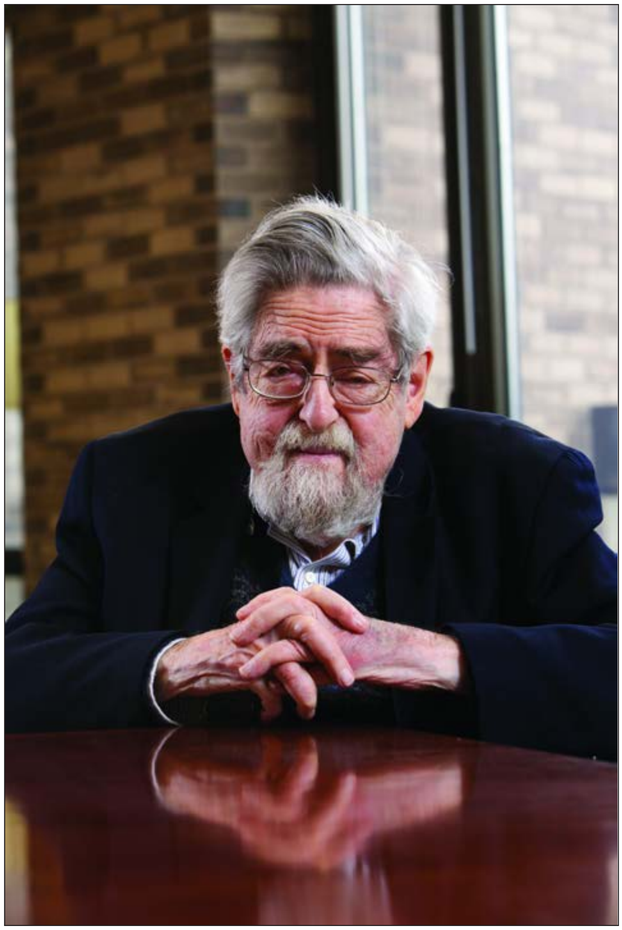
AD HONOREM LOUIS NIRENBERG



On May 19, 2015, Louis Nirenberg and John F. Nash Jr. received the 6 million NOK (about USD 750,000) Abel Prize in mathematics, Norway's response to Sweden's Nobel Prize in Physics. Initially proposed by Norwegian mathematician Sophus Lie for the hundredth anniversary in 1902 of the birth of Norwegian mathematician Niels Henrik Abel, it was established on the two hundreth anniversary in 2002. In a brilliant citation, John Rognes, chair of the Abel Committee, began with Newton and differential equations. Paradoxically it's easier to prove solutions exist if you allow them to be possibly nondifferentiable "weak" solutions, such as the generalized functions or distributions of de Rham, defined by their integrals against smooth functions. Then you can hope to prove the solution is regular (differentiable) after finding it. Rognes said:

"At first, a weak solution only exists in a virtual sense, through its interaction with other quantities. To become useful for applications, and to be accessible through numerical calculations with a computer, it is necessary to know that the weak solution is real, and that is generalized rates of change are actual rates of change.

"The regularity results of Nirenberg and Nash provide this kind of knowledge, with mathematical certainty."

The awarding of the prize by His Majesty King Harald V was followed by acceptance speeches from the laureates. Nirenberg charmed the assembly, starting by admitting that he was nervous, and then relaxing into jokes and stories. He celebrated his joy in collaboration and his love of inequalities.

During the Abel Lectures the next day, Nirenberg generously talked not only about his own work but also about the milestone progress on the Twin Primes Conjecture by Yitang Zhang. In his lecture on Nirenberg's work, Tristan Riviére said that, "Louis Nirenberg's scientific endeavor is an exemplary reminder to all of us that research is first and foremost a collective venture, in which debating, discussing, and exchanging ideas play a decisive role." In the Science Lecture on "Soap Bubbles and Mathematics," Frank Morgan observed that it was Nirenberg, with collaborators David Kinderlehrer and Joel Spruck, who proved that the singular curves in soap bubble clusters are real analytic.

The full text of Rognes's citation may be found at `www.abelprize.no/binfil/download.php?tid=63672`. Morgan's Science Lecture may be found in the September 2015 issue of the *Newsletter of the European Mathematical Society*.

Exploring the Unknown: The Work of Louis Nirenberg on Partial Differential Equations

This is a condensed write-up of Tristan Rivière's Abel Symposium lecture, May 20, 2015. For details and references see the original text at `arxiv.org/abs/1505.04930`*.*

Tristan Rivière

Partial differential equations are central objects in the mathematical modeling of the natural and social sciences: in sound propagation, heat diffusion, thermodynamics, electromagnetism, elasticity, fluid dynamics, quantum mechanics, population growth, and finance, for example. The theory entered its golden age in the second half of the twentieth century.

Since the early 1950s the mathematical work of Louis Nirenberg has made large contributions to this fundamental area of human knowledge. The name Nirenberg is associated with many of the milestones in the study of PDEs. The awarding of the Abel Prize to Nirenberg marks a special occasion for us to revisit the development of the field of PDEs and the work of one of the main actors in its exploration.

Sidebar 1. *Notices* articles on Nirenberg

Nash and Nirenberg Awarded 2015 Abel Prize, June/July, 2015

On the Work of Louis Nirenberg by Simon Donaldson, March, 2011

Interview with Louis Nirenberg, April, 2002

Louis Nirenberg Receives National Medal of Science, October, 1996

Tristan Rivière is professor of mathematics at ETH Zürich. His email address is `tristan.riviere@math.ethz.ch`*.*

DOI: http://dx.doi.org/10.1090/noti1328

Louis Nirenberg has liked to describe the field of partial differential equations as being "messy" and often acknowledges his special taste for this messiness. We'll start by presenting the original attempts made mostly in the nineteenth century to see PDEs as a whole and the limits and inadequacies of this approach.

Public domain.

In 1874 Sofia Kowalevski proved a general nonlinear version of the Cauchy-Kowalevski theorem on the existence and uniqueness of solutions to real-analytic partial differential equations.

A General Existence Result: The Cauchy-Kowalevski Theorem

Perhaps the first general systematic study of partial differential equations goes back to the work of Augustin-Louis Cauchy and his existence and uniqueness theorem for quasilinear first-order PDEs with real analytic data. In 1874, Sofia Kowalevski, apparently unaware of Cauchy's work, proved in her thesis a general nonlinear version of the result.

Some Inadequacies of the Cauchy-Kowalevski Theory

The Cauchy-Kowalevski theorem requires an analytic framework. The historical proof consists of an argument based on the convergence of power series. The question of whether there could be other nonanalytic versions has stimulated much research, and although there are uniqueness theorems for some classes of linear PDEs, there are also counterexamples to uniqueness. The general question remains to be settled.

If one seeks global solutions, which are expected to exist in physical problems, there is a need to relax the analytic framework, since singularities can appear in finite time.

Local Solvability

The Notion of Local Solvability and Lewy's Counterexample

The first attempt to go beyond the Cauchy-Kowalevski theory sacrifices uniqueness requirements and looks at "germs" of PDEs at a point. We consider the linear framework and ask whether one can enlarge the class of possible solutions from the analytic class to the C^∞ class or even to the much larger space of generalized functions called distributions. Outside the analytic framework, is a linear PDE always locally solvable?

Around 1955, Leon Ehrenpreis and Bernard Malgrange proved independently the local solvability of any linear PDE with constant coefficients. Using the Laurent Schwartz theory of tempered distributions, such a PDE can be converted into a convolution equation, and the problem is reduced to a division problem in function algebra. Encouraged by this result, the conjecture asserting that any linear operator with more general coefficients should be locally solvable became notorious in the PDE community. In 1957, Hans Lewy came up with a spectacular counterexample, namely, a C^∞ linear PDE in three variables with no C^1 solution in a neighborhood of the origin. This counterexample triggered intense research that involved many prominent analysts of that time, such as Lars Hörmander, Nirenberg, François Treves, and Yuri Vladimirovich Egorov. They were seeking necessary and sufficient conditions for local solvability.

The Nirenberg-Treves Local Solvability Condition

In three fundamental papers, Louis Nirenberg and François Treves proposed a necessary and sufficient condition for local solvability, the so-called (P) condition, established for a growing number of cases in successive works by Nirenberg and Treves themselves, by Beals and Fefferman, and by Hörmander, until in 2006 Nils Dencker proved the sufficiency of a generalized so-called Nirenberg-Treves (Ψ) condition for the very general class of pseudo-differential operators. Pseudo-differential operators were introduced by Nirenberg in collaboration with Joseph J. Kohn in 1965 and by Lars Hörmander.

Cauchy Problems and Global Solvability for Linear PDEs

The Notion of a Cauchy Problem

Understanding the local solvability of a PDE is certainly one important question, but one might argue that it should not come into play in physical applications where one expects global solutions. The question of global solvability is traditionally coupled with that of uniqueness, and together they form what is called a Cauchy problem. A Cauchy problem, or well-posed problem, consists of a linear PDE, a function space E in which the data (the input) makes sense, and a function space F to which the expected solution (the output, also called the unknown) should belong, along with the requirements that:

i) there exists exactly one solution in F for any given data in the function space E;

ii) the dependence of the solution on the data is continuous from E into F.

Finding the appropriate spaces E and F for linear PDEs has generated a tremendous amount of mathematical activity in the last century. It also gives the "asymptotics", or "constraints at the horizon", for solving many nonlinear problems. One can illustrate the notion of a Cauchy problem by looking at examples of ill-posed problems. One of them seeks a holomorphic extension in the disc $\mathbb{D}^2$ of prescribed C^1 boundary data. This is an ill-posed problem, because not every boundary data admits a C^1 holomorphic extension in $\mathbb{D}^2$ (for instance $1/z$). The ill-posedness can, however, be thwarted by replacing all C^1 functions on the boundary with the subspace of C^1 functions that are L^2-orthogonal to $1/z^k$ for all positive integers k.

The Fragmentation of the Analysis of PDEs

The search for Cauchy problems has imposed a fragmentation of the field of PDEs into multiple areas of analysis which have often developed independently of each other. The analysis of PDEs cannot be encapsulated into a single theory. It is in fact a field that might seem disorderly from the outside. But this messiness, which sometimes discourages would-be analysts, is in the very nature of PDEs, and it is the source of an infinite diversity of phenomena, arguments, and results. One may nevertheless attempt to put some order in this diversity by singling out three main families of operators: elliptic, such as Laplace's equation; parabolic, such as the heat equation; and hyperbolic, such as the wave equation. This (overly) simplified classification leaves out many equations, including physical ones, such as the Schrödinger equation or the Korteweg-deVries water-wave equation. Nonetheless, understanding how the three basic families differ from one another constitutes a first step in the study of hybrid and more complicated PDEs. Parabolic equations can be understood as elliptic equations with time propagation, and thus these two families enjoy many similar properties, such as smoothing and infinite speed of propagation. Hyperbolic equations, on the other hand, are very different, and they involve, for example, finite speed propagation and the transport of singularities. The work of Nirenberg has a barycenter closer to the first two families, so we shall focus on elliptic and parabolic PDEs.

The Agmon-Douglis-Nirenberg Elliptic Cauchy Problems in the Banach L^p spaces

In two fundamental papers published in 1959 and in 1964, Schmuel Agmon, Avron Douglis, and Nirenberg solved the elliptic Cauchy problems and the invertibility of elliptic operators of arbitrary orders. They worked in the context of Banach L^p spaces. They obtained a series of optimal results that opened the way to explore not only linear but also nonlinear PDEs, which were previously completely out of reach.

Inequalities and A Priori Estimates

Gagliardo-Nirenberg Interpolation Inequalities

The field of PDEs is structured by inequalities. They are the workhorses of the field. In the mid-1950s, functional analysis was already rich in inequalities: Hölder, Minkowski, Poincaré, Poincaré-Wirtinger, Young, Hausdorff-Young, Hardy, Hardy-Littlewood, etc., not to mention the more recent Sobolev inequalities. Deep scaling considerations (a common trait in Nirenberg's work) led Nirenberg around 1959—and independently Emilio Gagliardo—to discover a large family of inequalities for Sobolev norms partway between two others. For example, for a function u of compact support on $\mathbb{R}^n$, as t varies from 0 to 1, one gets a family of inequalities between the extreme cases:

$$\|u\|_{L^q} \leq C\|u\|_{L^p}^t \left(\int (|u|^n + |Du|^n)\right)^{(1-t)/n} (1 \leq p \leq q < +\infty),$$

where $t = p/q$.

The Use of Gagliardo-Nirenberg Inequalities for Proving A Priori Estimates

The notion of a priori estimates is central in PDEs. Roughly speaking, it deals with bounding the norm of an assumed solution in some Banach space E before we know that such a solution exists. In concrete situations, looking at a given nonlinear PDE problem, one establishes such an a priori bound in order to perform one of the numerous available analytical methods to finally prove the existence of a solution satisfying that bound: a fixed point argument in a perturbation approach, the continuity method, topological techniques such as Leray-Schauder theory, functional analysis approaches such as monotone operator theory, successive approximation, penalization approaches such as elliptization or the viscosity method, and variational approaches (such as minimization, min-max methods, or Morse theory). Gagliardo-Nirenberg inequalities are mostly used to control nonlinear terms and establish a priori estimates. There are countless applications for these inequalities.

The John-Nirenberg BMO Space: When Elasticity Meets Harmonic Analysis

The analysis of PDEs has evolved and keeps evolving in close partnership with the development of functional analysis and function space theory. Many linear and nonlinear problems in PDEs have stimulated the introduction of new function spaces, such as Sobolev spaces for solving the Dirichlet problem. The converse is also true: knowledge and properties of certain function spaces can trigger a new understanding of PDEs.

In 1961, the mathematician Fritz John was studying a rigidity problem from elasticity. The strain exerted on a perfect elastic solid can be measured by the distance of the gradient of the resulting deformation from the orthogonal group. He asked the following question:

If the gradient of a transformation from Euclidean space into itself is "close to" the group of rotations at every point, is it globally close to one single rotation?

John gave a counterexample but was able to give control on the rate of failure. In a subsequent collaboration, which has since become a milestone in analysis, John and Nirenberg systematically studied the subspace of locally integrable functions whose elements satisfy such estimates, the so-called space of functions of bounded mean oscillation (**BMO**). They proved that it is strictly larger than L^∞ but smaller than L^p_{loc} for any $p < +\infty$. The space **BMO**, which naturally arose in the context of elasticity in 1960, was apparently unknown to functional analysts. It was therefore a big surprise to discover, after the remarkable work of Elias Stein and Charles Fefferman in 1972, that BMO was the Banach dual of a famous space introduced in complex function theory some forty years earlier by Friedrich Riesz and named "Hardy space" H^1 after a famous work by Godfrey Hardy from 1915.

The dual spaces H^1 and **BMO** play a fundamental role in PDEs. Empirically, one could say that they are the "natural replacements" for L^1 and L^∞ which are not compatible with Calderón-Zygmund theory; in fact, the Agmon-Douglis-Nirenberg results do not hold either for L^1 or L^∞. In contrast, H^1 and **BMO** are well behaved in these theories.

The Maximum Principle

Nirenberg's Strong Maximum Principle for Parabolic Equations

It would be impossible to speak about the work of Nirenberg without mentioning the maximum principle. The contrast between the immense range of applications of this principle and the simplicity of the heuristic idea behind it is amazing.

In one dimension, the maximum principle states that a continuously twice differentiable function on the segment $[0,1]$ satisfying $u \geq 0$ on $[0,1]$ achieves its maximum value on the boundary of the segment. In higher dimensions, the maximum principle was known to Gauss since 1839 for solutions of Laplace's equation, owing to the mean value theorem for harmonic functions:

A solution to the Laplace equation on a bounded smooth domain achieves its extremal values on the boundary of the domain.

This formulation requires u to solve a PDE, but it was only at the beginning of the twentieth century that the idea of a general principle for elliptic partial differential inequalities emerged in a five-page paper of Eberhard Hopf (1927). In the early 1950s, a weak version of this principle was known to hold for some parabolic operators. In 1953, Louis Nirenberg proved a corresponding strong version, the subject of the next section.

The Notion of "Barriers", the "Moving Plane" Method, and the Gidas-Ni-Nirenberg Symmetry Principle

The heuristic idea behind the strong maximum principle, at least in the simpler elliptic framework, has an interesting geometric representation. We say that a linear elliptic operator satisfies the strong maximum principle if the following holds. Consider the graphs of a subsolution $\mathbb{u}$ satisfying a related inequality and a supersolution $\mathbb{v}$ satisfying the opposite inequality over a bounded domain, with one of them sitting above the other (i.e., $u \geq v$). The strong maximum principle says that if they touch at some interior point, then they are necessarily identical.

A subsolution satisfying $\mathbb{L}u \geq 0$ is then said to be a "barrier" with respect to a supersolution satisfying $\mathbb{L}v \leq 0$ and vice versa. This geometric interpretation is an incentive to manufacture barriers with respect to solutions, subsolutions, and supersolutions in order to prove pointwise inequalities via the maximum principle. This fruitful technique has become a classic in analysis, where it is known as a comparison argument. Devising suitable barriers is nothing short of being an art in itself. It requires deep intuition and thorough experience of the problems considered.

The geometric interpretation of the maximum principle in both its strong and weak formulations was probably first used in the 1955 work of Alexander Danilowitsch Alexandrow. He used a comparison argument between the solution itself and some reflections of it in order to prove that embedded constant mean curvature closed surfaces in $\mathbb{R}^3$ are necessarily round spheres. Following an important paper by James Serrin, Nirenberg, in collaboration with Basilis Gidas and Wei Ming Ni, converted Alexandrow's original idea for constant mean curvature surfaces into a general method, as beautiful as it is efficient, nowadays known as the "moving plane method". With it, one can prove symmetry and uniqueness results for positive solutions of semilinear scalar equations of the form

$$-\Delta u = \mathbb{f}(\mathbb{u}).$$

These symmetry and uniqueness results are of utmost importance, since they extend to a nonlinear framework a fundamental principle in quantum mechanics and in spectral theory, stating that the ground state of the Laplace operator, which is necessarily positive, enjoys special symmetries and has multiplicity one (i.e., it is unique—the Krein-Rutman theorem).

This method and the ensuing symmetry results have important applications to diverse areas of science: the study of ground states of nonlinear Schrödinger models in quantum mechanics (Figure 1), the vortex theory of Onsager in thermodynamics, turbulence in statistical physics, phase-transitions in Van der Walls fluids, the Yamabe problem in differential geometry, etc.

The moving plane method consists of comparing an arbitrary solution $\mathbb{u}$ for the semilinear equation with its successive reflections across a continuous family of parallel hyperplanes. These reflections are used as barrier functions for $\mathbb{u}$. A key ingredient of the method is the strong version of the maximum principle discovered by Hopf in the 1950s, now known as the Hopf boundary lemma.

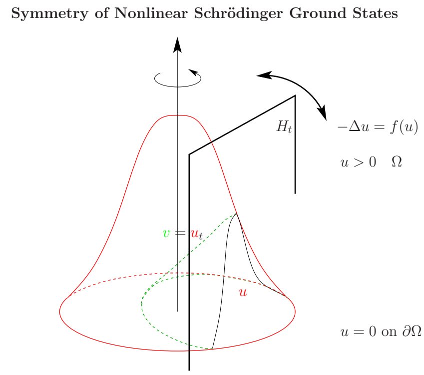


Figure 1. The Gidas-Ni-Nirenberg moving plane method proves, for example, the symmetry of the Schrödinger ground state.

Later on, Nirenberg, in collaboration with Henri Berestycki, introduced a new "sliding method", devised to prove various pointwise estimates and asymptotic behaviors for solutions in cylindrical domains to semilinear equations, as well as their parabolic counterparts. The sliding method has numerous important applications to traveling-front problems in the mathematical modeling of combustion and flame propagation. Its novel idea consists of comparing the solution with its translations along the axis of the cylinder rather than using the images by successive symmetries of the solution as barriers.

The Dirichlet Problem for Nonlinear Second-Order Elliptic Equations

We have stressed the importance of a priori estimates for solving the Dirichlet problem of semilinear equations. Proofs combine the Agmon-Douglis-Nirenberg L^p-theory for boundary-value problems with Gagliardo-Nirenberg estimates in various Banach spaces. For many scalar equations of elliptic type which are more nonlinear and more degenerate than semilinear equations, the maximum principle is an additional tool that can be added to the mix to obtain the desired estimates.

In a series of five fundamental papers written in collaboration with Luis Caffarelli and Joel Spruck, Nirenberg identified the maximum principle as a fundamental device to obtain a priori estimates and solve the Dirichlet problem for so-called fully nonlinear PDEs. An example of such an equation is the Monge-Ampère equation, which appears in problems related to optimal transport as well as in geometric problems of prescribed curvature.

These five papers by Caffarelli, Nirenberg, Spruck, and Joseph Kohn (on one of them) have stimulated a tremendous amount of research activity on fully nonlinear PDEs. These equations have an immense range of applications in many fields of science, including material sciences, finance, and computer vision. The original ideas of Nirenberg et al. have influenced the development of a whole branch of analysis, called viscosity theory for PDEs, where the maximum principle plays a central and decisive role. The viscosity theory for PDEs was introduced

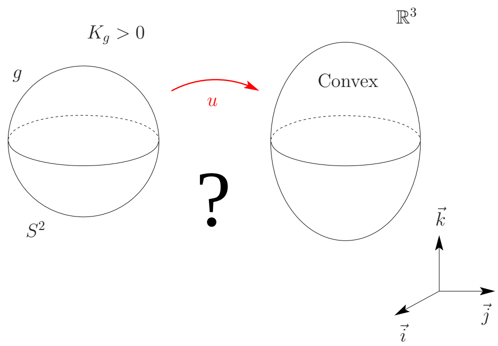


Figure 2. Can an abstract sphere with smooth prescribed positive Gauss curvature be realized as a convex surface in $\mathbb{R}^3$?

by Lawrence Craig Evans (1980) and by Michael Crandall and Pierre-Louis Lions (1983).

Solving Problems from Geometry

The analysis of PDEs and differential geometry are intimately intertwined. The central roles played by the Laplace operator and by the ∂^{bar}-operator in Riemann surface theory constitute the simplest illustrations. The second half of the twentieth century saw a dramatic acceleration of the transfer of techniques from nonlinear PDEs to the resolution of problems that seem a priori confined to geometry. A spectacular example of the might of the PDE approach in geometry is the recent proof of the Poincaré conjecture by Grigori Perelman, which relies heavily on the parabolic Ricci flow of Richard Hamilton.

Nirenberg's Resolution of the Weyl Problem

The taste for geometry and the influence of geometric questions are manifest in Nirenberg's work. He is among the pioneers who introduced elaborate analysis tools for solving questions pertaining to embeddings, tensors, curvature, and complex structures. His doctoral work itself dealt with geometry. Following the invitation from his advisors James Stoker and Kurt Friedrichs, he solved a problem originally posed by Hermann Weyl:

Given a metric of positive Gauss curvature on the two-dimensional sphere, can it be isometrically embedded as a convex surface in $\mathbb{R}^3$*?*

Nirenberg proved that it can if the given metric is four times continuously differentiable (1953). In this first work, the general philosophy is already present: one looks for a priori estimates and combines them with suitable continuity methods that leave the a priori estimates unchanged along the deformation.

The Nirenberg Problem

The original "Nirenberg problem" can be stated as follows:

Which functions K *on the two-sphere can be realized as the Gauss curvature of metrics conformal to the standard metric?*

This simply formulated question has brought forth an enormous amount of work since the early 1970s, not only because it is the simplest instance of a wide range of similar questions (higher dimension, different curvature tensor, scalar curvature, Q-curvature, σ_k-curvatures, fractional curvatures) but also because it gives rise to the major issues faced by conformal geometric analysts in their study of "critical nonlinear PDEs", such as concentration of compactness, Palais-Smale sequences, and Morse theory. These issues appear as well in many celebrated problems: the Yamabe problem, harmonic map theory, Yang-Mills equations, and constant mean curvature surfaces, to name a few. The apparent simplicity of the Nirenberg problem fosters the universal difficulties arising in conformal geometric analysis.

It would be much beyond the scope of the present report to give a detailed account of the various arguments and creativity which have flourished in the quest for solving Nirenberg's problem. We content ourselves with mentioning that not every choice for K gives rise to a solution. This is seen, for example, using the Gauss-Bonnet theorem and the now well-known Kazdan-Warner necessary condition.

The Newlander-Nirenberg Complex Frobenius Theorem

Nirenberg has made important contributions to complex geometry and complex analysis. Once again, PDE techniques lie at the heart of the approach he favored to tackle various geometric questions.

An important one deals with the integrability of almost complex structures. Nirenberg and his student A. Newlander succeeded in proving that a certain complex Frobenius-type condition is sufficient for local solvability in the C^∞ framework.

Conclusion

At the end of these notes, one feels somewhat frustrated to have only presented a small part of the prolific and monumental work of Louis Nirenberg. Many important contributions have been omitted, such as the analyticity of solutions to analytic PDEs (in collaboration with Charles Morrey), the regularity of free-boundary problems (in collaboration with David Kinderlehrer and Joel Spruck), and the partial regularity of solutions to the Navier-Stokes equation (in collaboration with Luis Caffarelli and Robert Kohn), which to this day remains the optimal step towards solving the Millennium problem.

Nirenberg's scientific endeavor is an exemplary reminder to all of us that research is first and foremost a collective venture in which debating, discussing, and exchanging ideas play a decisive role. It is the result of no coincidence that Nirenberg made his professional home at the Courant Institute at New York University, a prestigious institution that has fostered, since its very creation, a unique laboratory for the free exchange of scientific ideas.

Although there are still many important theoretical questions to answer, the analysis of partial differential equations is nowadays mostly aimed at better understanding other fields of science, with applications in geometry, physics, mechanics, chemistry, biology, social sciences, technology. These developments, and the ones to come, anchor their roots in the immense efforts deployed in the last century by human intelligence in this area of mathematics. Mathematical knowledge is, however, not only made of an accumulation of truths and results confined to papers and books and transmitted in this form to future generations. A large and immaterial share of mathematical knowledge resides in the "living part" of mathematics, in mathematicians themselves, with their intuitions, their hesitations, their perseverance, and most importantly, with their quest and search for beauty (as surprising as it may sound to nonmathematicians!). Hermann Weyl once said, "My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful." We do not know whether Nirenberg would agree with this quote, but we would nonetheless like to thank him for the beautiful mathematics he has produced and for generously sharing it with us all for so many years.

Photo: Eirik Furu Baardsen.

Tristan Rivière speaking at the lively Abel Symposium in Norway, May 2015.

Twenty Years Ago in the *Notices*

February 1996

Shadows of the Mind: A Search for the Missing Science of Consciousness, by William Faris.

Faris reviews the controversial book by Roger Penrose, and along the way provides lucid explanations of quantum mechanics.

www.ams.org/notices/199602/faris.pdf