1 A Simpler Proof of Theorem 10.35

Theorem 1 (Absolute Continuity on Lines) *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$. A function $u \in L^p(\Omega)$ belongs to the space $W^{1,p}(\Omega)$ if and only if it has a representative $\overline{u}$ that is absolutely continuous on $\mathcal{L}^{N-1}$ a.e. line segments of Ω that are parallel to the coordinate axes, and whose first order (classical) partial derivatives belong to $L^p(\Omega)$. Moreover the (classical) partial derivatives of $\overline{u}$ agree $\mathcal{L}^N$ a.e. with the weak derivatives of u.*

Proof. Step 1: Assume that $u \in W^{1,p}(\Omega)$. Consider a sequence of standard mollifiers $\{\varphi_\varepsilon\}_{\varepsilon>0}$ and for every $\varepsilon > 0$ define $u_\varepsilon := u * \varphi_\varepsilon$ in $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. By Lemma 10.16,

$$\lim_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x) - \nabla u(x)|^p \, dx = 0.$$

It follows by Fubini's theorem that for all $i = 1, \dots, N$,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^{N-1}} \left(\int_{(\Omega_\varepsilon)_{x_i'}} |\nabla u_\varepsilon(x_i', x_i) - \nabla u(x_i', x_i)|^p \, dx_i \right) dx_i' = 0,$$

where $(\Omega_\varepsilon)_{x_i'} := \{x_i \in \mathbb{R} : (x_i', x_i) \in \Omega_\varepsilon\}$, and so we may find a subsequence $\{\varepsilon_n\}$ such that for all $i = 1, \dots, N$ and for $\mathcal{L}^{N-1}$ a.e. $x_i' \in \mathbb{R}^{N-1}$,

$$\lim_{n\to\infty} \int_{(\Omega_{\varepsilon_n})_{x_i'}} |\nabla u_{\varepsilon_n}(x_i', x_i) - \nabla u(x_i', x_i)|^p \, dx_i = 0. \tag{1}$$

Set $u_n := u_{\varepsilon_n}$ and

$$E := \left\{ x \in \Omega : \lim_{n\to\infty} u_n(x) \text{ exists in } \mathbb{R} \right\}.$$

Since E contains every Lebesgue points of u, we have that $\mathcal{L}^N(\Omega \setminus E) = 0$. Define

$$\overline{u}(x) := \begin{cases} \lim_{n\to\infty} u_n(x) & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\overline{u}$ is a representative of u, since by Theorem C.19, $\{u_n\}$ converges pointwise at every Lebesgue point of u. It remains to prove that $\overline{u}$ has the desired properties.

By Fubini's theorem for every $i = 1, \dots, N$ we have that

$$\int_{\mathbb{R}^{N-1}} \left(\int_{\Omega_{x_i'}} |\nabla u(x_i', x_i)|^p \, dx_i \right) dx_i' < \infty$$

and

$$\int_{\mathbb{R}^{N-1}} \mathcal{L}^1\left(\left\{ x_i \in \Omega_{x_i'} : (x_i', x_i) \notin E \right\} \right) dx_i' = 0,$$

where $\Omega_{x_i'} := \{x_i \in \mathbb{R} : (x_i', x_i) \in \Omega\}$, and so we may find a set $N_i \subset \mathbb{R}^{N-1}$, with $\mathcal{L}^{N-1}(N_i) = 0$, such that for all $x_i' \in \mathbb{R}^{N-1} \setminus N_i$ for which $\Omega_{x_i'}$ is nonempty we have that

$$\int_{\Omega_{x_i'}} |\nabla u(x_i', x_i)|^p \, dx_i < \infty, \tag{2}$$

(1) holds for all $i = 1, \dots, N$ and $(x_i', x_i) \in E$ for $\mathcal{L}^1$ a.e. $x_i \in \Omega_{x_i'}$.

Fix any such x_i' and let $I \subseteq \Omega_{x_i'}$ be a maximal interval. Fix $t_0 \in I$ such that $(x_i', t_0) \in E$ and let $t \in I$. For all n large, the interval of endpoints t and t_0 is contained in $(\Omega_{\varepsilon_n})_{x_i'}$ and so, since $u_n \in C^\infty(\Omega_{\varepsilon_n})$, by the fundamental theorem of calculus,

$$u_n(x_i', t) = u_n(x_i', t_0) + \int_{t_0}^{t} \frac{\partial u_n}{\partial x_i}(x_i', s)\, ds.$$

Since $(x_i', t_0) \in E$. Then$u_n(x_i', t_0) \to \overline{u}(x_i', t_0) \in \mathbb{R}$. On the other hand, by (1)

$$\lim_{n\to\infty} \int_{t_0}^{t} \left| \frac{\partial u_n}{\partial x_i}(x_i', s) - \frac{\partial u_n}{\partial x_i}(x_i', s) \right| ds = 0. \tag{3}$$

Hencewe have that there exists the limit

$$\begin{aligned} \lim_{n\to\infty} u_n(x_i', t) &= \lim_{n\to\infty} \left(u_n(x_i', t_0) + \int_{t_0}^{t} \frac{\partial u_n}{\partial x_i}(x_i', s)\, ds \right) \\ &= \overline{u}(x_i', t_0) + \int_{t_0}^{t} \frac{\partial u}{\partial x_i}(x_i', s)\, ds. \end{aligned}$$

Note that by the definition of E and $\overline{u}$, this implies, in particular, that

$$(x_i', t) \in E \tag{4}$$

and that

$$\overline{u}(x_i', t) = \overline{u}(x_i', t_0) + \int_{t_0}^{t} \frac{\partial u}{\partial x_i}(x_i', s)\, ds \tag{5}$$

for *all* $t \in I$. Hence, by Lemma 3.31, the function $\overline{u}(x_i', \cdot)$ is absolutely continuous in I and $\frac{\partial \overline{u}}{\partial x_N}(x_i', t) = \frac{\partial u}{\partial x_i}(x_i', t)$ for $\mathcal{L}^1$ a.e. $t \in I$.

Step 2: Assume that u admits a representative $\overline{u}$ that is absolutely continuous on $\mathcal{L}^{N-1}$ a.e. line segments of Ω that are parallel to the coordinate axes, and whose first order (classical) partial derivatives belong to $L^p(\Omega)$. Fix $i = 1, \dots, N$ and let $x_i' \in \mathbb{R}^{N-1}$ be such that $\overline{u}(x_i', \cdot)$ is absolutely continuous on the open set $\Omega_{x_i'}$. Then for every function $\varphi \in C_c^\infty(\Omega)$, by the integration by parts formula for absolutely continuous functions, we have

$$\int_{\Omega_{x_i'}} \overline{u}(x_i', t) \frac{\partial \varphi}{\partial x_i}(x_i', t)\, dt = -\int_{\Omega_{x_i'}} \frac{\partial \overline{u}}{\partial x_i}(x_i', t)\, \varphi(x_i', t)\, dt.$$

Since this holds for $\mathcal{L}^{N-1}$ a.e. $x_i' \in \mathbb{R}^{N-1}$, integrating over $\mathbb{R}^{N-1}$ and using Fubini's theorem yields

$$\int_\Omega \overline{u}(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx = -\int_\Omega \frac{\partial \overline{u}}{\partial x_i}(x)\, \varphi(x)\, dx,$$

which implies that $\frac{\partial \overline{u}}{\partial x_i} \in L^p(\Omega)$ is the weak partial derivative of $\overline{u}$ with respect to x_i . This shows that $u \in W^{1,p}(\Omega)$. ■