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On Diffusion-Induced Grain-Boundary Motion

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Abstract. We consider a sharp interface model which describes diffusion-induced grain-boundary motion in a poly-crystalline material. This model leads to a fully nonlinear coupled system of partial differential equations. We show existence and uniqueness of smooth solutions.

1. Introduction

In this paper we consider a model which describes diffusion-induced grain-boundary motion of a surface which separates different grains in a poly-crystalline material. Let Γ_0 be a compact closed hypersurface in $\mathbb{R}^n$ which is the boundary of an open domain, and let $u_0 : \Gamma_0 \to \mathbb{R}$ be a given function. Then we are looking for a family $\Gamma := \{\Gamma(t)\,;\, t \geq 0\}$ of hypersurfaces and a family of functions $\{u(\cdot, t) : \Gamma(t) \to \mathbb{R}\,;\, t \geq 0\}$ such that the following system of equations holds:

$$\begin{aligned} &V = -H_\Gamma - f(u), && \Gamma(0) = \Gamma_0, \\ &\dot{u} = \Delta_\Gamma u - VH_\Gamma u + Vu + g(u), && u(0) = u_0. \end{aligned} \tag{1.1}$$

Here $V(t)$ denotes the normal velocity of Γ at time t, while $H_{\Gamma(t)}$ and $\Delta_{\Gamma(t)}$ stand for the mean curvature and the Laplace-Beltrami of $\Gamma(t)$, respectively. The symbol $\dot{u}$ denotes the derivative of u along flow lines which are orthogonal to $\Gamma(t)$, see the definition in (2.6). We assume that

$$f, g \in C^\infty(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad f(0) = 0, \quad g(0) = U.$$

In two dimensions, the interface $\Gamma(t)$ represents the boundary of a grain of a thin poly-crystalline material with vapor on top (in the third dimension). The vapor in the third dimension contains a certain solute which is absorbed by the interface and which diffuses along the interface. Furthermore, as the interface moves, some of the solute will be deposited in the bulk through which the interface has passed. The chemical composition of the newly created crystal behind the advancing grain will be different from that in front, because atoms of the solute have been deposited there. For this physical background we consider only convex curves, and we choose the signs so that a family of shrinking curves has negative normal velocity. A high concentration u of the solute in the interface increases the velocity, because the

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interface tries to reduce that concentration by depositing the solute in the regions it passes through. In addition, the stretching or shrinking of Γ during its motion induces a change in the concentration of the solute.

This situation results in the following terms: $V = -H_\Gamma$ is the usual motion by mean curvature that models motion driven purely by surface tension, and the term $f(u)$ results from the deposition effect. Here, $f(u) = u^2$ is reasonable [**9**]. As for the second equation, $\Delta_\Gamma u$ describes diffusion on a manifold, $-VH_\Gamma u$ indicates the concentration change due to the change of the length of the interface, Vu describes the reduction of the solute due to deposition, and $g(u)$ results from absorption of the solute from the vapor. Physically, $g(u) = U - u$ is meaningful, where U is the concentration of the solute in the vapor [**9**].

DIGM is known to be an important component of many complicated diffusion processes in which there are moving grain boundaries; see [**3**] and the references cited therein. In this type of phenomenon, the free energy of the system can be reduced by the incorporation of some of the solute into one or both of the grains separated by the grain boundary. In the DIGM mechanism, this transfer is accomplished by the disintegration of one grain and the simultaneous building up of the adjacent grain, the solute being added during the build-up process. This results in the migration of the grain boundary [**9**]. The possibility of reducing the free energy this way does not automatically imply that migration actually takes place; mechanisms for this to happen have been proposed, including the recent one in [**3**].

In [**3**], a thermodynamically consistent phase-field model for DIGM is suggested. This model has two phase fields, one being the concentration of the solute, and the other one being an order parameter which distinguishes the two crystal grains by the values $+1$ and -1, and which takes intermediate values in the grain boundary.

In this paper we consider a sharp interface model for DIGM. The same model has been studied in [**14**], where existence and uniqueness of classical Hölder solutions is proved. Here we improve this result considerably. Namely, we prove that solutions are in fact smooth in space and time.

Let $\Gamma := \{\Gamma(t)\,;\, t \in [0,T)\}$ be a family of closed compact embedded hypersurfaces in $\mathbb{R}^n$ and let

$$\mathcal{M}_0 := \bigcup_{t\in[0,T)} \Gamma(t) \times \{t\}\,, \qquad \mathcal{M} := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}\,. \tag{1.2}$$

Finally, let u be a function on $\mathcal{M}_0$. Then we call (Γ, u) a smooth C^∞-solution of (1.1) on $[0,T)$ if the following properties hold:

- $\mathcal{M}$ is an n-dimensional manifold of class C^∞ in $\mathbb{R}^{n+1}$ and $u|_{\mathcal{M}} \in C^\infty(\mathcal{M})$,
- $\mathcal{M}_0$ is a C^1-manifold with boundary $\mathcal{M}_0 \cap (\mathbb{R}^n \times \{0\})$ and $u \in C^1(\mathcal{M}_0)$,
- $\mathcal{M}_0 \cap (\mathbb{R}^n \times \{0\}) = \Gamma_0 \times \{0\} \equiv \Gamma_0$ and $u|_{\Gamma_0} = u_0$,
- the pair (Γ, u) satisfies system (1.1).

We are now ready to state our main theorem on existence and uniqueness of smooth solutions for (1.1)

Theorem 1.1. *Let $\beta \in (0,1)$ be given and suppose that $\Gamma_0 \in C^{2+\beta}$ and that $u_0 \in C^{2+\beta}(\Gamma_0)$. Then system* (1.1) *has a smooth solution (Γ, u) on $[0,T)$ for some $T > 0$. The solution is unique in the class* (4.1).

A detailed analysis shows that (1.1) is a fully nonlinear coupled system, where the fully nonlinear character comes in through the term $VH_\Gamma u$. It is shown in [**14**] that (1.1) admits classical solutions which are smooth in time and $C^{2+\alpha}$ in space for given initial data in $C^{2+\beta}$, where $0<\alpha<\beta<1$.

In order to investigate system (1.1) we represent the moving hypersurface $\Gamma(t)$ as a graph over a fixed reference manifold Σ and then transform (1.1) to an evolution equation over Σ. This leads to a fully nonlinear system which is parabolic (in the sense that the linearization generates an analytic semigroup on an appropriate function space), as is shown in [**14**]. Since the fully nonlinear term occurs on the cross diagonal we will be able to combine maximal regularity results and bootstrapping arguments to show that solutions immediately regularize for positive times.

System (1.1) reduces to the well-known mean curvature flow

$$V=-H_\Gamma, \qquad \Gamma(0)=\Gamma_0\,, \tag{1.3}$$

if $u_0=0$ and $U=0$, since $u\equiv 0$ then solves the second equation of (1.1). It is well-known that solutions of the mean curvature flow (1.3) remain strictly convex if Γ_0 is strictly convex, and that $\Gamma(t)$ shrinks to a point in finite time [**10, 12**]. Moreover, embedded curves in the plane always become convex before they shrink to a point [**11**]. We do not know if similar properties hold true for system (1.1).

2. Motion of the Interface

In this section we briefly introduce the mathematical setting in order to reformulate (1.1) as an evolution equation over a fixed reference manifold. Here we follow [**14**], see also [**5, 6, 7, 8**] for a similar situation.

Let Σ be a smooth compact closed hypersurface in $\mathbb{R}^n$, and assume that Γ_0 is close in a C^1 sense to this fixed reference manifold Σ. Let ν be the unit normal field on Σ. We choose $a>0$ such that

$$X:\Sigma\times(-a,a)\to\mathbb{R}^n\,, \qquad X(s,r):=s+r\nu(s)$$

is a smooth diffeomorphism onto its image $\mathcal{R}:=\mathrm{im}(X)$, that is,

$$X\in \mathit{Diff}^{\infty}(\Sigma\times(-a,a),\mathcal{R}).$$

This can be done by taking $a>0$ sufficiently small so that Σ has a tubular neighborhood of radius a. It is convenient to decompose the inverse of X into $X^{-1}=(S,\Lambda)$, where

$$S\in C^\infty(\mathcal{R},\Sigma) \qquad \text{and} \qquad \Lambda\in C^\infty(\mathcal{R},(-a,a)).$$

$S(x)$ is the nearest point on Σ to $x\in\mathcal{R}$, and $\Lambda(x)$ is the signed distance from x to Σ, that is, to $S(x)$. Moreover, $\mathcal{R}$ consists of those points in $\mathbb{R}^n$ with distance less than a to Σ.

Let $T>0$ be a fixed number. In the sequel we assume that $\Gamma:=\{\Gamma(t),\, t\in[0,T)\}$ is a family of graphs in normal direction over Σ. To be precise, we ask that there is a function $\rho:\Sigma\times[0,T)\to(-a,a)$ such that

$$\Gamma(t)=\mathrm{im}\big([s\mapsto X(s,\rho(s,t))]\big), \quad t\in[0,T).$$

$\Gamma(t)$ can then also be described as the zero-level set of the function

$$\Phi_\rho\,:\,\mathcal{R}\times[0,T)\to\mathbb{R}\,, \qquad \Phi_\rho(x,t):=\Lambda(x)-\rho(S(x),t); \tag{2.1}$$

one has $\Gamma(t) = \Phi_\rho(\cdot,t)^{-1}(0)$ for any fixed $t \in [0,T)$. Hence, the unit normal field $N(x,t)$ on $\Gamma(t)$ at x can be expressed as

$$N(x,t) = \frac{\nabla_x \Phi_\rho(x,t)}{|\nabla_x \Phi_\rho(x,t)|}\,, \tag{2.2}$$

and the normal velocity V of Γ at time t and at the point $x = X(s,\rho(s,t))$ is given by

$$V(x,t) = \frac{\partial_t \rho(s,t)}{|\nabla_x \Phi_\rho(x,t)|}\,. \tag{2.3}$$

We can now explain the precise meaning of the derivative $\dot u(x,t)$ for $x \in \Gamma(t)$. Given $x \in \Gamma(t)$, let $\{z(\tau,x) \in \mathbb{R}^n\,;\, \tau \in (-\varepsilon,\varepsilon)\}$ be a flow line through x such that

$$z(\tau,x) \in \Gamma(t+\tau), \quad \dot z(\tau) = (VN)(z(\tau),t+\tau), \quad \tau \in (-\varepsilon,\varepsilon), \quad z(0) = x. \tag{2.4}$$

The existence of a unique trajectory $\{z(\tau,x) \in \mathbb{R}^n\,;\, \tau \in (-\varepsilon,\varepsilon)\}$ with the above properties is established in the next result.

Lemma 2.1. *Suppose $\rho \in C^2(\Sigma \times (0,T))$ and let $\Gamma(t) := \Phi_\rho(\cdot,t)^{-1}(0)$ for t in $(0,T)$. Then for every $x \in \Gamma(t)$ there exist an $\varepsilon > 0$ and a unique solution $z(\cdot,x) \in C^1((-\varepsilon,\varepsilon),\mathbb{R}^n)$ of (2.4).*

Proof. This result is proved in [**14**, Lemma 2.1]. For the reader's convenience we include a short proof. Observe that (2.4) is equivalent to the ordinary differential equation

$$(\dot z, \dot t) = ((VN)(z,t),1)\,, \quad (z(0),t(0)) = (x,t) \tag{2.5}$$

on the manifold $\mathcal{M} = \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. We show that

$$((VN)(x,t),1) \in T_{(x,t)}(\mathcal{M}) \quad \text{for any} \quad (x,t) \in \mathcal{M}\,.$$

For this let $\Psi_\rho := \Phi_\rho|_{\mathcal{R}\times(0,T)}$ and observe that $\mathcal{M} = \Psi_\rho^{-1}(0)$, so that the vector

$$(\nabla_x \Phi_\rho(x,t), -\partial_t \rho(S(x),t))$$

is orthogonal to $\mathcal{M}$ at $(x,t) \in \mathcal{M}$. Using the definition of Φ_ρ it can easily be seen that $\partial_\nu \Phi_\rho = 1$, and hence the vector displayed above is nonzero. By (2.2) and (2.3) we have

$$\big(((VN)(x,t),1)|(\nabla_x \Phi_\rho(x,t), -\partial_t \rho(S(x),t))\big) = 0\,, \quad (x,t) \in \mathcal{M}\,,$$

showing that $((VN)(x,t),1)$ is tangential to $\mathcal{M}$ at (x,t). We can now conclude that there is an $\varepsilon > 0$ such that (2.5) has a unique solution

$$[\tau \mapsto (z(\tau,x),t+\tau)] \in C^1((-\varepsilon,\varepsilon),\mathcal{M})\,.$$

It follows that $[\tau \mapsto z(\tau,x)] \in C^1((-\varepsilon,\varepsilon),\mathbb{R}^n)$ is the unique solution of (2.4). $\square$

Let $(x,t) \in \mathcal{M}$ be given. Then we define

$$\dot u(x,t) := \frac{d}{d\tau} u(z(\tau,x),t+\tau)\Big|_{\tau=0}\,. \tag{2.6}$$

We now introduce the pull-back function v of u,

$$v : \Sigma \times [0,T) \to \mathbb{R}, \quad v(s,t) := u(X(s,\rho(s,t)),t). \tag{2.7}$$

Since $u(x,t) = v(S(x),t)$, it follows from (2.4) and (2.6) that

$$\dot u(x,t) = \frac{d}{d\tau} v(S(z(\tau,x)),t+\tau)\Big|_{\tau=0} = (\nabla_x v(S(x),t)|N(x,t))V(x,t) + \frac{dv}{dt}(S(x),t).$$

Note that this formula also makes sense if $t = 0$ and $x \in \Gamma(0)$, whereas we required $t > 0$ in (2.6). We take this last formula as new definition for $\dot{u}$, that is, we set

$$\dot{u}(x,t) := (\nabla_x v(S(x),t)|N(x,t))V(x,t) + \frac{dv}{dt}(S(x),t), \quad (x,t) \in \mathcal{M}_0\,. \tag{2.8}$$

Finally, we set

$$\begin{aligned} L(\rho)(s,t) \ &:= |\nabla_x \Phi_\rho(x,t)|\Big|_{x=X(s,\rho(s,t))}, \\ I(\rho,v)(s,t) &:= (\nabla_x v(S(x),t)|N(x,t))\Big|_{x=X(s,\rho(s,t))}, \end{aligned} \tag{2.9}$$

for $(s,t) \in \Sigma \times [0,T)$ and we obtain

$$\dot{u}(x,t)\Big|_{x=X(s,\rho(s,t))} = \frac{dv}{dt}(s,t) + I(\rho,v)(s,t)V(x,t)\Big|_{x=X(s,\rho(s,t))}. \tag{2.10}$$

3. The Transformed Equations

Given an open set $U \subset \mathbb{R}^n$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^s(U)$, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order s. If Σ is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for Σ. It is known that $BUC^t(\Sigma)$ is continuously embedded in $h^s(\Sigma)$ whenever $t > s$. Moreover, the little Hölder spaces have the interpolation property

$$(h^s(\Sigma), h^t(\Sigma))_\theta = h^{(1-\theta)s+\theta t}(\Sigma), \quad \theta \in (0,1), \tag{3.1}$$

whenever $s, t, (1-\theta)s + \theta t \in \mathbb{R}^+ \setminus \mathbb{N}$, and where $(\cdot,\cdot)_\theta$ denotes the continuous interpolation method of DaPrato and Grisvard [**4**], see also [**1, 2, 13**].

In the following we fix $t \in (0,T)$ and drop it in our notation. Given $\alpha \in (0,1)$ and $k \in \mathbb{N}$ we set

$$\begin{aligned} U(k,\alpha) &: = \{\rho \in h^{k+\alpha}(\Sigma)\,;\, \|\rho\|_{C(\Sigma)} < a\} \\ \mathbb{U}(k,\alpha) &: = U(k,\alpha) \times h^{k+\alpha}(\Sigma). \end{aligned} \tag{3.2}$$

Clearly, the sets $U(k,\alpha)$ and $\mathbb{U}(k,\alpha)$ are open in $h^{k+\alpha}(\Sigma)$ and in $(h^{k+\alpha}(\Sigma))^2$, respectively. Given $\rho \in U(k,\alpha)$, we introduce the mapping

$$\theta_\rho \,:\, \Sigma \to \mathbb{R}^n, \qquad \theta_\rho(s) := X(s,\rho(s)) \text{ for } s \in \Sigma\,, \qquad \rho \in U.$$

It follows that θ_ρ is a well-defined $(k+\alpha)$-diffeomorphism from Σ onto $\Gamma_\rho := \mathrm{im}(\theta_\rho)$. Let

$$\theta_\rho^* u := u \circ \theta_\rho \quad \text{for} \quad u \in C(\Gamma_\rho), \quad \theta_*^\rho v := v \circ \theta_\rho^{-1} \quad \text{for} \quad v \in C(\Sigma),$$

be the pull-back and the push-forward operator, respectively. Given $\rho \in U(k,\alpha)$, $k \geq 2$, we denote by Δ_{Γ_ρ} and H_{Γ_ρ} the Laplace–Beltrami operator and the mean curvature, respectively, of Γ_ρ. Finally we set

$$\Delta_\rho := \theta_\rho^* \Delta_{\Gamma_\rho} \theta_*^\rho, \qquad H(\rho) := \theta_\rho^* H_{\Gamma_\rho}.$$

We will now consider the smoothness properties the substitution operators induced by the local functions f and g, and of the operators L and I introduced in (2.9). Moreover we investigate the structure of the transformed operators Δ_ρ and $H(\rho)$.

For this, we introduce the set $\mathcal{H}(E_1, E_0)$ of generators of analytic semigroups. To be more precise, we assume that E_0 and E_1 are Banach spaces such that E_1

is densely injected in E_0, and we use the symbol $\mathcal{H}(E_1, E_0)$ to denote the set of all linear operators $A \in \mathcal{L}(E_1, E_0)$ such that $-A$ is the generator of a strongly continuous analytic semigroup on E_0. It is known that $\mathcal{H}(E_1, E_0)$ is an open subset of $\mathcal{L}(E_1, E_0)$, which will be given the relative topology of $\mathcal{L}(E_1, E_0)$.

Lemma 3.1. *Assume that* $\alpha \in (0,1)$ *and* $k \in \mathbb{N}$.

(a) $[v \mapsto (f(v), g(v))] \in C^\infty(h^{k+\alpha}(\Sigma), h^{k+\alpha}(\Sigma) \times h^{k+\alpha}(\Sigma))$.

(b) $[\rho \mapsto L(\rho)] \in C^\infty(U(k+1,\alpha), h^{k+\alpha}(\Sigma))$.

(c) $[(\rho, v) \mapsto I(\rho, v)] \in C^\infty(\mathbb{U}(k+1,\alpha), h^{k+\alpha}(\Sigma))$.

(d) *There exists a function*

$$C \in C^\infty(U(k+2,\alpha), \mathcal{H}(h^{k+2+\alpha}(\Sigma), h^{k+\alpha}(\Sigma)))$$

such that $\Delta_\rho v = -C(\rho)v$ *for* $(\rho, v) \in \mathbb{U}(k+2,\alpha)$.

(e) *There exist functions*

$$\begin{aligned} &P \in C^\infty(U(k+1,\alpha), \mathcal{H}(h^{k+2+\alpha}(\Sigma), h^{k+\alpha}(\Sigma))), \\ &K \in C^\infty(U(k+1,\alpha), h^{k+\alpha}(\Sigma)) \end{aligned}$$

such that $H(\rho) = P(\rho)\rho + K(\rho)$ *for* $\rho \in U(k+2,\alpha)$. *Furthermore,*

(f) $[\rho \mapsto L(\rho)P(\rho)] \in C^\infty(U(k+1,\alpha), \mathcal{H}(h^{k+2+\alpha}(\Sigma), h^{k+\alpha}(\Sigma)))$.

Proof. This follows by similar arguments as in the proofs of [**14**, Lemmas 3.1–3.3], and of [**5**, Section 2]. □

We are now ready to investigate the transformed system of equations

$$\begin{aligned} &\frac{d\rho}{dt} = -L(\rho)P(\rho)\rho - L(\rho)K(\rho) - L(\rho)f(v), && \rho(0) = \rho_0, \\ &\frac{dv}{dt} = \Delta_\rho v + \Big(I(\rho, v) + H(\rho)v - v\Big)\Big(H(\rho) + f(v)\Big) + g(v), && v(0) = v_0. \end{aligned} \tag{3.3}$$

In the following, we call (ρ, v) a smooth solution of (3.3) on $[0, T)$ if

$$(\rho, v) \in C^1(\Sigma \times [0,T), \mathbb{R}^2) \cap C^\infty(\Sigma \times (0,T), \mathbb{R}^2), \tag{3.4}$$

and if (ρ, v) satisfies system (3.3).

Lemma 3.2. (1.1) *and* (3.3) *are equivalent: Smooth solutions of* (1.1) *give rise to smooth solutions of* (3.3), *and vice-versa.*

Proof. This can be proved similarly as in [**14**, Lemma 4.1]. □

4. Existence and Uniqueness of Smooth Solutions

Theorem 4.1. *Let* $\mathbb{V} := \mathbb{U}(2,\alpha)$. *Given any* $w_0 := (\rho_0, v_0) \in \mathbb{V}$ *there exists a number* $T = T(w_0) > 0$ *such that system* (3.3) *has a unique maximal smooth solution*

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C([0,T), \mathbb{V}) \cap C^1([0,T), h^\alpha(\Sigma) \times h^\alpha(\Sigma)) \cap C^\infty(\Sigma \times (0,T), \mathbb{R}^2).$$

The map $[w_0 \mapsto (\rho(\cdot, w_0), v(\cdot, w_0))]$ *defines a smooth semiflow on* $\mathbb{V}$.

PROOF. It follows from [**14**, Theorem 4.3] that (3.3) has a unique maximal solution

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C([0,T), \mathbb{V}) \cap C^1([0,T), h^\alpha(\Sigma)). \tag{4.1}$$

Moreover, [**14**, Equation (4.7)] shows that the solution has the additional smoothness property

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C^\infty((0,T), h^{2+\alpha}(\Sigma) \times h^{2+\alpha}(\Sigma)). \tag{4.2}$$

Let $T_0 \in (0,T)$ be fixed and choose $\tau \in [0, T_0)$. We consider the linear parabolic equation

$$\frac{d\rho}{dt} + A(t)\rho = F(t), \quad \tau < t \le T_0, \quad \rho(\tau) = \rho(\tau, w_0), \tag{4.3}$$

on $h^{1+\alpha}(\Sigma)$, with

$$\begin{aligned} A(t) &:= L(\bar\rho(t))P(\bar\rho(t)), \\ F(t) &:= -L(\bar\rho(t))K(\bar\rho(t)) - L(\bar\rho(t))f(\bar v(t)), \end{aligned}$$

for $t \in [\tau, T_0]$, where $\bar\rho(t) := \rho(t, w_0)$ and $\bar v(t) := v(t, w_0)$. If follows from (4.1) that $\rho(\tau, w_0) \in h^{2+\alpha}(\Sigma)$. Moreover, (4.1) and Lemma 3.1 with $k = 1$ yield

$$(A, F) \in C([\tau, T_0], \mathcal{H}(h^{3+\alpha}(\Sigma), h^{1+\alpha}(\Sigma)) \times h^{1+\alpha}(\Sigma)). \tag{4.4}$$

Let $X_0 := h^{1+\gamma}(\Sigma)$ and $X_1 := h^{3+\gamma}(\Sigma)$ for some fixed $\gamma \in (0, \alpha)$. It follows with the same arguments as in Lemma 3.1 that $A(t) \in \mathcal{H}(X_1, X_0)$ for $t \in [\tau, T_0]$. Next, note that the interpolation result (3.1) implies that

$$X_\theta := (X_0, X_1)_\theta \doteq h^{1+\alpha}(\Sigma) \quad \text{if} \quad \theta = (\alpha - \gamma)/2\,,$$

where $\doteq$ indicates that the spaces are equal, except for equivalent norms. Let $A_\theta(t)$ denote the maximal X_θ-realization of $A(t)$, where $A(t)$ is considered as an operator in $\mathcal{L}(X_1, X_0)$, and let $X_{1+\theta}(A(t))$ denote its domain, equipped with the graph norm. Using

$$A(t) \in \mathcal{H}(h^{3+\alpha}(\Sigma), h^{1+\alpha}(\Sigma)) \text{ and } A_\theta(t) \in \mathcal{H}(X_{1+\theta}(A(t)), X_\theta),$$

we readily infer that

$$X_{1+\theta}(A(t)) \doteq X_{1+\theta}(A(\tau)) \doteq h^{3+\alpha}(\Sigma) \quad \text{for} \quad t \in [\tau, T_0].$$

It follows from the maximal regularity result [**1**, Remark III.3.4.2.(c)], from (3.1) and [**1**, Theorem III.2.3.3] with $E_0 := h^{1+\alpha}(\Sigma)$, $E_1 := h^{3+\alpha}(\Sigma)$ and $(\rho, \mu) = (0, 1/2)$, and from [**1**, Proposition III.2.1.1] that equation (4.3) admits a unique solution

$$\rho \in C([\tau, T_0], h^{2+\alpha}(\Sigma)) \cap C((\tau, T_0], h^{3+\alpha}(\Sigma)) \cap C^1((\tau, T_0], h^{1+\alpha}(\Sigma)). \tag{4.5}$$

It is a consequence of (4.5) and of (4.3) that ρ satisfies

$$\rho \in C([\tau, T_0], h^{2+\alpha}(\Sigma)) \cap C^1([\tau, T_0], h^\alpha(\Sigma)),$$

so that ρ has at least the same regularity as $\rho(\cdot, w_0)$. Moreover, ρ solves the same equation on $h^\alpha(\Sigma)$ as $\rho(\cdot, w_0)$ for $t \in [\tau, T_0]$, and we conclude that $\rho = \rho(\cdot, w_0)|_{[\tau, T_0]}$. Since τ and T_0 can be chosen arbitrarily we obtain

$$\rho(\cdot, w_0) \in C((0,T), h^{3+\alpha}(\Sigma)) \cap C^1((0,T), h^{1+\alpha}(\Sigma)). \tag{4.6}$$

Now we use (4.6) to show that $v(\cdot, w_0)$ also enjoys better regularity properties in the space variable than stated in (4.1). Let $\tau \in (0, T_0)$ be fixed and consider the linear parabolic equation

$$\frac{dv}{dt} + B(t)v = G(t), \quad \tau < t \le T_0, \quad v(\tau) = v(\tau, w_0), \tag{4.7}$$

on $h^{1+\alpha}(\Sigma)$, with

$$B(t) := C(\bar{\rho}(t)),$$
$$G(t) := \Big(I(\bar{\rho}(t), \bar{v}(t)) + H(\bar{\rho}(t))\bar{v}(t) - \bar{v}(t)\Big)\Big(H(\bar{\rho}(t)) + f(\bar{v}(t))\Big) + g(\bar{v}(t)),$$

for $t \in [\tau, T_0]$, where $\bar{\rho}(t) := \rho(t, w_0)$ and $\bar{v}(t) := v(t, w_0)$. It is a consequence of (4.1), (4.6), and of Lemma 3.1 with $k = 1$, that

$$(B, G) \in C([\tau, T_0], \mathcal{H}(h^{3+\alpha}(\Sigma), h^{1+\alpha}(\Sigma)) \times h^{1+\alpha}(\Sigma)), \tag{4.8}$$

and that $v(\tau, w_0) \in h^{2+\alpha}(\Sigma)$. As above we infer that (4.7) has a unique solution

$$v \in C([\tau, T_0], h^{2+\alpha}(\Sigma)) \cap C((\tau, T_0], h^{3+\alpha}(\Sigma)) \cap C^1((\tau, T_0], h^{1+\alpha}(\Sigma)). \tag{4.9}$$

This allows us to conclude, once again, that

$$v(\cdot, w_0) \in C((0, T), h^{3+\alpha}(\Sigma)) \cap C^1((0, T), h^{1+\alpha}(\Sigma)). \tag{4.10}$$

In a next step we use (4.6) and (4.10) to deduce that $\rho(\cdot, \rho_0)$ has more regularity than noted in (4.6). It should be observed that this time we need to choose $\tau \in (0, T_0)$, whereas $\tau = 0$ was admissible in (4.3)–(4.5). To be more precise, we consider

$$\frac{d\rho}{dt} + A(t)\rho = F(t), \quad \tau < t \le T_0, \quad \rho(\tau) = \rho(\tau, w_0),$$

as an evolution equation on $h^{2+\alpha}(\Sigma)$. It follows from (4.6), (4.10) and Lemma 3.1 with $k = 2$ that

$$(A, F) \in C([\tau, T_0], \mathcal{H}(h^{4+\alpha}(\Sigma), h^{2+\alpha}(\Sigma)) \times h^{2+\alpha}(\Sigma)),$$

and that $\rho(\tau, w_0) \in h^{3+\alpha}(\Sigma)$. We conclude by similar arguments as above—involving maximal regularity—that the solution of (4.3) satisfies

$$\rho \in C([\tau, T_0], h^{3+\alpha}(\Sigma)) \cap C((\tau, T_0], h^{4+\alpha}(\Sigma)) \cap C^1((\tau, T), h^{2+\alpha}(\Sigma)),$$

and that $\rho = \rho(\cdot, w_0)|_{[\tau, T_0]}$. Since τ and T_0 are arbitrary we get

$$\rho(\cdot, w_0) \in C((0, T), h^{4+\alpha}(\Sigma)) \cap C^1((0, T), h^{2+\alpha}(\Sigma)). \tag{4.11}$$

We can repeat the arguments and we arrive, after m steps, to the conclusion

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C((0, T), (h^{m+2+\alpha}(\Sigma))^2) \cap C^1((0, T), (h^{m+\alpha}(\Sigma))^2). \tag{4.12}$$

Let $j \in \mathbb{N}$ be a number such that $2j \le m$. Then one also obtains

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C^j((0, T), (h^{m-2j+\alpha}(\Sigma))^2). \tag{4.13}$$

In order to show (4.13), let us assume that we have already verified that

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C^{j-1}((0, T), (h^{m-2(j-1)+\alpha}(\Sigma))^2)$$

for some j in $\{2, \cdots, m\}$. Then it follows from Lemma 3.1 with $k = m - 2j$ that

$$(A, F),\ (B, G) \in C^{j-1}((0, T), \mathcal{L}(h^{m+2-2j+\alpha}(\Sigma), h^{m-2j+\alpha}(\Sigma)) \times h^{m-2j+\alpha}(\Sigma)).$$

Now we go back to the evolution equation (4.3) and (4.7), respectively, and conclude that

$$\Big(\frac{d}{dt}\rho(\cdot,w_0),\frac{d}{dt}v(\cdot,w_0)\Big)\in C^{j-1}((0,T),(h^{m-2j+\alpha}(\Sigma))^2),$$

and hence $(\rho(\cdot,w_0),v(\cdot,w_0))\in C^j((0,T),(h^{m-2j+\alpha}(\Sigma))^2)$. Since $m\in\mathbb{N}$ can be chosen arbitrarily we have proved that

$$(\rho(\cdot,w_0),v(\cdot,w_0))\in C^\infty((0,T),C^\infty(\Sigma)\times C^\infty(\Sigma)). \tag{4.14}$$

This completes the proof of Theorem 4.1 □

Remarks 4.2. **(a)** The strategy for the bootstrapping arguments in the proof of Theorem 4.1 relies on the following observation: The equation for ρ in (3.3), while coupled, is quasilinear in ρ and involves no derivatives of v. Therefore, if we insert $v(\cdot,w_0)$ into the first equation, we can take advantage of the regularizing effect to establish more regularity for $\rho(\cdot,w_0)$. As the equation for v is also quasilinear once ρ is frozen, we can now use that $\rho(\cdot,w_0)$ has more regularity to improve the regularity of $v(\cdot,w_0)$. These steps can then be repeated.

(b) The bootstrapping arguments used in the proof of Theorem 4.1 could also be based on [**1**, Theorem II.1.2.1]. Indeed, it follows from equations (3.1), (4.1), and from [**1**, Proposition II.1.1.2] that

$$(\rho(\cdot,w_0),v(\cdot,w_0))\in C^{1-\theta}([0,T),h^{2+\gamma}(\Sigma))\text{ if }\gamma\in(0,\alpha)\text{ and }\theta=1-(\alpha-\gamma)/2\,.$$

A slightly modified version of Lemma 3.1 then yields

$$(A,F)\in C^{1-\theta}([0,T),\mathcal{H}(h^{3+\gamma}(\Sigma),h^{1+\gamma}(\Sigma))\times h^{1+\gamma}(\Sigma)).$$

Theorem II.1.2.1 in [**1**] shows that the solution of the linear parabolic equation (4.3) has better regularity properties than stated in (4.1). One can then go on and reiterate the arguments.

(c) It is important to note that system (1.1) or (3.3), respectively, is fully nonlinear. This indicates that one needs maximal regularity results—which compensate for the loss of derivatives—in order to get a solution via a fixed point argument. This has been achieved in [**14**].

(d) Theorem 1.1 allows to construct solutions even if they are not represented as graphs over the initially fixed reference manifold Σ. Indeed, we can take $\Gamma(t_1)$ for some $t_1\in[0,T)$ as new reference manifold and then get solutions on a time interval $[t_1,t_2]$. Thus we are not restricted to hypersurfaces which are graphs over a fixed manifold.

Proof of Theorem 1.1. Let Γ_0 be a given compact, closed $C^{2+\beta}$-manifold in $\mathbb{R}^n$. As in Section 2 we find a smooth reference manifold Σ and a function $\rho_0\in C^{2+\beta}(\Sigma)$ such that

$$\Gamma_0=\mathrm{im}([s\mapsto X(s,\rho_0(s))])\,.$$

Since $C^{2+\beta}(\Sigma)\subset h^{2+\alpha}(\Sigma)$ for $\alpha\in(0,\beta)$ we also have that $\rho_0\in U(2,\alpha)$. Given $u_0\in C^{2+\beta}(\Gamma_0)$, let $v_0:\Sigma\to\mathbb{R}$ be defined by $v_0(s):=u_0(X(s,\rho_0(s)))$ for $s\in\Sigma$. We can conclude that $v_0\in h^{2+\alpha}(\Sigma)$. Theorem 4.1 yields the existence of a unique solution

$$(\rho(\cdot,w_0),v(\cdot,w_0))\in C([0,T),\mathbb{V})\cap C^1([0,T),h^\alpha(\Sigma)\times h^\alpha(\Sigma))\cap C^\infty(\Sigma\times(0,T))$$

for system (3.3), where we have set $w_0=(\rho_0,v_0)$. Clearly, this solution also satisfies the regularity assumptions in (3.4). Lemma 3.2 then shows that (1.1) has a classical solution on $[0, T)$. The solution is unique in the class (4.1), as follows from Theorem 4.1, and the proof of Theorem 1.1 is now completed. $\square$

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