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L_1 Stability for Systems of Hyperbolic Conservation Laws

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Abstract. In this paper, we summarize our results on constructing a nonlinear functional which is equivalent to L_1 distance between two weak solutions to systems of hyperbolic conservation laws and non-increasing in time. The weak solutions are constructed by Glimm scheme through the wave tracing methed. Therefore, such an explicit functional depending only on the two wave patterns of the solutions yields directly the uniqueness of solutions by Glimm scheme and reveals the effects of nonlinear interaction and coupling on the L_1 topology.

1. Introduction

Consider the Cauchy problem for a system of conservation laws,

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \quad t\geq 0,\ -\infty<x<\infty, \tag{1.1}$$

$$u(x,0)=u_0(x), \qquad -\infty<x<\infty, \tag{1.2}$$

where u and $f(u)$ are n-vectors. We assume that the system is strictly hyperbolic, i.e. the matrix $\partial f(u)/\partial u$ has real and distinct eigenvalues $\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_n(u)$ for all u under consideration, with the corresponding right eigenvectors $r_i(u),\ i=1,2,\ldots,n$. Each characteristic field is assumed to be either linearly degenerate or genuinely nonlinear, [13], i.e. $r_i(u)\cdot\nabla\lambda_i(u)\equiv 0$ or $r_i(u)\cdot\nabla\lambda_i(u)\neq 0$, $i=1,2,\ldots,n$.

The purpose of our research is to construct a nonlinear functional $H(t)=H[u(\cdot,t),v(\cdot,t)]$, which is equivalent to $\|u-v\|_{L_1}$ of two weak solutions $u(x,t)$ and $v(x,t)$ to (1.1) and (1.2) and is non-increasing in time. It also depends explicitly on the wave patterns of these two solutions. In general, the functional $H[u(\cdot,t),v(\cdot,t)]$ consists three parts: the first part is the product of the Glimm's functional and $L(t)$ representing the L_1 distance between $u(x,t)$ and $v(x,t)$, which reveals the interaction effects of nonlinear waves on the L_1 topology; the second part is $Q_d(t)$, which registers the effect of nonlinear coupling of waves in different families on $\|(u-v)(x,t)\|_{L_1(x)}$, making use of the strict hyperbolicity of the system; the third part,

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which is called the generalised entropy functional, captures the genuine nonlinearity of the characteristic fields. If the distance between two states in the phase plane is measured by rarefaction wave curves instead of Hugoniot wave curves, there is one more main component in the nonlinear functional $H[u(\cdot,t),v(\cdot,t)]$ denoted by $L_h(t)$. This functional is needed because the generalised entropy functional can be used only to control the difference of the third orders of shocks pertaining to different solutions instead of shock waves strengths to the cubic power. The summation of $L(t)$ and $L_h(t)$ shows that some of the effect of a shock on the L_1 distance is conservative.

The weak solutions in consideration are constructed by Glimm scheme through the wave tracing method, [11, 16]. Therefore the nonlinear functional $H[u(\cdot,t), v(\cdot,t)]$ immediately yields the uniqueness of weak solutions obtained by Glimm scheme. Furthermore, measuring the L_1 distance between the solutions by the nonlinear functional $H[u(\cdot,t),v(\cdot,t)]$ is robust and therefore it does not require any particular approximation scheme. In [5], this functional is defined for weak solutions obtained by the wave front tracking method, [1, 3]. In fact, our analysis would be applied to any approximate scheme based on the characterisitc method, c.f. [7, 10].

There has been much progress on the well-posedness problem. In [2], the problem when the two solutions are infinitesimally close is studied by making uses of the fact that the topology of the shock waves are close in this case. This analysis is used to study the continuous dependence of the solutions on its initial data for 2×2 systems in [1] and for $n\times n$ systems in [3]. By homotopically deforming one solution to the other to construct a Riemann semigroup, this line of approach requires the monitoring of the changes of the topology of shocks in the approximate solutions. Hence the nonlinear functional thus defined depends not only on the two wave patterns of $u(x,t)$ and $v(x,t)$.

Glimm's nonlinear functional can be defined for any BV entropy solutions and was proved to be non-increasing in time for piecewise smooth solutions [5]. It will be interesting to define the above nonlinear functional $H[u(\cdot,t),v(\cdot,t)]$ for general BV entropy solutions and show that it is non-increasing in time. Also it will be interesting to study the more general case without the assumption of genuine nonlinearity.

Some general uniqueness formulations have been formulated by various authors, [3, 4] and references therein. For attempts on the uniqueness based on the L_2-norm, see [8, 9, 15, 17, 24]. For comments on non-contractiveness in the L_1-norm, see [27].

2. Glimm's Functional and Wave Tracing

Glimm scheme uses Riemann solutions as building blocks and consists of constructing Riemann data by using a random sequence. At each time step, a nonlinear functional corresponding to the interaction potential is used to control the increase in the new waves strengths after interaction. The Riemann problem for (1.1) with initial value

$$u(x,0)=\begin{cases} u_l, & x<0,\\ u_r, & x>0,\end{cases} \tag{2.1}$$

has the scattering property which represents the large time behaviour of the solution. The rarefaction curve is the integral curve $R_i(u_0)$ of $r_i(u)$ through a given state u_0, and the Hugoniot curves through a state u_0 are

$$H(u_0) \equiv \{u \ : \ u - u_0 = \sigma(u_0, u)(f(u) - f(u_0))\} \tag{2.2}$$

for some scalar $\sigma = \sigma(u_0, u)$. The following basic theorem is from [13].

THEOREM 2.1. *Suppose that system* (1.1) *is strictly hyperbolic. Then, in a small neighborhood of a state* u_0*, the Hugoniot curves consists of* n *curves* $H_i(u_0)$*,* $i = 1, 2, \dots, n$*, with the following properties:*
(i) *The Hugoniot curve* $H_i(u_0)$ *and the rarefaction curve* $R_i(u_0)$ *have second order contact at* $u = u_0$*,*
(ii) *The shock speed* $\sigma(u_0, u)$*,* $u \in H_i(u_0)$ *satisfies:*

$$\sigma(u_0, u) = \frac{1}{2}(\lambda_i(u) + \lambda_i(u_0)) + O(1)|u - u_0|^2. \tag{2.3}$$

(iii) *For a genuinely nonlinear field,* (u_0, u)*,* $u \in R_i^+(u_0)$*, is a rarefaction wave; and* $(u_-, u_+) = (u_0, u)$*,* $u \in H_i^-(u_0)$*, is a shock wave satisfying the following Lax entropy condition*

$$\lambda_i(u_-) > s > \lambda_i(u_+). \tag{2.4}$$

(iv) *For a linearly degenerate field* $H_i(u_0) = R_i(u_0)$ *and* (u_0, u)*,* $u \in R_i(u_0)$*, forms a contact discontinuity with speed* s*:*

$$s = \lambda_i(u) = \lambda_i(u_0). \tag{2.5}$$

We construct the wave curves $W_i(u_0)$ as follows:

$$W_i(u_0) \equiv \begin{cases} R_i^+(u_0) \ U \ H_i^-(u_0), & i\text{-th characteristic g.nl.;} \\ R_i(u_0) = H_i(u_0), & i\text{-th characteristic l.dg.} \end{cases} \tag{2.6}$$

where for each genuinely nonlinear field, we let $\lambda_i(u) < \lambda_i(u_0)$ for the states u on $H_i^-(u_0)$ and $R_i^-(u_0)$ and $\lambda_i(u) \geq \lambda_i(u_0)$ for u on $H_i^+(u_0)$ and $R_i^+(u_0)$. Thus (u_0, u) forms an elementary i-wave when $u \in W_i(u_0)$. These waves are the building blocks for the solution of the Riemann problem.

By using strict hyperbolicity and the inverse function theorem, the Riemann problem can be solved in the class of elementary waves [13]:

THEOREM 2.2. *Suppose that each characteristic field is either genuinely nonlinear or linearly degenerate. Then the Riemann problem for* (1.1) *has a unique solution in the class of elementary waves provided that the states are in a small neighborhood of a given state.*

The Glimm scheme is a finite difference scheme involving a random sequence a_i, $i = 0, 1, \dots$, $0 < a_i < 1$. Let $r = \triangle x$, $s = \triangle t$ be the mesh sizes satisfying the (C-F-L) condition

$$\frac{r}{s} > \ 2|\lambda_i(u)|, \qquad 1 \leq i \leq n, \tag{2.7}$$

for all states u under consideration. The approximate solutions $u(x, t) = u_r(x, t)$ depends on the random sequence $\{a_k\}$ and is defined inductively in time as follows:

$$u(x, 0) = u_0((h + a_0)r), \qquad hr < x < (h + 1)r, \tag{2.8}$$

$$u(x, ks) = u((h + a_i)r - 0, ks - 0), \quad hr < x < (h + 1)r, \quad k = 1, 2, \dots . \tag{2.9}$$

Due to (C-F-L) condition (2.6) these elementary waves do not interact within the layer. Thus the approximation solution is an exact solution except at the interfaces $t = ks$, $k = 1, 2, \dots$. The following theorem is from Glimm [11]:

THEOREM 2.3. *Suppose that the initial data $u_0(x)$ is of small total variation T.V. Then the approximate solutions $u(x,t)$ are of small total variation $O(1)T.V.$ in x for all time t. Moreover, for almost all choices of the sequence $\{a_k\}_{k=1}^{\infty}$, the approximate solutions tend to an exact solution for a sequence of the mesh sizes r, s tending to zero with r/s fixed and r,s satisfying (C-F-L) condition. The exact solution $u(x,t)$ is of bounded variation in x for any time $t \geq 0$:*

$$\text{variation}_{-\infty<x<\infty}\, u(x,t) = O(1)T.V. \tag{2.10}$$

and is continuous in $L_1(x)$-norm:

$$\int_{-\infty}^{\infty} |u(x,t_1) - u(x,t_2)|dx = O(1)|t_1 - t_2|, \quad t_1,\ t_2 \geq 0. \tag{2.11}$$

The proof of the above Theorem is based on the proof of the non-increasingness of the Glimm's functional $F(u)(t)$ defined as follows:

$$\begin{aligned}
J(t) &\equiv \sum\{|\alpha| :\ \alpha \text{ any wave at time } t\},\\
D_d(t) &\equiv \sum\{|\alpha|\ |\beta| :\ \alpha \text{ and } \beta \text{ interacting waves of distinct}\\
&\qquad \text{characteristic families at time } t\},\\
D_s(t) &\equiv \sum_{i=1}^{n} D_s^i, \qquad (2.12)\\
D_s^i(t) &\equiv \sum\{|\alpha||\beta|(-\min\{\Theta(\alpha,\beta),0\}) :\ \alpha \text{ and } \beta \text{ interacting}\\
&\qquad i\text{-waves at time } t\},\\
D(t) &\equiv D_d(t) + D_s(t),\\
F(u)(t) &\equiv J(t) + MD(t).
\end{aligned}$$

Here M is a sufficiently large constant and $\Theta(\alpha,\beta)$ is the interacting angle between α and β, cf. [11, 23].

It was shown in [16] that in fact the approximate solutions constructed by Glimm's scheme converges to a weak solution as long as the random sequence is equidistributed. In this wave tracing method, the waves are classified into the following three categories: surviving ones, canceled ones, and those produced by interactions. We summarize the result in [16] as follows:

THEOREM 2.4. *The waves in an approximate solution in a given a time zone $\Lambda = \{(x,t) : -\infty < x < \infty, K_1 s \leq t < K_2 s\}$ can be partitioned into subwaves of categories I, II or III with the following properties up to an error due to the random sequence:*
(i) *The subwaves in I are surviving. Given a subwave $\alpha(t)$, $K_1 s \leq t < K_2 s$ in I, write $\alpha \equiv \alpha(K_1 s)$ and denote by $|\alpha(t)|$ its strength and $\lambda(\alpha(t)$ its speed at time t, by$[\lambda(\alpha)]$ the variation of its speed and by $[\alpha]$ the variation of the jump of the states across it over the time interval $K_1 s \leq t < K_2 s$. Then*

$$\sum_{\alpha\in I}([\alpha] + |\alpha(K_1 s)|[\lambda(\alpha)]) = O(1)D(\Lambda). \tag{2.13}$$

(ii) *A subwave $\alpha(t)$ has positive initial strength $|\alpha(K_1 s)| > 0$, but is canceled in the zone Λ, $|\alpha(K_2 s)| = 0$. Moreover, the total strength and variation of the wave shape* *satisfy*

$$\sum_{\alpha\in II}([\alpha]+|\alpha(K_1 s)|)\le C(\Lambda)+O(1)D(\Lambda). \tag{2.14}$$

(iii) *A subwave in III has zero initial strength $|\alpha(K_1s)|=0$, and is created in the zone Λ, $|\alpha(K_2s)>0$. Moreover, the total variation satisfies*

$$\sum_{\alpha\in III}([\alpha]+|\alpha(t)|)=O(1)D(\Lambda),\ \ K_1s\le t<K_2s, \tag{2.15}$$

where $C(\Lambda)$ and $D(\Lambda)$ represent the cancellation and interaction potential change in the region Λ respectively.

An application of the above theorem gives the following deterministic version of the Glimm scheme, cf. [16]:

THEOREM 2.5. *Suppose that the random sequence a_k, $k=1,2,\dots$ is equidistributed. Then the limiting function $u(x,t)$ of the Glimm scheme is a weak solution of the hyperbolic conservation laws.*

Theorem 2.4 was also applied to the study of the regularity and large time behaviour of the solutions and the convergence rate of the Glimm scheme, cf. [6, 18] and reference therein.

The application of Theorem 2.4 to L_1 stability of weak solutions is that waves can be viewed as linearly superimposed in each region $(p-1)Ns<t<pNs$ in the wave tracing method.

3. Nonlinear Functionals and Main Theorems

Given two solutions $u(x,t)$ and $v(x,t)$ of the system (1.1), we define their pointwise distance along the Hugoniot curves: solve the Riemann problem $(u(x,t), v(x,t))$ by discontinuity waves:

$$u_0=u(x,t),\ u_i\in H_i(u_{i-1}),\quad i=0,1,\dots,n,\ u_n=v(x,t). \tag{3.1}$$

Without loss of generality, we assume that the i-th component u^i of the vector u is a non-singular parameter along the i-th Hugoniot and rarefaction curves. We set

$$q_i(x,t)\equiv(u_i-u_{i-1})^i,\qquad \lambda_i(x)\equiv\lambda_i(u_{i-1}(x),u_i(x)), \tag{3.2}$$

This way of assigning the distance is convenient in that u^i is a conservative quantity and so it satisfies simple wave interaction estimates. Another advantage over choosing the Euclidean distance is that the strength of a shock (u_-,u_+) is the same as that of the rarefaction shock (u_+,u_-) in our measurement.

For an i-wave α^i in the solutions $u(x,t)$ or $v(x,t)$, we denote by $x(\alpha^i)=x(\alpha^i(t))$ its location at time t, and $q_j^{\pm}(\alpha^i)$ for $q_j(x(\alpha^i)\pm,t)$, $1\le j\le n$. For $j=i$ we also use the abbreviated notations $q^{\pm}(\alpha^i)=q_i^{\pm}(\alpha^i)$. The linear part $L[u,v]$ of the nonlinear functional $H[u,v]$ is equivalent to the $L_1(x)$ distance of the solutions:

$$L[u(\cdot,t),v(\cdot,t)]\equiv\sum_{i=1}^n L_i[u(\cdot,t),v(\cdot,t)]$$

$$L_i[u(\cdot,t),v(\cdot,t)]\equiv\int_{-\infty}^{\infty}|q_i(x,t)|dx. \tag{3.3}$$

Without any ambiguity, we will use u and v to denote the approximate solutions in the Glimm scheme and also the corresponding weak solutions when the mesh sizes tend to zero. As in [5, 23], we will use the notations $J(u)$ and $J(v)$ to denote the waves in the solutions u and v at a given time, respectively. And $J \equiv J(u) \cup J(v)$. Moreover, α^i denotes a i-wave in J. The other two components of the nonlinear functional $H[u, v]$, the quadratic $Q_d(t)$ and the generalized entropies $E(t)$, are defined as follows:

$$Q_d(t) \equiv Q_d[u(\cdot,t), v(\cdot,t)] = \sum_{\alpha^i \in J} Q_d(\alpha^i)$$

$$Q_d(\alpha^i) = |\alpha^i| \Big(\sum_{j>i} \int_{-\infty}^{x(\alpha^i)} |q_j(x,t)|dx + \sum_{j<i} \int_{x(\alpha^i)}^{\infty} |q_j(x,t)|dx \Big) \tag{3.4}$$

$$E(t) \equiv E[u(\cdot,t), v(\cdot,t)] = \sum_{i=1}^{n} E^i(t),$$

$$\begin{aligned} E^i(t) = &\sum_{\alpha^i \in J(u)} |\alpha^i| \Big(\int_{x(\alpha^i)}^{\infty} |\min\{0, q_i(x,t)\}|dx + \int_{-\infty}^{x(\alpha^i)} \max\{0, q_i(x,t)\}dx \Big) \\ &+ \sum_{\alpha^i \in J(v)} |\alpha^i| \Big(\int_{x(\alpha^i)}^{\infty} \max\{0, q_i(x,t)\}dx + \int_{-\infty}^{x(\alpha^i)} |\min\{0, q_i(x,t)\}|dx \Big). \end{aligned} \tag{3.5}$$

For any given time $T = MNs$ in the Glimm scheme through the wave tracing method, we define the main nonlinear functional $H(t)$ as follows:

$$H(t) \equiv H[u(\cdot,t), v(\cdot,t)] \equiv (1 + K_1 F(p-1)Ns))L(t) + K_2(Q_d(t) + E(t)),$$

for $t \in ((p-1)Ns, pNs)$, $p = 1, \ldots, M$. Notice here that the Glimm's functional $F = F(u) + F(v)$ is valued at the end time $t = (p-1)Ns$. The jump of the functionals $L(t)$, $Q_d(t)$ and $E(t)$ at each time step $t = pNs$, $p = 1, 2, \ldots, M$ due to wave interaction can be controlled by $O(1)[F(pNs) - F((p-1)Ns)]L(pNs)$.

Remark 3.1. If the distance between the two solutions is not measured by the Hugoniot curves but rarefaction wave curves as was done in Liu-Yang [22] for 2×2 system, then the functional $H[u(\cdot,t), v(\cdot,t)]$ has more components. One of which is called $L_h(t)$ which is used to control the error caused by the bifurcation of the shock wave curve from the rarefaction curve, and it is of the third order of the shock wave strength. The functional can be generalized to the general $n \times n$ system in the following form

$$H[u(\cdot,t), v(\cdot,t)] \equiv (1 + K_1 F)(L + L_h) + K_2(Q_d(t) + E(t)) + k_3 D(t),$$

where $D(t)$ is used to control the jumps of L_h due to the introduction of the 'domain of influence' for shock waves. The 'domain of influence' for shock wave can be defined when we consider the following two sets of n scalar functions:

$$\theta^i(x,t) \equiv \sum_{\alpha^i \in J(u), x(\alpha^i) < x} (q_i(x(\alpha^i)+, t) - q_i(x(\alpha^i)-, t)),$$

$$\eta^i(x,t) \equiv \sum_{\alpha^i \in J(v), x(\alpha^i) < x} (q_i(x(\alpha^i)+, t) - q_i(x(\alpha^i)-, t)),$$

where $q_i(x,t)$, $i = 1, 2, \ldots, n$, is defined as in (3.1) using rarefaction wave curves instead of Hugoniot curves.

To estimate $dL(t)/dt$ inside each region $(p-1)Ns < t < pNs$, the following two lemmas are needed.

LEMMA 3.1. *Let* $\bar{u} \in \Omega$, $\xi, \xi' \in \mathbf{R}$, $k \in \{1, \ldots, n\}$. *Define the states and the wave speeds*

$$u = H_k(\xi)(\bar{u}), \quad u' = H_k(\xi')(u), \quad u'' = H_k(\xi+\xi')(\bar{u}),$$
$$\lambda = \lambda_k(\bar{u}, u), \quad \lambda' = \lambda_k(u, u'), \quad \lambda'' = \lambda_k(\bar{u}, u'').$$

Then we have

$$|(\xi+\xi')(\lambda''-\lambda') - \xi(\lambda-\lambda')| = O(1)\cdot|\xi\xi'|\,|\xi+\xi'|.$$

LEMMA 3.2. *If the values* ξ, ξ_j, ξ'_j, $j = 1, 2, \ldots, n$, *satisfy*

$$H_n(\xi_n)\circ\cdots\circ H_1(\xi_1)(u) = \begin{cases} H_n(\xi'_n)\circ\cdots\circ H_1(\xi'_1)\circ H_i(\xi)(u), & or \\ H_i(\xi)\circ H_n(\xi'_n)\circ\cdots\circ H_1(\xi'_1)(u), \end{cases} \tag{3.6}$$

then

$$|\xi_i - \xi'_i - \xi| + \sum_{j\neq i}|\xi_j - \xi'_j| = O(1)|\xi|\left(|\xi'_i||\xi'_i+\xi| + \sum_{j\neq i}|\xi'_j|\right).$$

For the particular case where $\alpha^i = (u_-, u_+)$ is a shock in v with jump $[\alpha^i] \equiv (u_+ - u_-)^i$, the first part of Lemma 3.2 becomes

$$\begin{aligned} &\left|q^+(\alpha^i) - q^-(\alpha^i) - [\alpha^i]\right| + \sum_{j\neq i}|q_j^+(\alpha^i) - q_j^-(\alpha^i)| \\ &\quad = O(1)\cdot\left(|q^-(\alpha^i)||q^-(\alpha^i)+[\alpha^i]| + \sum_{j\neq i}\left|q_j^-(\alpha^i)\right|\right)|\alpha^i|, \\ &\quad = O(1)\cdot\left(|q^+(\alpha^i)||q^+(\alpha^i)+[\alpha^i]| + \sum_{j\neq i}\left|q_j^+(\alpha^i)\right|\right)|\alpha^i|. \end{aligned} \tag{3.7}$$

For definiteness, we set $[\alpha^i] < 0$ if α^i is a shock, and $[\alpha^i] > 0$ if α^i is a rarefaction wave. Recalling that $[\alpha^i] \in]0, \epsilon]$ when α^i is a rarefaction wave, using both parts of Lemma 3.2, we have the estimates

$$\begin{aligned} &\left|q^+(\alpha^i) - q^-(\alpha^i) - [\alpha^i]\right| + \sum_{j\neq i}|q_j^+(\alpha^i) - q_j^-(\alpha^i)| \\ &\quad = O(1)\cdot\left(\epsilon + |q^-(\alpha^i)||q^-(k_\alpha^i)+[\alpha^i]| + \sum_{j\neq i}\left|q_j^-(\alpha^i)\right|\right)|\alpha^i|, \\ &\quad = O(1)\cdot\left(\epsilon + |q^+(\alpha^i)|\,|q^+(\alpha^i)+[\alpha^i]| + \sum_{j\neq i}\left|q_j^+(\alpha^i)\right|\right)|\alpha^i|. \end{aligned} \tag{3.8}$$

The error $O(1)\epsilon$ due to rarefaction shocks and the one due to the random sequence tend to zero as the grid size tends to zero. Besides these errors, there are two main errors of the following order in estimating $dL(t)/dt$ when $t \in ((p-1)Ns, pNs)$:

$$E_1 = \sum_{\alpha^i\in J}|\alpha^i|\sum_{j\neq i}\left|q_j^\pm(\alpha^i)\right|; \quad \text{and} \quad E_2 = \sum_{\alpha^i\in J}|\alpha^i|\max\{q^+(\alpha^i)q^-(\alpha^i), 0\}.$$

By using the strict hyperbolicity of the system, it can be shown that the error term E_1 can be controlled by the good terms in $dQ_d(t)/dt$. And the error term E_2 can be controlled by the good terms from $dE(t)/dt$ by the genuine nonlinearity of the charateristic field. The reason that the cubic order error term E_2 can be controlled by the generalised entropy functional comes from the following theorem for convex scalar conservation laws.

THEOREM 3.1. *For a convex scalar conservation law, the generalized entropy functional is defined as follows:*

$$E(t) = \sum_{\alpha\in J_1} |\alpha| \Big(\int_{x(\alpha)}^{\infty} (u-v)_+(x,t)dx + \int_{-\infty}^{x(\alpha)} (u-v)_-(x,t)dx \Big)$$

$$+ \sum_{\alpha\in J_2} |\alpha| \Big(\int_{x(\alpha)}^{\infty} (v-u)_+(x,t)dx + \int_{-\infty}^{x(\alpha)} (v-u)_-(x,t)dx \Big), \tag{3.9}$$

for any two approximate solutions $u(x,t)$ and $v(x,t)$ in the Glimm's scheme through the wave tracing method with total variations bounded by T.V.. The generalised entropy functional satisfies

$$\frac{d}{dt}E(t) \leq -C_1 \sum_{\alpha\in J} |\alpha| \max\{q_-(\alpha)q_+(\alpha), 0\} + O(1)T.V.\epsilon, \tag{3.10}$$

where

$$q_\pm(\alpha) = q_\pm(\alpha(t)) \equiv (u_1 - u_2)(x(\alpha(t)\pm, t)). \tag{3.11}$$

Here the summation is over all waves α at time t in both solutions.

REMARK 3.2. Since the derivative of the integral of a convex entropy with respect to time gives a negative of all shock waves strengths to the cubic power, the L_2 norm of a solution can be used when we consider the case when one of the solution is a constant. In fact, for $u(x,0) \in L_1(x)$, the nonlinear functional $H[u(x,t)]$ takes a form [21]:

$$H[u(\cdot,t)] \equiv (1 + K_1F)L(t) + K_2(Q_d(t) + ||u(\cdot,t)||^2_{L_2}).$$

REMARK 3.3. For the case when the Hugoniot curves coincide with the rarefaction wave curves, i.e. the Temple's class [28], the nonlinear functional $H[u(x,t), v(x,t)]$ takes a very simple form [19]:

$$H[u(\cdot,t), v(\cdot,t)] \equiv (1 + K_1F)L(t) + K_2Q_d(t).$$

We conclude the above discussion into the following main theorem.

THEOREM 3.2. *Suppose that the total variation of the initial data of the solutions is sufficiently small and bounded by T.V., and that $u_0(x) - v_0(x) \in L_1(R)$. Then, for the exact weak solutions $u(x,t)$ and $v(x,t)$ of (1.1) constructed by Glimm's scheme, there exists a constant G independent of time such that*

$$\|u(x,t) - v(x,t)\}_{L_1} \leq G\|u(x,s) - v(x,s)\|_{L_1},$$

for any s, t, $0 \leq s \leq t < \infty$.

This theorem immediately implies the following theorem on uniqueness of the weak solution constructed by Glimm scheme.

THEOREM 3.3. *For any given initial data with total variation sufficiently small, the whole sequence of the approximate solutions contructed by the Glimm scheme converges to a unique weak solution of* (1.1) *as the mesh sizes tend to zero.*

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