1 A Simpler Proof of Theorem C.19

Remark 1 *Note that if* $u \in L^p(\Omega)$, $1 \le p \le \infty$, *or* u *is nonnegative or nonpositive, then* $(C.6)$ *is defined* for all $\boldsymbol{x} \in \mathbb{R}^N$.

The first main result of this subsection is the following theorem.

Theorem 2 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set, let* $\varphi \in L^1(\mathbb{R}^N)$ *be a nonnegative bounded function satisfying* $(C.5)$, *and let* $u \in L^1_{\text{loc}}(\Omega)$.

(i) If $u \in C(\Omega)$, *then* $u_\varepsilon \to u$ *as* $\varepsilon \to 0^+$ *uniformly on compact subsets of* Ω.

(ii) For every Lebesgue point $\boldsymbol{x} \in \Omega$ *(and so for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \Omega$*),* $u_\varepsilon(\boldsymbol{x}) \to u(\boldsymbol{x})$ *as* $\varepsilon \to 0^+$. *Moreover, if* $u \in L^p(\Omega)$, $1 \le p \le \infty$, *then* $u_\varepsilon(\boldsymbol{x}) \to 0$ *for every* $\boldsymbol{x} \in \mathbb{R}^N \setminus \overline{\Omega}$.

(iii) If $u \in L^p(\Omega)$, $1 \le p \le \infty$, *then*

$$\|u_\varepsilon\|_{L^p(\mathbb{R}^N)} \le \|u\|_{L^p(\Omega)} \tag{1}$$

for every $\varepsilon > 0$ *and*

$$\|u_\varepsilon\|_{L^p(\mathbb{R}^N)} \to \|u\|_{L^p(\Omega)} \quad \text{as } \varepsilon \to 0^+. \tag{2}$$

(iv) If $u \in L^p(\Omega)$, $1 \le p < \infty$, *then*

$$\lim_{\varepsilon \to 0^+} \left(\int_\Omega |u_\varepsilon - u|^p \, d\boldsymbol{x} \right)^{\frac{1}{p}} = 0.$$

Proof. (i) Let $K \subset \Omega$ be a compact set. For any fixed

$$0 < \eta < \operatorname{dist}(K, \partial\Omega)$$

let

$$K_\eta := \left\{ \boldsymbol{x} \in \mathbb{R}^N : \operatorname{dist}(\boldsymbol{x}, K) \le \eta \right\}.$$

so that $K_\eta \subset \Omega$. Note that for $\varepsilon > 0$ sufficiently small we have that $K_\eta \subset \Omega_\varepsilon$. Since K_η is compact and u is uniformly continuous on K_η, for every $\rho > 0$ there exists $\delta = \delta(\eta, K, \rho) > 0$ such that

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| \le \rho \tag{3}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in K_\eta$, with $|\boldsymbol{x} - \boldsymbol{y}| \le \delta$. Let $0 < \varepsilon < \min\{\delta, \eta\}$. Then for all $\boldsymbol{x} \in K$,

$$\begin{aligned} |u_\varepsilon(\boldsymbol{x}) - u(\boldsymbol{x})| &= \left| \int_\Omega \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) u(\boldsymbol{y}) \, d\boldsymbol{y} - u(\boldsymbol{x}) \right| \\ &= \frac{1}{\varepsilon^N} \left| \int_{B(\boldsymbol{x},\varepsilon)} \varphi\left(\frac{\boldsymbol{x} - \boldsymbol{y}}{\varepsilon} \right) [u(\boldsymbol{y}) - u(\boldsymbol{x})] \, d\boldsymbol{y} \right| \\ &\le \|\varphi\|_\infty \frac{1}{\varepsilon^N} \int_{B(\boldsymbol{x},\varepsilon)} |u(\boldsymbol{y}) - u(\boldsymbol{x})| \, d\boldsymbol{y}, \end{aligned} \tag{4}$$

where we have used (C.5) and the fact that $\operatorname{supp}\varphi_\varepsilon(\cdot-\boldsymbol{y})\subseteq\overline{B(\boldsymbol{x},\varepsilon)}$. It follows by (3) that

$$|u_\varepsilon(\boldsymbol{x})-u(\boldsymbol{x})|\le\rho\alpha_N\|\varphi\|_\infty$$

for all $\boldsymbol{x}\in K$, and so $\|u_\varepsilon-u\|_{C(K)}\le\rho\alpha_N\|\varphi\|_\infty$.
(ii) Let $\boldsymbol{x}\in\Omega$ be a Lebesgue point of u, that is

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{B(\boldsymbol{x},\varepsilon)}|u(\boldsymbol{y})-u(\boldsymbol{x})|\,d\boldsymbol{y}=0,$$

then from (4) it follows that $u_\varepsilon(\boldsymbol{x})\to u(\boldsymbol{x})$ as $\varepsilon\to0^+$.
(iii) To prove (1) it is enough to assume that $u\in L^p(\Omega)$. If $1\le p<\infty$, then by Hölder's inequality and (C.5) for all $\boldsymbol{x}\in\mathbb{R}^N$,

$$\begin{aligned}|u_\varepsilon(\boldsymbol{x})|&=\left|\int_\Omega(\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y}))^{\frac{1}{p'}}(\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y}))^{\frac{1}{p}}u(\boldsymbol{y})\,d\boldsymbol{y}\right|\\&\le\left(\int_\Omega\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}\right)^{\frac{1}{p'}}\left(\int_\Omega\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})|u(\boldsymbol{y})|^p\,d\boldsymbol{y}\right)^{\frac{1}{p}}\qquad(5)\\&\le\left(\int_\Omega\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})|u(\boldsymbol{y})|^p\,d\boldsymbol{y}\right)^{\frac{1}{p}}\end{aligned}$$

and so by Fubini's theorem and (C.5) once more

$$\begin{aligned}\int_{\mathbb{R}^N}|u_\varepsilon(\boldsymbol{x})|^p\,d\boldsymbol{x}&\le\int_{\mathbb{R}^N}\int_\Omega\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})|u(\boldsymbol{y})|^p\,d\boldsymbol{y}d\boldsymbol{x}\\&=\int_\Omega|u(\boldsymbol{y})|^p\left(\int_{\mathbb{R}^N}\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{x}\right)d\boldsymbol{y}\\&=\int_\Omega|u(\boldsymbol{y})|^p\,d\boldsymbol{y}.\end{aligned}$$

On the other hand, if $p=\infty$, then for every $\boldsymbol{x}\in\mathbb{R}^N$,

$$\begin{aligned}|u_\varepsilon(\boldsymbol{x})|&\le\int_\Omega\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})|u(\boldsymbol{y})|\,d\boldsymbol{y}\\&\le\|u\|_{L^\infty(\Omega)}\int_\Omega\varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}\le\|u\|_{L^\infty(\Omega)}\end{aligned}$$

again by (C.5), and so (1) holds for all $1\le p\le\infty$.

In particular,

$$\limsup_{\varepsilon\to0^+}\|u_\varepsilon\|_{L^p(\mathbb{R}^N)}\le\|u\|_{L^p(\Omega)}.$$

To prove the opposite inequality, assume first that $1\le p<\infty$. By part (ii), $u_\varepsilon(\boldsymbol{x})\to u(\boldsymbol{x})$ as $\varepsilon\to0^+$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x}\in\Omega$, and so by Fatou's lemma

$$\int_\Omega|u(\boldsymbol{x})|^p\,d\boldsymbol{x}=\int_\Omega\lim_{\varepsilon\to0^+}|u_\varepsilon(\boldsymbol{x})|^p\,d\boldsymbol{x}\le\liminf_{\varepsilon\to0^+}\int_{\mathbb{R}^N}|u_\varepsilon(\boldsymbol{x})|^p\,d\boldsymbol{x}.$$

If $p=\infty$, then again by part (ii) $u_\varepsilon(\boldsymbol{x}) \to u(\boldsymbol{x})$ as $\varepsilon \to 0^+$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$. Hence

$$|u(\boldsymbol{x})| = \lim_{\varepsilon \to 0^+} |u_\varepsilon(\boldsymbol{x})| \leq \liminf_{\varepsilon \to 0^+} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$. It follows that

$$\|u\|_{L^\infty(\Omega)} \leq \liminf_{\varepsilon \to 0^+} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}.$$

Hence, (2) holds also in this case.
(iv) Fix $\rho > 0$ and find a function $v \in C_c(\Omega)$ such that

$$\|u - v\|_{L^p(\Omega)} \leq \rho.$$

Since $K := \operatorname{supp} v$ is compact, it follows from part (i) that for every $0 < \eta < \operatorname{dist}(K, \partial\Omega)$, the mollification v_ε of v converges to v uniformly in the compact set

$$K_\eta := \left\{\boldsymbol{x} \in \mathbb{R}^N : \operatorname{dist}(\boldsymbol{x}, K) \leq \eta\right\}.$$

Since $v_\varepsilon = v = 0$ in $\Omega \setminus K_\eta$ for $0 < \varepsilon < \eta$, we have that

$$\int_\Omega |v_\varepsilon - v|^p \, d\boldsymbol{x} = \int_{K_\eta} |v_\varepsilon - v|^p \, d\boldsymbol{x} \leq \left(\|v_\varepsilon - v\|_{C(K_\eta)}\right)^p |K_\eta| \leq \rho,$$

provided $\varepsilon > 0$ is sufficiently small. By Minkowski's inequality

$$\begin{aligned}\|u_\varepsilon - u\|_{L^p(\Omega)} &\leq \|u_\varepsilon - v_\varepsilon\|_{L^p(\Omega)} + \|v_\varepsilon - v\|_{L^p(\Omega)} + \|v - u\|_{L^p(\Omega)} \\ &\leq 2\|u - v\|_{L^p(\Omega)} + \|v_\varepsilon - v\|_{L^p(\Omega)} \leq 3\rho,\end{aligned}$$

where we have used (1) for the function $u - v$. ■