UPDATED REFERENCES WITH ANNOTATION TO: NEW EXAMPLES OF FROBENIUS EXTENSIONS

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1. New Developments related to Frobenius Extensions since 1998

A number of topics related to Frobenius extensions have appeared in mathematical journals since the time around which *New Examples of Frobenius Extensions*, referred to as NEFE, was submitted to the A.M.S. in 1998. They are briefly described below together with references for further reading. We adopt the notation in NEFE to describe these developments.

(1) A Frobenius algebra in a tensor category $\mathcal{C}$ is an abstract algebra-coalgebra where (co)multiplication and (co)unit are arrows in $\mathcal{C}$. A Frobenius extension $A|S$ may be viewed as a Frobenius algebra in the category $\mathcal{M}$ of S-bimodules under ordinary tensor product over S, as described in connection with corings in NEFE, chapter 4. The notion of a **special Frobenius algebra in a category** is a basic notion to mathematical physicists doing rational conformal field theory [15, 16, 35]: for example, a strongly separable extension $A|S$ (cf. ch. 3, NEFE) is a special Frobenius algebra in the category $\mathcal{M}$.

(2) **Weak Hopf algebras** are replacing Hopf algebras to some extent as objects of interest to mathematicians doing tensor categories [41, 13] and mathematical physicists doing algebraic quantum field theory [1, 12]. They are associative algebras and coassociative coalgebras where comultiplication is multiplicative but not unit-preserving, the counit is no longer multiplicative, and the antipode satisfies several modified axioms. There are separable subalgebras anti-isomorphic under the antipode

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and with projections from the weak Hopf algebra that are important to the axiomatic construction of weak Hopf algebra theory [2, 3, 40]. Examples of weak Hopf algebras, which are also known as **quantum groupoids**, come from the Temperley-Loeb algebras, full matrix algebras and more generally groupoid algebras [37, 38, 39]. Finite-dimensional weak Hopf algebras are Frobenius algebras [43]; so we predict weak Hopf subalgebras are twisted Frobenius extensions based on [27, 6.1]. There is also a Radford formula for the fourth power of the antipode in weak Hopf algebras [2, 36, 43, 14]. In addition, there is a version of the Radford formula for Frobenius algebras with anti-automorphism in [22] applied to quasi-Hopf algebras, which are almost associative Hopf algebras which deviate from associativity by a three cocycle. Manin predicts that weak Hopf algebras will have a role with Hopf algebras and groups acting on C^*-algebras in a description of arithmetical applications of real multiplication [33].

(3) A **depth two** extension $A|S$ is a ring extension more general than an H-separable extension having a defining condition like in section 2.6, NEFE, where A-A-bimodules are replaced by the weaker conditions S-A-bimodules (left D2) and A-S-bimodules (right D2) [30]. The H-theory in NEFE section 2.6 applies as well as to depth two extensions to establish that they generalize the notion in von Neumann subfactor theory [18], which is related to the level at which the Bratteli diagrams of inclusions of semisimple algebras of the derived tower begin to reflect (as in a basic construction). Another example of depth two extension is a finite Hopf-Galois extension [21]. A depth two, split, separable extension is a quasi-Frobenius (QF) extension [30, 3.1], a partial answer to the open problem described in section 2.5 NEFE and further explained in [9, 8].

Of most interest is a depth two Frobenius extension $A|S$, since they have a certain Galois structure coming from the obvious action of $\mathrm{End}\,{}_SA_S$, which may be given a Hopf algebroid structure over the centralizer $C_A(S) := R$ [30, 32, 4]. This action has the subalgebra S as its invariants if the module A_S is balanced. There are a number of theorems of the form, given a Frobenius extension $A|S$ with centralizer R, $A|S$ is a Hopf R-algebroid-Galois extension if and only if $A|S$ is depth two and balanced extension. For example, if R is identifiable with the ground field for a split irreducible algebra extension $A|S$, then $A|S$ is finite Hopf-Galois iff $A|S$ is depth two [25, 24, 26]. This is in analogy with the well-known theorem for finite field extensions, that they are Galois extensions iff they are separable and normal. If R is a separable subalgebra of A, which is a depth two Frobenius extension of S, then $\mathrm{End}\, A_S$ is a smash product of A with the weak Hopf algebra $\mathrm{End}\, {}_S A_S$: the references are from most to least number of hypotheses [37, 26, 30]. If we restrict ourselves to H-separable extensions of simple rings, there is a Sugano-Galois correspondence between subrings and subHopf algebroids [19, 20, 21]. In the most general case of a depth two extension, $\mathrm{End}\, {}_S A_S$ and $(A \otimes_S A)^S$ have left and right R-bialgebroid structures dual to one another, and acting on A and $\mathrm{End}\, {}_S A$, respectively [30]. If $A|S$ is additionally a Frobenius extension, the natural anti-isomorphism between $\mathrm{End}\, A_S$ and $\mathrm{End}\, {}_S A$ restricts to the antipode on $\mathrm{End}\, {}_S A_S$ studied in [4] as a **Hopf algebroid** and in [42] as a double algebra. For example, the double algebra structure on $(A \otimes_S A)^S$ comes from the "E-multiplication" and the algebra structure induced by the isomorphism with the endomorphism ring $\mathrm{End}\, {}_A A \otimes_S A_A$ [42].

In modular representations of finite groups [31] there are various useful simplications using Frobenius, separable, split and other ring extensions taking place in the induced representation-theoretical method of attacking problems of for example Brauer, Broué and Donovan. Depth two complex group algebra extensions turn out to be identical with normal subgroups [23]. There are also interesting problems of when Hecke algebras of finite symmetry groups are separable extensions of their parabolic subgroups.

(4) As touched upon in chapter four of NEFE, there is the **coring** point of view on Frobenius extensions as well as on entwined modules, descent and quantum principal bundles (generalized Hopf-Galois extensions and Galois corings) [5, 7, 10]. There are towers of corings [6] made up of Frobenius corings and an equivalent formulation of the biseparability problem in [8]. The coring structure of the R-bialgebroids of a depth two extension control some aspects of the extension itself [8, 21], just as a semisimple or cosemisimple Hopf algebra acting in a finite

Hopf-Galois extension leads to a split or separable Frobenius extension. There is also a quantum Yang-Baxter equational-type point of view on Frobenius and separable algebras in [11]. There is an alternative way (to that in chapter five NEFE) to associate solutions to the Yang-Baxter equation via associative algebras which leads to the Alexander knot polynomials [34].

If you have corrections or additional topics, please send me an email.

References

[1] G. Böhm, K. Szlachányi, A coassociative C^*-quantum group with nonintegral dimensions, *Lett. Math. Phys.* **35** (1996), 437–456

[2] G. Böhm, F. Nill and K. Szlachányi, Weak Hopf algebras, I. Integral theory and C^*-structure, *J. Algebra* **221** (1999), 385-438.

[3] G. Böhm, K. Szlachányi, Weak Hopf Algebras, II. Representation theory, dimensions, and the Markov trace, *J. Algebra* **233** (2000) 156–212.

[4] G. Böhm and K. Szlachányi, Hopf algebroids with bijective antipode: axioms, integrals, and duals, *J. Algebra* **274** (2004), 708–750.

[5] T. Brzeziński. The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties. *Alg. Rep. Theory* **5** (2002), 389–410.

[6] T. Brzeziński, Towers of corings, *Commun. Alg.* **31** (2002), 2015–2026.

[7] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, LMS **309**, Cambridge University Press, 2003.

[8] T. Brzeziński, L. Kadison and R. Wisbauer, On coseparable and biseparable corings, to appear in: *Hopf Algebras and Noncommutative Geometry* (Proc. Conf. Brussels, May 2002) eds. S. Caenepeel and F. Van Oystaeyen, Lect. Notes in Pure & Applied Math., vol. **239**, Marcel Dekker, New York, 2004, 71–89. `arXiv.math/RA/0208122`.

[9] S. Caenepeel and L. Kadison, Are biseparable extensions Frobenius? *K-Theory* **24** (2001), 361–383.

[10] S. Caenepeel and E. De Groot, Galois theory for weak Hopf algebras, preprint, 2004.

[11] S. Caenepeel, B. Ion and G. Militaru, The structure of Frobenius algebras and separable algebras, *K-Theory* **19** (2000), 365–402.

[12] S. Doplicher and J.E. Roberts, A new duality theory for compact groups, *Inventiones* **98** (1989), 157–218.

[13] P. Etingof and V. Ostrik, Finite tensor categories, preprint (2003) `math.QA/0301027`.

[14] P. Etingof, D. Nikshych and V. Ostrik, An analogue of Radford's S^4-formula for finite tensor categories, *Int. J. Math.*, to appear.

[15] J. Fuchs and C. Schweigert, Category theory for conformal boundary conditions, `arXiv.math.CT/0106050`.

[16] J. Fuchs, I. Runkel and C. Schweigert, Boundaries, defects and Frobenius algebras, *Ann. Henri Poincar* **4** (2003), 175–182.

[17] K. Hirata, Some types of separable extensions of rings, *Nagoya Math. J.* **33** (1968), 107–115.

[18] V.F.R. Jones and V.S. Sunder, *Introduction to Subfactors*, LMS **234**, Cambridge Univ. Press, 1997.

[19] L. Kadison, The Miyashita-Ulbrich action and H-separable extensions, *Hokkaido Math. J.* **30** (2001), 689–695.

[20] L. Kadison, Hopf algebroids and H-separable extensions, *Proc. A.M.S.* **131** (2003), 2993–3002.

[21] L. Kadison, Hopf algebroids and Galois extensions, *Bull. Belg. Math. Soc. - Simon Stevin*, to appear, www.mathpreprints.com `20020426/4`.

[22] L. Kadison, An approach to quasi-Hopf algebras via Frobenius coordinates, preprint (2003), www.mathpreprints.com `20040105/1`.

[23] L. Kadison and B. Külshammer, Depth two, normality and a trace ideal condition for Frobenius extensions, preprint (2004), www.mathpreprints.com `20040519/1`.

[24] L. Kadison and D. Nikshych, Outer actions of Hopf algebra centralizers on separable extensions, *Comm. Alg.* **30**, no. 1 (2002), 383–410.

[25] L. Kadison and D. Nikshych, Hopf algebra actions on strongly separable extensions of depth two, *Advances in Math.* **164** (2001), 258–286.

[26] L. Kadison and D. Nikshych, Frobenius extensions and weak Hopf algebras, *J. Algebra* **244** (2001), 312–342. `RA/0102010`.

[27] L. Kadison and A.A. Stolin, An approach to Hopf algebras via Frobenius coordinates, *Beiträge zur Alg. und Geometrie* **42** (2001), 359–384.

[28] L. Kadison and A.A. Stolin, Separability and Hopf algebras, in: *Algebra and its Applications* (Athens, Ohio, March 1999), eds. D.V. Huynh, S.K. Jain and S. Lòpez-Permouth, Contemporary Math., vol. **259**, A.M.S., Providence, (2000), 279–298.

[29] L. Kadison and A.A. Stolin, An approach to Hopf algebras via Frobenius coordinates II, *J. Pure Appl. Alg.* **176** (2002), 127–152.

[30] L. Kadison and K. Szlachányi, Bialgebroid actions on depth two extensions and duality, *Advances in Math.* **179** (2003), 75–121. `RA/0108067`

[31] B. Külshammer, *Lectures on Block Theory*, LMS **161**, Cambridge University Press, 1991.

[32] J.-H. Lu, Hopf algebroids and quantum groupoids, *Int. J. Math.* **7** (1996), 47–70.

[33] Y. Manin, Real multiplication and noncommutative geometry, in: *The Legacy of Niels Henrik Abel* (Proc. Abel Bicentennial conference, Oslo, June 20002), eds. O.A. Laudal and R. Piene, Springer, New York (2004), 685–727.

[34] G. Massuyeau and F. Nichita, Yang-Baxter operators arising from algebra structures and the Alexander polynomial of knots, *Comm. Alg.*, to appear.

[35] M. Müger, From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories, *J. Pure Appl. Alg.* **180** (2003), 91–123.

[36] D. Nikshych, On the structure of weak Hopf algebras, *Advances in Math.* **170** (2002), 257–286.

[37] D. Nikshych and L. Vainerman, A characterization of depth 2 subfactors of II_1 factors, *J. Func. Analysis* **171** (2000), 278–307.

[38] D. Nikshych and L. Vainerman, A Galois correspondence for actions of quantum groupoids on II_1-factors, *J. Func. Analysis*, **178** (2000), 113-142.

[39] D. Nikshych and L. Vainerman, Finite dimensional quantum groupoids and their applications, in: Proceedings of Hopf Algebras workshop, MSRI Publications, 2000; `QA/0006057`.

[40] F. Nill, Axioms for weak bialgebras, preprint, 1998, `QA/9805104`.

[41] K. Szlachányi, Finite quantum groupoids and inclusions of finite type, *Fields Inst. Comm. Ser.*, to appear. `math.QA/0011036`.

[42] K. Szlachányi, The double algebraic view of finite quantum groupoids, preprint, 2004, `QA/0402151`.

[43] P. Vecsernyés, Larson-Sweedler theorem and the role of grouplike elements in weak Hopf algebras, *J. Alg.* **270** (2003), 471–520.

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