)\|^2+\|z(\tau)\|^2\right)d\tau\\
&=O(1)\left(\|V_0\|_1^2+\|z_0\|^2+\delta\right). \qquad (3.2)
\end{aligned}
$$

Proof. The system (2.14) can be rewritten as a hyperbolic equation for the function V

$$V_{tt} - \operatorname{div}\left[\vartheta\begin{pmatrix} V_x + \widetilde{v} \\ V_y \end{pmatrix} - \widetilde{\vartheta}\begin{pmatrix} \widetilde{v} \\ 0 \end{pmatrix}\right] + \alpha V_t = 0, \tag{3.3}$$

which can be linearized as follows

$$V_{tt} - \operatorname{div}\left[\widetilde{\vartheta}\begin{pmatrix} V_x \\ V_y \end{pmatrix}\right] + \alpha V_t = \operatorname{div}\left[(\vartheta - \widetilde{\vartheta})\begin{pmatrix} V_x + \widetilde{v} \\ V_y \end{pmatrix}\right] = F. \tag{3.4}$$

Multiplying (3.4) for $V_t + \lambda V$ and integrating on $dxdy$ one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\int\left[V_t^2 + 2\lambda V V_t + \widetilde{\vartheta}(V_x^2 + V_y^2) + \alpha\lambda V^2\right]dxdy \\ &+\int\int\left[(\alpha-\lambda)V_t^2 + \lambda\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy \\ &= \frac{1}{2}\int\int\widetilde{\vartheta}_t(V_x^2+V_y^2)dxdy + \int\int(V_t+\lambda V)Fdxdy. \end{aligned} \tag{3.5}$$

The left hand side of the above relation clearly gives the quantity we have to estimate, once we control the product VV_t in terms of V^2 and V_t^2, which is possible by choosing an appropriate small value for the constant λ. Therefore, we have to estimate the right hand side of (3.5) to conclude the proof. We start by investigating the term $\int\int zFdxdy$. To this end, we introduce the function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$H(|P|) = \int_{\widetilde{v}}^{|P|} s\vartheta(s)ds, \quad P \in \mathbb{R}^2.$$

With the notation

$$G = \begin{pmatrix} V_x + \widetilde{v} \\ V_y \end{pmatrix},$$

this term becomes

$$\begin{aligned} \int\int zFdxdy &= -\int\int\begin{pmatrix} V_x \\ V_y \end{pmatrix}_t \cdot \left[\vartheta\begin{pmatrix} V_x+\widetilde{v} \\ V_y \end{pmatrix} - \widetilde{\vartheta}\begin{pmatrix} \widetilde{v} \\ 0 \end{pmatrix}\right. \\ &\quad \left. - \widetilde{\vartheta}\begin{pmatrix} V_x \\ V_y \end{pmatrix}\right]dxdy \\ &= -\frac{d}{dt}\int\int\left[H(|G|) - \widetilde{\vartheta}\widetilde{v}V_x - \frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy \\ &\quad + \int\int\begin{pmatrix} \widetilde{v} \\ 0 \end{pmatrix}_t \cdot \left[\vartheta\begin{pmatrix} V_x+\widetilde{v} \\ V_y \end{pmatrix} - \widetilde{\vartheta}\begin{pmatrix} \widetilde{v} \\ 0 \end{pmatrix}\right]dxdy \\ &\quad - \int\int V_x(\widetilde{\vartheta}_t\widetilde{v} - \widetilde{v}_t\widetilde{\vartheta})dxdy - \frac{1}{2}\int\int\widetilde{\vartheta}_t(V_x^2+V_y^2)dxdy \\ &= -\frac{d}{dt}\int\int\left[H(|G|) - \widetilde{\vartheta}\widetilde{v}V_x - \frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy \\ &\quad - \frac{1}{2}\int\int\widetilde{\vartheta}_t(V_x^2+V_y^2)dxdy + \int\int\begin{pmatrix} \widetilde{v} \\ 0 \end{pmatrix}_t \cdot \left[\vartheta\begin{pmatrix} V_x+\widetilde{v} \\ V_y \end{pmatrix}\right. \\ &\quad \left. - \widetilde{\vartheta}\begin{pmatrix} \widetilde{v} \\ 0 \end{pmatrix} - \widetilde{\vartheta}'\begin{pmatrix} \widetilde{v} \\ 0 \end{pmatrix}V_x - \widetilde{\vartheta}\begin{pmatrix} V_x \\ V_y \end{pmatrix}\right]dxdy. \end{aligned}$$

With the above equality, we can rewrite the right hand side of (3.5) as follows

$$
\begin{aligned}
&-\lambda\int\int(\vartheta-\widetilde{\vartheta})\begin{pmatrix}V_x\\V_y\end{pmatrix}\cdot\begin{pmatrix}V_x+\widetilde{v}\\V_y\end{pmatrix}dxdy\\
&-\frac{d}{dt}\int\int\left[H(|G|)-\widetilde{\vartheta}\widetilde{v}V_x-\frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy\\
&+\int\int\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}_t\cdot\left[\vartheta\begin{pmatrix}V_x+\widetilde{v}\\V_y\end{pmatrix}-\widetilde{\vartheta}\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}-\widetilde{\vartheta}'\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}V_x\right.\\
&\left.-\widetilde{\vartheta}\begin{pmatrix}V_x\\V_y\end{pmatrix}\right]dxdy=I_1+I_2+I_3. \qquad (3.6)
\end{aligned}
$$

Thus, we have to estimate the terms $I_1+I_2+I_3$ in (3.6). Since

$$\vartheta-\widetilde{\vartheta}=\widetilde{\vartheta}'V_x+O\left(V_x^2+V_y^2\right),$$

combining the result of lemma 3.1 and the a priori assumption (3.1), we get

$$
\begin{aligned}
I_1&=-\lambda O(1)\int\int\widetilde{\vartheta}'\widetilde{v}V_x^2dxdy+\lambda\varepsilon O(1)\int\int(V_x^2+V_y^2)dxdy\\
&\le\lambda\varepsilon O(1)\int\int(V_x^2+V_y^2)dxdy.
\end{aligned}
$$

The Taylor expansion of the functions $H(|P|)$ and $\vartheta(|P|)P$ yields

$$H(|G|)-\widetilde{\vartheta}\widetilde{v}V_x-\frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)=\frac{1}{2}\widetilde{\vartheta}'\widetilde{v}V_x^2+O\left(V_x^3+V_y^3\right) \qquad (3.7)$$

and

$$\vartheta\begin{pmatrix}V_x+\widetilde{v}\\V_y\end{pmatrix}-\widetilde{\vartheta}\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}-\widetilde{\vartheta}'\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}V_x-\widetilde{\vartheta}\begin{pmatrix}V_x\\V_y\end{pmatrix}=O\left(V_x^2+V_y^2\right).$$

Hence, in view of (2.4), (2.5), (2.10) and (3.1), the previous relation implies

$$|I_3|=O(1)\int\int|\widetilde{v}_t|\left(|V_x|^2+|V_y^2|\right)dxdy=\frac{O(1)\delta}{(1+t)^2}.$$

Finally, integrating (3.7) in dt and using again lemma 3.1 and (3.1), we get

$$
\begin{aligned}
\int_0^t I_2ds&=\frac{1}{2}\int\int\widetilde{\vartheta}'\widetilde{v}V_x^2dxdy+\varepsilon O(1)\int\int\left(V_x^2+V_y^2\right)dxdy+O(1)\|V_0\|_1\\
&\le\varepsilon O(1)\int\int\left(V_x^2+V_y^2\right)dxdy+O(1)\|V_0\|_1.
\end{aligned}
$$

Therefore, integrating (3.5) in dt, for δ, ε and λ small enough, it follows (3.2). □

The previous lemma gives a bound of the H^1 norm of V, without any decay property.We can improve the estimate (3.2) by showing the first convergence result for the functions V and z. The proof of such property is based essentially on the decays of the 1-D solution $\widetilde{v}$ contained in (2.4), (2.5) and (2.10).

LEMMA 3.3. *Suppose ε, δ and $\|V_0\|_7^2+\|z_0\|_6^2$ are sufficiently small. Then*

$$
\begin{aligned}
(1+t)\left(\|V_x(t)\|^2+\|V_y(t)\|^2+\|z(t)\|^2\right)&+\int_0^t(1+\tau)\|z(\tau)\|^2d\tau\\
&=O(1)\left(\|V_0\|_1^2+\|z_0\|^2+\delta\right).
\end{aligned}
$$

PROOF. We multiply the linearized equation (3.4) by $(1+t)V_t$. Therefore, after integrating in $dxdy$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\int(1+t)\left[V_t^2+\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy+\int\int\alpha(1+t)V_t^2dxdy\\
&=\frac{1}{2}\int\int(1+t)\widetilde{\vartheta}_t(V_x^2+V_y^2)dxdy+\frac{1}{2}\int\int\left[V_t^2+\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy\\
&+\int\int(1+t)V_tFdxdy=I_1+I_2+I_3.
\end{aligned}\tag{3.8}$$

The results of lemma 3.2 yield

$$I_2=O(1)\left(\|V_0\|_1^2+\|z_0\|^2+\delta\right).$$

With the previous notations, we have

$$\begin{aligned}
I_3=&-\frac{d}{dt}\int\int(1+t)\left[H(|G|)-\widetilde{\vartheta}\widetilde{v}V_x-\frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy\\
&+\int\int\left[H(|G|)-\widetilde{\vartheta}\widetilde{v}V_x-\frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy\\
&-\frac{1}{2}(1+t)\int\int\widetilde{\vartheta}_t(V_x^2+V_y^2)dxdy+\int\int(1+t)\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}_t\cdot\left[\vartheta\begin{pmatrix}V_x+\widetilde{v}\\V_y\end{pmatrix}\right.\\
&\left.-\widetilde{\vartheta}\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}-\widetilde{\vartheta}'\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}V_x-\widetilde{\vartheta}\begin{pmatrix}V_x\\V_y\end{pmatrix}\right]dxdy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_1+I_3=&-\frac{d}{dt}\int\int(1+t)\left[H(|G|)-\widetilde{\vartheta}\widetilde{v}V_x-\frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy\\
&+\int\int\left[H(|G|)-\widetilde{\vartheta}\widetilde{v}V_x-\frac{1}{2}\widetilde{\vartheta}(V_x^2+V_y^2)\right]dxdy\\
&+\int\int(1+t)\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}_t\cdot\left[\vartheta\begin{pmatrix}V_x+\widetilde{v}\\V_y\end{pmatrix}-\widetilde{\vartheta}\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}\right.\\
&\left.-\widetilde{\vartheta}'\begin{pmatrix}\widetilde{v}\\0\end{pmatrix}V_x-\widetilde{\vartheta}\begin{pmatrix}V_x\\V_y\end{pmatrix}\right]dxdy=J_1+J_2+J_3.
\end{aligned}$$

Moreover, proceeding as in the proof of lemma 3.2, we have

$$\int_0^t J_1ds\le O(1)\varepsilon\int\int(1+t)\left(V_x^2+V_y^2\right)dxdy+O(1)\|V_0\|_1;$$

$$J_2\le O(1)\varepsilon\int\int\left(V_x^2+V_y^2\right)dxdy.$$

Thus, (3.2) implies

$$\int_0^t J_2ds=O(1)\left(\|V_0\|_1^2+\|z_0\|^2+\delta\right).$$

Finally, the last term can be bounded by using again (3.2) and the decay of $\widetilde{v}_t$

$$\begin{aligned}
J_3&=O(1)\int\int(1+t)|\widetilde{v}_t|\left(V_x^2+V_y^2\right)dxdy=O(1)\int\int\left(V_x^2+V_y^2\right)dxdy\\
&=O(1)\left(\|V_0\|_1^2+\|z_0\|^2+\delta\right).
\end{aligned}$$

As before, we conclude the proof integrating (3.8) in dt and choosing δ, ε and λ small enough. □

Now we turn to the study of the estimates for the higher derivatives of V and z. In the next lemma, we prove the first H^2 result, regarding essentially the x and y derivatives of V and z.

Lemma 3.4. *Suppose ε, δ and $\|V_0\|_7^2 + \|z_0\|_6^2$ are sufficiently small. Then*

$$\begin{aligned}
&(1+t)^2(\|V_{xx}(t)\|^2 + \|V_{xy}(t)\|^2 + \|V_{yy}(t)\|^2 + \|z_x(t)\|^2 + \|z_y(t)\|^2)\\
&+\int_0^t (1+\tau)\left[\|V_{xx}(\tau)\|^2 + \|V_{xy}(\tau)\|^2 + \|V_{yy}(\tau)\|^2\right] d\tau\\
&+\int_0^t (1+\tau)^2\left[\|z_x(\tau)\|^2 + \|z_y(\tau)\|^2\right] d\tau\\
&= O(1)(\|V_0\|_2^2 + \|z_0\|_1^2 + \delta).
\end{aligned}$$

Proof. We start by differentiating the linearized equation (3.4) in x and y in order to have

$$\mathcal{Z}_{tt} - \operatorname{div}\left[\widetilde{\vartheta}\begin{pmatrix}\mathcal{Z}_x\\ \mathcal{Z}_y\end{pmatrix}\right] + \alpha\mathcal{Z}_t = F_x + \operatorname{div}\left[\widetilde{\vartheta}_x\begin{pmatrix}\mathcal{Z}\\ \mathcal{W}\end{pmatrix}\right] \tag{3.9}$$

and

$$\mathcal{W}_{tt} - \operatorname{div}\left[\widetilde{\vartheta}\begin{pmatrix}\mathcal{W}_x\\ \mathcal{W}_y\end{pmatrix}\right] + \alpha\mathcal{W}_t = F_y, \tag{3.10}$$

where $\mathcal{Z} = V_x$ and $\mathcal{W} = V_y$. We multiply (3.9) for $\mathcal{Z}_t$ and we integrate on $dxdy$ and we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\int\left[\mathcal{Z}_t^2 + \widetilde{\vartheta}\left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2\right)\right]dxdy + \alpha\int\int\mathcal{Z}_t^2 dxdy\\
&= \frac{1}{2}\int\int\widetilde{\vartheta}_t\left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2\right)dxdy + \int\int\mathcal{Z}_t F_x dxdy\\
&+\int\int\mathcal{Z}_t\operatorname{div}\left[\widetilde{\vartheta}_x\begin{pmatrix}\mathcal{Z}\\ \mathcal{W}\end{pmatrix}\right]dxdy = I_1 + I_2 + I_3.
\end{aligned}\tag{3.11}$$

Due to (2.4), (2.5) and (2.10), the first term is estimated as follows

$$|I_1| = O(1)\delta\int\int\left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2\right)dxdy.$$

Moreover, the last term can be bounded by using also the Young inequality

$$\begin{aligned}
I_3 &= \int\int\left[\mathcal{Z}_t\left(\widetilde{\vartheta}_{xx}V_x + \widetilde{\vartheta}_x\mathcal{Z}_x + \widetilde{\vartheta}_x\mathcal{W}_y\right)\right]dxdy \le E_\alpha\int\int\mathcal{Z}_t^2 dxdy\\
&+\; O(1)(1+t)^{-2}\int\int V_x^2 dxdy + O(1)\delta\int\int\left(\mathcal{Z}_x^2 + \mathcal{W}_y^2\right)dxdy,
\end{aligned}$$

where E_α is a small positive constant (depending only on α) which will be chosen afterwards. Now, let us consider the second term in the right-hand-side of (3.11).

Integration by parts yields

$$
\begin{aligned}
I_2 &= -\int\int \begin{pmatrix} V_{xxt} \\ v_{xyt} \end{pmatrix} \cdot \left((\vartheta - \widetilde{\vartheta}) \begin{pmatrix} V_x + \widetilde{v} \\ V_y \end{pmatrix} \right)_x dxdy \\
&= -\int\int V_{xxt} \left[(\vartheta - \widetilde{\vartheta})_x (V_x - \widetilde{v}) + (\vartheta - \widetilde{\vartheta})(V_{xx} - \widetilde{v}_x) \right] dxdy \\
&\quad -\int\int V_{xyt} \left[(\vartheta - \widetilde{\vartheta})_x V_y + (\vartheta - \widetilde{\vartheta}) V_{xy} \right] dxdy \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

Developing the x-derivative of $\vartheta - \widetilde{\vartheta}$, J_1 becomes

$$
\begin{aligned}
J_1 = -\int\int V_{xxt} &\left[\frac{\vartheta'(V_x - \widetilde{v})}{|G|} V_{xx} + \frac{\vartheta' V_y}{|G|} V_{xy} \right. \\
&\left. + \left(\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' \right) \widetilde{v}_x \right] (V_x - \widetilde{v}) dxdy,
\end{aligned}
$$

where G represent again the vector $\begin{pmatrix} V_x + \widetilde{v} \\ V_y \end{pmatrix}$. We examine the terms in J_1 one by one.

$$
\begin{aligned}
-\int\int V_{xxt} \frac{\vartheta'(V_x - \widetilde{v})^2}{|G|} V_{xx} dxdy = -\frac{1}{2}\frac{d}{dt} \int\int \frac{\vartheta'(V_x - \widetilde{v})^2}{|G|} \mathcal{Z}_x^2 dxdy \\
+ O(1)(\varepsilon + \delta) \int\int \mathcal{Z}_x^2 dxdy.
\end{aligned}
$$

We emphasize that the total derivative with respect to t in the above relation has the "right" sign for ε small enough, because

$$
\left. \frac{\vartheta'(V_x - \widetilde{v})^2}{|G|} \right|_{V_x = V_y = 0} = \widetilde{\vartheta}' \widetilde{v} > 0,
$$

thanks to lemma 3.1. The last term in J_1 can be treated as follows

$$
\begin{aligned}
&-\int\int V_{xxt} (V_x - \widetilde{v}) \left(\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' \right) \widetilde{v}_x dxdy \\
&= \int\int V_{xt} (V_x - \widetilde{v})_x \left(\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' \right) \widetilde{v}_x dxdy \\
&+ \int\int V_{xt} (V_x - \widetilde{v}) \left(\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' \right) \widetilde{v}_{xx} dxdy \\
&+ \int\int V_{xt} (V_x - \widetilde{v}) \left(\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' \right)_x \widetilde{v}_x dxdy.
\end{aligned} \tag{3.12}
$$

Since

$$
\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' = O\left(|V_x| + |V_y| \right),
$$

in view of (2.4), (2.5) and (2.10) and using the Young inequality, the first two terms in (3.12) are bounded by

$$
\begin{aligned}
O(1) \int\int |\mathcal{Z}_t| |V_x| \left(|\widetilde{v}_{xx}| + |\widetilde{v}_x| |V_{xx} + \widetilde{v}_x| \right) dxdy \leq E_\alpha \int\int \mathcal{Z}_t^2 dxdy \\
+ O(1)(1+t)^{-2} \int\int V_x^2 dxdy.
\end{aligned}
$$

Evaluating the x-derivative in the last part of (3.12) we prove that this quantity is controlled by

$$O(1)\int\int |\mathcal{Z}_t||\widetilde{v}_x|\left[|V_{xx}|+|\widetilde{v}_x|\left(|V_x|+|V_y|\right)+|V_y||V_{xy}|+|V_{xx}|\left(|V_x|+|V_y|\right)\right]dxdy$$
$$\leq E_\alpha\int\int \mathcal{Z}_t^2 dxdy+O(1)(1+t)^{-2}\int\int\left(V_x^2+V_y^2\right)dxdy$$
$$+O(1)\delta\int\int\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right)dxdy.$$

The remaining term of J_1

$$-\int\int V_{xxt}\frac{\vartheta'(V_x+\widetilde{v})}{|G|}V_yV_{xy}dxdy \tag{3.13}$$

can not be bounded for the moment: it will become a part of a total derivative with respect to t. Let us turn now on

$$J_2=\int\int V_{xxt}(\vartheta-\widetilde{\vartheta})V_{xx}dxdy-\int\int V_{xxt}(\vartheta-\widetilde{\vartheta})\widetilde{v}_x dxdy.$$

An integration by part in the last term gives

$$\int\int \mathcal{Z}_t\left[(\vartheta-\widetilde{\vartheta})_x\widetilde{v}_x+(\vartheta-\widetilde{\vartheta})\widetilde{v}_{xx}\right]dxdy$$
$$=O(1)\int\int|\mathcal{Z}_t|\left[|\widetilde{v}_x||V_{xx}|+|\widetilde{v}_x||V_y||V_{xy}|+|\widetilde{v}_x|^2\left(|V_x|+|V_y|\right)\right.$$
$$\left.+\,|\widetilde{v}_x||\widetilde{v}_{xx}|\left(|V_x|+|V_y|\right)\right]dxdy$$
$$\leq E_\alpha\int\int \mathcal{Z}_t^2 dxdy+O(1)(1+t)^{-2}\int\int\left(V_x^2+V_y^2\right)dxdy$$
$$+O(1)\delta\int\int\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right)dxdy,$$

by using also $\vartheta-\widetilde{\vartheta}=O\left(|V_x|+|V_y|\right)$. Moreover, the first term is equal to

$$-\frac{1}{2}\frac{d}{dt}\int\int(\vartheta-\widetilde{\vartheta})\mathcal{Z}_x^2dxdy+\frac{1}{2}\int\int(\vartheta-\widetilde{\vartheta})_t\mathcal{Z}_x^2dxdy$$
$$=-\frac{1}{2}\frac{d}{dt}\int\int(\vartheta-\widetilde{\vartheta})\mathcal{Z}_x^2dxdy+O(1)(\varepsilon+\delta)\int\int\mathcal{Z}_x^2dxdy.$$

Proceeding in the same way, we bound J_4

$$J_4=-\frac{1}{2}\frac{d}{dt}\int\int(\vartheta-\widetilde{\vartheta})\mathcal{Z}_y^2dxdy+\frac{1}{2}\int\int(\vartheta-\widetilde{\vartheta})_t\mathcal{Z}_y^2dxdy$$
$$=-\frac{1}{2}\frac{d}{dt}\int\int(\vartheta-\widetilde{\vartheta})\mathcal{Z}_y^2dxdy+O(1)(\varepsilon+\delta)\int\int\mathcal{Z}_y^2dxdy.$$

Finally, the last term is

$$J_3=-\int\int V_{xyt}\left[\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_{xx}+\frac{\vartheta' V_y}{|G|}V_{xy}\right.$$
$$\left.+\left(\frac{\vartheta'(V_x-\widetilde{v})}{|G|}-\widetilde{\vartheta}'\right)\widetilde{v}_x\right]V_ydxdy.$$

As before, the term

$$-\int\int V_{xyt}\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_{xx}V_y dxdy \tag{3.14}$$

will be considered later. By using arguments similar to the previous ones, we get

$$\begin{aligned}
-\int\int V_{xyt}\frac{\vartheta' V_y^2}{|G|}V_{xy}dxdy &= -\frac{1}{2}\frac{d}{dt}\int\int\frac{\vartheta' V_y^2}{|G|}\mathcal{Z}_y^2 dxdy\\
&\quad + O(1)(\varepsilon+\delta)\int\int\mathcal{Z}_y^2 dxdy\\
&\quad -\int\int V_{xyt}V_y\widetilde{v}_x\left(\frac{\vartheta'(V_x+\widetilde{v})}{|G|}-\widetilde{\vartheta}'\right)dxdy\\
&= -\int\int V_{xt}V_{yy}\widetilde{v}_x\left(\frac{\vartheta'(V_x+\widetilde{v})}{|G|}-\widetilde{\vartheta}'\right)dxdy\\
&\quad -\int\int V_{xt}V_y\widetilde{v}_x\left(\frac{\vartheta'(V_x+\widetilde{v})}{|G|}-\widetilde{\vartheta}'\right)_y dxdy.
\end{aligned}\tag{3.15}$$

As before, the first term of (3.15) is bounded by

$$\begin{aligned}
O(1)\int\int|\mathcal{Z}_t||\widetilde{v}_x||\mathcal{W}_y|\left(|V_x|+|V_y|\right)dxdy &\le E_\alpha\int\int\mathcal{Z}_t^2 dxdy\\
&\quad + O(1)\delta\int\int\mathcal{W}_y^2 dxdy,
\end{aligned}$$

while the second is studied by developing the y-derivative

$$\begin{aligned}
&\int\int V_{xt}V_y\widetilde{v}_x\left(\frac{\vartheta'(V_x+\widetilde{v})}{|G|}-\widetilde{\vartheta}'\right)_y dxdy\\
&= O(1)\int\int|\mathcal{Z}_t||\widetilde{v}_x||V_y|\left(|\mathcal{Z}_y|+|\mathcal{W}_y||V_y|\right)dxdy\\
&\le E_\alpha\int\int\mathcal{Z}_t^2 dxdy + O(1)\delta\int\int\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy.
\end{aligned}$$

Grouping together (3.13) and (3.14) we get

$$\begin{aligned}
&-\int\int\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\left(V_{xxt}V_{xy}+V_{xyt}V_{xx}\right)dxdy\\
&= -\frac{d}{dt}\int\int\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\mathcal{Z}_x\mathcal{Z}_y dxdy + O(1)(\varepsilon+\delta)\int\int\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right)dxdy.
\end{aligned}$$

Therefore, the relation (3.11) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\int\left[\mathcal{Z}_t^2+\widetilde{\vartheta}\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right)\right]dxdy+\alpha\int\int\mathcal{Z}_t^2 dxdy\\
&\le E_\alpha\int\int\mathcal{Z}_t^2 dxdy + O(1)(\varepsilon+\delta)\int\int\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy\\
&\quad -\frac{1}{2}\frac{d}{dt}\int\int\left[\frac{\vartheta'(V_x-\widetilde{v})^2}{|G|}\mathcal{Z}_x^2+(\vartheta-\widetilde{\vartheta})(\mathcal{Z}_x^2+\mathcal{Z}_y^2)+\frac{\vartheta' V_y^2}{|G|}\mathcal{Z}_y^2\right.\\
&\quad \left.+2\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\mathcal{Z}_x\mathcal{Z}_y\right]dxdy + O(1)(1+t)^{-2}\int\int\left(V_x^2+V_y^2\right)dxdy.
\end{aligned}\tag{3.16}$$

We pass now to the estimates regarding the quantity $\mathcal{W} = V_y$. Multiplying (3.10) by $\mathcal{W}_t$ and integrating by part one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\int\left[\mathcal{W}_t^2+\widetilde{\vartheta}\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)\right]dxdy+\alpha\int\int\mathcal{W}_t^2dxdy\\ &=\frac{1}{2}\int\int\widetilde{\vartheta}\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy+\int\int\mathcal{W}_tF_ydxdy=I_1+I_2.\end{aligned}\tag{3.17}$$

As we did in the previous estimate, the first term is easily bounded in the following way

$$|I_1| = O(1)\delta\int\int\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy.$$

Moreover,

$$\begin{aligned}I_2&=-\int\int\begin{pmatrix}V_{xyt}\\ v_{yyt}\end{pmatrix}\cdot\left((\vartheta-\widetilde{\vartheta})\begin{pmatrix}V_x+\widetilde{v}\\ V_y\end{pmatrix}\right)_y dxdy\\ &=-\int\int V_{xyt}\left[\vartheta_y(V_x-\widetilde{v})+(\vartheta-\widetilde{\vartheta})V_{xy}\right]dxdy\\ &\quad-\int\int V_{yyt}\left[\vartheta_yV_y+(\vartheta-\widetilde{\vartheta})V_{yy}\right]dxdy\\ &=J_1+J_2+J_3+J_4.\end{aligned}$$

Hence,

$$\begin{aligned}J_1&=-\int\int V_{xyt}(V_x+\widetilde{v})\left[\frac{\vartheta'(V_x+\widetilde{v})}{|G|}V_{xy}+\frac{\vartheta'V_y}{|G|}V_{yy}\right]dxdy\\ &=-\frac{1}{2}\frac{d}{dt}\int\int\mathcal{Z}_y^2\frac{\vartheta'(V_x+\widetilde{v})^2}{|G|}dxdy+O(1)(\varepsilon+\delta)\int\int\mathcal{Z}_y^2dxdy\\ &\quad-\int\int\frac{\vartheta'(V_x+\widetilde{v})}{|G|}V_yV_{xyt}V_{yy}dxdy,\end{aligned}$$

where, as before, the first term has the "right" sign, while the last term will be studied in the sequel. The terms J_2 and J_4 are similar to those we considered above

$$\begin{aligned}J_2&=-\frac{1}{2}\frac{d}{dt}\int\int(\vartheta-\widetilde{\vartheta})\mathcal{Z}_y^2dxdy+O(1)(\varepsilon+\delta)\int\int\mathcal{Z}_y^2dxdy;\\ J_4&=-\frac{1}{2}\frac{d}{dt}\int\int(\vartheta-\widetilde{\vartheta})\mathcal{W}_y^2dxdy+O(1)(\varepsilon+\delta)\int\int\mathcal{W}_y^2dxdy.\end{aligned}$$

Evaluating ϑ_y in J_3, we get

$$\begin{aligned}J_3&=-\int\int V_{yyt}V_y\left[\frac{\vartheta'(V_x+\widetilde{v})}{|G|}V_{xy}+\frac{\vartheta'V_y}{|G|}V_{yy}\right]dxdy\\ &=-\frac{1}{2}\frac{d}{dt}\int\int\mathcal{W}_y^2\frac{\vartheta'V_y^2}{|G|}dxdy+O(1)(\varepsilon+\delta)\int\int\mathcal{W}_y^2dxdy\\ &\quad-\int\int\frac{\vartheta'(V_x+\widetilde{v})}{|G|}V_yV_{yyt}V_{xy}dxdy.\end{aligned}$$

Finally,

$$
\begin{aligned}
&-\int\int \frac{\vartheta'(V_x-\widetilde{v})}{|G|} V_y\left(V_{xyt}V_{yy}+V_{yyt}V_{xy}\right) dxdy \\
&= -\frac{d}{dt}\int\int \frac{\vartheta'(V_x-\widetilde{v})}{|G|} V_y \mathcal{Z}_y \mathcal{W}_y dxdy + O(1)(\varepsilon+\delta)\int\int\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right) dxdy.
\end{aligned}
$$

Thus, (3.17) can be rewritten as follows

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\int\left[\mathcal{W}_t^2+\widetilde{\vartheta}\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)\right] dxdy + \alpha\int\int \mathcal{W}_t^2 dxdy \\
&= O(1)(\varepsilon+\delta)\int\int\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right) dxdy \\
&-\frac{1}{2}\frac{d}{dt}\int\int\Bigg[\frac{\vartheta'(V_x-\widetilde{v})^2}{|G|}\mathcal{Z}_y^2+(\vartheta-\widetilde{\vartheta})(\mathcal{Z}_y^2+\mathcal{W}_y^2)+\frac{\vartheta' V_y^2}{|G|}\mathcal{W}_y^2 \\
&+2\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\mathcal{Z}_y\mathcal{W}_y\Bigg] dxdy. \qquad (3.18)
\end{aligned}
$$

Now we multiply (3.9) for $\lambda\mathcal{Z}$, where, as in lemma 3.2, λ is a small, nonnegative constant which will be chosen at the end. Integration in $dxdy$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\int\left[\lambda\alpha\mathcal{Z}^2+2\lambda\mathcal{Z}\mathcal{Z}_t\right] dxdy + \lambda\int\int\widetilde{\vartheta}\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right) dxdy \\
&-\lambda\int\int \mathcal{Z}_t^2 dxdy = \lambda\int\int \mathcal{Z}\,\mathrm{div}\left[\widetilde{\vartheta}_x\begin{pmatrix}\mathcal{Z}\\ \mathcal{W}\end{pmatrix}\right] dxdy \\
&+\lambda\int\int \mathcal{Z}F_x dxdy = I_1+I_2. \qquad (3.19)
\end{aligned}
$$

Young inequality implies

$$
\begin{aligned}
I_2 &= -\lambda\int\int \mathcal{Z}_x\widetilde{\vartheta}_x V_x dxdy + \lambda\int\int \mathcal{W}_y\widetilde{\vartheta}_x V_x dxdy \\
&\le \lambda E_{\widetilde{\vartheta}}\int\int\left(\mathcal{Z}_x^2+\mathcal{W}_y^2\right) dxdy + \lambda O(1)(1+t)^{-1}\int\int V_x^2 dxdy,
\end{aligned}
$$

where $E_{\widetilde{\vartheta}}$ is a small, positive constant, depending only on the (positive) quantity

$$
\min\left\{\vartheta(v) : v\in[-\|\widetilde{v}\|_\infty, +\|\widetilde{v}\|_\infty]\right\},
$$

which will be chosen afterwards. Moreover,

$$
\begin{aligned}
I_2 &= -\lambda\int\int\begin{pmatrix}V_{xx}\\ v_{xy}\end{pmatrix}\cdot\left((\vartheta-\widetilde{\vartheta})\begin{pmatrix}V_x+\widetilde{v}\\ V_y\end{pmatrix}\right)_x dxdy \\
&= -\lambda\int\int V_{xx}\left[(\vartheta-\widetilde{\vartheta})_x(V_x-\widetilde{v})+(\vartheta-\widetilde{\vartheta})(V_{xx}-\widetilde{v}_x)\right] dxdy \\
&-\lambda\int\int V_{xy}\left[(\vartheta-\widetilde{\vartheta})_x V_y+(\vartheta-\widetilde{\vartheta})V_{xy}\right] dxdy \\
&= J_1+J_2+J_3+J_4.
\end{aligned}
$$

As in the previous calculations,

$$
\begin{aligned}
J_1 &= -\lambda \int\int V_{xx} \left[\frac{\vartheta'(V_x - \widetilde{v})}{|G|} V_{xx} + \frac{\vartheta' V_y}{|G|} V_{xy} \right. \\
&\quad \left. + \left(\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' \right) \widetilde{v}_x \right] (V_x - \widetilde{v}) dxdy \\
&\leq \lambda O(1)\varepsilon \int\int \left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2 \right) dxdy + \lambda O(1) \int\int |\mathcal{Z}_x||\widetilde{v}_x| \left(|V_x| + |V_y| \right) dxdy \\
&\leq \lambda O(1)\varepsilon \int\int \left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2 \right) dxdy + \lambda E_{\widetilde{\vartheta}} \int\int \mathcal{Z}_x^2 dxdy \\
&\quad + \lambda O(1)(1+t)^{-1} \int\int \left(V_x^2 + V_y^2 \right) dxdy,
\end{aligned}
$$

since the first term in J_1,

$$
-\lambda \int\int \frac{\vartheta'(V_x + \widetilde{v})^2}{|G|} V_{xx}^2 dxdy,
$$

is negative for ε sufficiently small, as we pointed out previously. Moreover,

$$
\begin{aligned}
J_2 &= -\lambda O(1)\varepsilon \int\int \mathcal{Z}_x^2 dxdy + \lambda E_{\widetilde{\vartheta}} \int\int \mathcal{Z}_x^2 dxdy \\
&\quad + \lambda O(1)(1+t)^{-1} \int\int \left(V_x^2 + V_y^2 \right) dxdy; \\
J_4 &= \lambda O(1) \int\int \mathcal{Z}_y^2 dxdy.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
J_3 &= -\lambda \int\int V_{xy} \left[\frac{\vartheta'(V_x - \widetilde{v})}{|G|} V_{xx} + \frac{\vartheta' V_y}{|G|} V_{xy} \right. \\
&\quad \left. + \left(\frac{\vartheta'(V_x - \widetilde{v})}{|G|} - \widetilde{\vartheta}' \right) \widetilde{v}_x \right] V_y dxdy \\
&\leq \lambda O(1)\varepsilon \int\int \left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2 \right) dxdy + \lambda O(1)\varepsilon \int\int \mathcal{Z}_y^2 dxdy \\
&\quad + \lambda O(1) \int\int |\mathcal{Z}_y||\widetilde{v}_x||V_y| \left(|V_x| + |V_y| \right) dxdy \\
&\leq \lambda O(1)\varepsilon \int\int \left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2 \right) dxdy + \lambda E_{\widetilde{\vartheta}} \int\int \mathcal{Z}_x^2 dxdy \\
&\quad + \lambda O(1)(1+t)^{-1} \int\int \left(V_x^2 + V_y^2 \right) dxdy.
\end{aligned}
$$

Thus, the relation (3.19) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int\int \left[\lambda\alpha \mathcal{Z}^2 + 2\lambda \mathcal{Z}\mathcal{Z}_t \right] dxdy + \lambda \int\int \widetilde{\vartheta} \left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2 \right) dxdy \\
&- \lambda \int\int \mathcal{Z}_t^2 dxdy \leq \lambda E_{\widetilde{\vartheta}} \int\int \left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2 + \mathcal{W}_y^2 \right) dxdy \\
&+ \lambda\varepsilon O(1) \int\int \left(\mathcal{Z}_x^2 + \mathcal{Z}_y^2 \right) dxdy \\
&+ \lambda O(1)(1+t)^{-1} \int\int \left(V_x^2 + V_y^2 \right) dxdy. \qquad (3.20)
\end{aligned}
$$

A similar estimate can be achieved for the quantity $\mathcal{W}$, by multiplying (3.10) by $\lambda\mathcal{W}$. Therefore, proceeding as before, we end up to a relation of the form

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\int\left[\lambda\alpha\mathcal{W}^2+2\lambda\mathcal{W}\mathcal{W}_t\right]dxdy+\lambda\int\int\widetilde{\vartheta}\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy\\&-\lambda\int\int\mathcal{W}_t^2dxdy\le\lambda O(1)\varepsilon\int\int\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy.\end{aligned}\tag{3.21}$$

Summing the estimates (3.16), (3.18), (3.20) and (3.21) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\int\left[\mathcal{Z}_t^2+\widetilde{\vartheta}\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right)\right]dxdy+(\alpha-\lambda)\int\int\mathcal{Z}_t^2dxdy\\&+\frac{1}{2}\frac{d}{dt}\int\int\left[\mathcal{W}_t^2+\widetilde{\vartheta}\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)\right]dxdy+(\alpha-\lambda)\int\int\mathcal{W}_t^2dxdy\\&+\frac{1}{2}\frac{d}{dt}\int\int\left[\lambda\alpha\mathcal{Z}^2+2\lambda\mathcal{Z}\mathcal{Z}_t\right]dxdy+\lambda\int\int\widetilde{\vartheta}\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right)dxdy\\&+\frac{1}{2}\frac{d}{dt}\int\int\left[\lambda\alpha\mathcal{W}^2+2\lambda\mathcal{W}\mathcal{W}_t\right]dxdy+\lambda\int\int\widetilde{\vartheta}\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy\\&\le E_\alpha\int\int\mathcal{Z}_t^2dxdy+O(1)(\varepsilon+\delta)\int\int\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy\\&-\frac{1}{2}\frac{d}{dt}\int\int\left[\frac{\vartheta'(V_x-\widetilde{v})^2}{|G|}\mathcal{Z}_x^2+(\vartheta-\widetilde{\vartheta})(\mathcal{Z}_x^2+\mathcal{Z}_y^2)+\frac{\vartheta' V_y^2}{|G|}\mathcal{Z}_y^2\right.\\&\left.+2\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\mathcal{Z}_x\mathcal{Z}_y\right]dxdy+O(1)(1+t)^{-2}\int\int\left(V_x^2+V_y^2\right)dxdy\\&+O(1)(\varepsilon+\delta)\int\int\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy\\&-\frac{1}{2}\frac{d}{dt}\int\int\left[\frac{\vartheta'(V_x-\widetilde{v})^2}{|G|}\mathcal{Z}_y^2+(\vartheta-\widetilde{\vartheta})(\mathcal{Z}_y^2+\mathcal{W}_y^2)+\frac{\vartheta' V_y^2}{|G|}\mathcal{W}_y^2\right.\\&\left.+2\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\mathcal{Z}_y\mathcal{W}_y\right]dxdy+\lambda E_{\widetilde{\vartheta}}\int\int\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy\\&+\lambda\varepsilon O(1)\int\int\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2\right)dxdy+\lambda O(1)(1+t)^{-1}\int\int\left(V_x^2+V_y^2\right)dxdy\\&+\lambda O(1)\varepsilon\int\int\left(\mathcal{Z}_y^2+\mathcal{W}_y^2\right)dxdy.\end{aligned}$$

At this point, we choose λ, ε, δ, E_α, $E_{\widetilde{\vartheta}}$ sufficiently small and we control the products $\mathcal{Z}\mathcal{Z}_t$ and $\mathcal{W}\mathcal{W}_t$ in order to have

$$\begin{aligned}&\frac{d}{dt}\int\int\left[\mathcal{Z}_t^2+\mathcal{W}_t^2+\widetilde{\vartheta}\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2+\mathcal{W}_y^2\right)+\lambda(\mathcal{Z}+\mathcal{W})\right]dxdy\\&+\int\int\left[\mathcal{Z}_t^2+\mathcal{W}_t^2+\lambda\widetilde{\vartheta}\left(\mathcal{Z}_x^2+\mathcal{Z}_y^2+\mathcal{W}_y^2\right)\right]dxdy\\&\le-\frac{1}{2}\frac{d}{dt}\int\int\left[\frac{\vartheta'(V_x-\widetilde{v})^2}{|G|}\mathcal{Z}_x^2+(\vartheta-\widetilde{\vartheta})(\mathcal{Z}_x^2+\mathcal{Z}_y^2)+\frac{\vartheta' V_y^2}{|G|}\mathcal{Z}_y^2\right.\end{aligned}$$

$$+2\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\mathcal{Z}_x\mathcal{Z}_y\Bigg]\,dxdy+O(1)(1+t)^{-2}\int\int\left(V_x^2+V_y^2\right)dxdy$$
$$-\frac{1}{2}\frac{d}{dt}\int\int\left[\frac{\vartheta'(V_x-\widetilde{v})^2}{|G|}\mathcal{Z}_y^2+(\vartheta-\widetilde{\vartheta})(\mathcal{Z}_y^2+\mathcal{W}_y^2)+\frac{\vartheta' V_y^2}{|G|}\mathcal{W}_y^2\right.$$
$$\left.+2\frac{\vartheta'(V_x-\widetilde{v})}{|G|}V_y\mathcal{Z}_y\mathcal{W}_y\right]dxdy$$
$$+\lambda O(1)(1+t)^{-1}\int\int\left(V_x^2+V_y^2\right)dxdy. \tag{3.22}$$

Hence, we integrate (3.22) in dt and, using the relations

$$|\vartheta-\widetilde{\vartheta}|=O(1)\varepsilon$$
$$|V_y|=O(1)\varepsilon$$
$$\frac{\vartheta'(V_x-\widetilde{v})^2}{|G|}>0 \quad \text{for } \varepsilon<<1,$$

we get the first H^2 estimate

$$\|V_{xx}(t)\|^2+\|V_{xy}(t)\|^2+\|V_{yy}(t)\|^2+\|z_x(t)\|^2+\|z_y(t)\|^2$$
$$+\int_0^t\left(\|V_{xx}(\tau)\|^2+\|V_{xy}(\tau)\|^2+\|V_{yy}(\tau)\|^2+\|z_x(\tau)\|^2+\|z_y(\tau)\|^2\right)d\tau$$
$$=O(1)\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).$$

Moreover, we first multiply (3.22) by $(1+t)$ and then we integrate in dt to get (using the relations and the estimate above)

$$(1+t)\left[\|V_{xx}(t)\|^2+\|V_{xy}(t)\|^2+\|V_{yy}(t)\|^2+\|z_x(t)\|^2+\|z_y(t)\|^2\right]$$
$$+\int_0^t(1+\tau)\left[\|V_{xx}(\tau)\|^2+\|V_{xy}(\tau)\|^2+\|V_{yy}(\tau)\|^2+\|z_x(\tau)\|^2+\|z_y(\tau)\|^2\right]d\tau$$
$$=O(1)\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).$$

Finally, we consider (3.22) for $\lambda=0$ and we multiply it by $(1+t)^2$. Integrating the relation obtained in dt and using all the relations above, we end up with the last estimate we need to conclude the proof. □

The differentiation of the equation (3.4) with respect to t gives a better asymptotic result, contained in the following lemma. This phenomenon follows from the fact that the t-derivatives of $\overline{v}$ (and hence of $\widetilde{v}$) have better asymptotic decays than the x-derivatives of $\overline{v}$.

Lemma 3.5. *Suppose ε, δ and $\|V_0\|_7^2+\|z_0\|_6^2$ are sufficiently small. Then*

$$(1+t)^2\|z(t)\|^2+(1+t)^3(\|z_t(t)\|^2+\|z_x(t)\|^2+\|z_y(t)\|^2)$$
$$+\int_0^t\left[(1+\tau)^2(\|z_x(\tau)\|^2+\|z_y(\tau)\|^2)+(1+\tau)^3\|z_t(\tau)\|^2\right]d\tau$$
$$=O(1)\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).$$

The proof of this lemma follows step by step the proof of lemma 3.4 and it is omitted. Finally, iterating the procedure, it is possible to prove the following lemmas.

LEMMA 3.6. *Suppose ε, δ and $\|V_0\|_7^2 + \|z_0\|_6^2$ are sufficiently small. Then, for any $k \le 6$,*

$$(1+t)^{k+1}\|D^{k+1}V(t)\|^2 + (1+t)^{k+1}\|D^k z(t)\|^2 + \int_0^t (1+\tau)^k \|D^{k+1}V(\tau)\|^2 d\tau + \int_0^t (1+\tau)^{k+1}\|D^k z(\tau)\|^2 d\tau = O(1)\left(\|V_0\|_{k+1}^2 + \|z_0\|_k^2 + \delta\right).$$

LEMMA 3.7. *Suppose ε, δ and $\|V_0\|_7^2 + \|z_0\|_6^2$ are sufficiently small. Then, for any $k \le 6$,*

$$(1+t)^{k+2}\|D^k z(t)\|^2 + (1+t)^{k+2}\|D^{k-1}z_t(t)\|^2 + \int_0^t (1+\tau)^{k+1}\|D^k z(\tau)\|^2 d\tau + \int_0^t (1+\tau)^{k+2}\|D^{k-1}z_t(\tau)\|^2 d\tau = O(1)\left(\|V_0\|_{k+1}^2 + \|z_0\|_k^2 + \delta\right).$$

REMARK 3.8. Since the nonlinear function ϑ depends on V_x and V_y, in order to compute the energy estimates, we have to bound the H^4 norm of V, so we bound, by Sobolev embedding (in 2-D), the L^∞ norm of V_x and V_y. However, the trilinear terms which appears in the energy estimates to achieve the H^4 bounds are of the form

$$D^\alpha V D^\beta V D^\gamma V,$$

with $|\alpha| + |\beta| + |\gamma| \le 10$. Therefore, since in all the terms of the H^4 estimate we must have α, β, $\gamma \le 4$, there are terms with the property α, β, $\gamma \ge 3$. Therefore, the H^4 bounds are not enough to close the estimate and we need at least H^6 to control third derivatives in L^∞. With a simple argument, we can prove that the H^7 norm is enough to close the proof. Indeed, in the H^7 case, the trilinear terms are of the form

$$D^\alpha V D^\beta V D^\gamma V,$$

with $|\alpha| + |\beta| + |\gamma| \le 16$. Thus, the terms with the maximum number of derivatives can be reduced, by integration by parts, in one of the two following forms:

$$\partial_t D^7 V D^7 V D^1 V$$
$$D^7 V D^\alpha V D^\beta V,$$

with $|\alpha| + |\beta| = 9$. The first term can be written as a total derivative with respect to t and it is controlled by the energy (using the smallness of $|D^1 V| = O(1)\varepsilon$). The second one is no longer trilinear, since now either α or β is less or equal to 4 and hence either $D^\alpha V$ or $D^\beta V$ is controlled in L^∞.

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