WHAT IS...

Gauss Curvature?

Editors

Gaussian curvature is a curvature intrinsic to a two-dimensional surface, something you'd never expect a surface to have. A bug living inside a curve cannot tell if it is curved or not; all the bug can do is walk forward and backward, measuring distance. But a very intelligent bug living on a surface can, by walking around, never leaving the surface, measure its Gaussian curvature.

The Definition. Assuming you know how to define the curvature of a curve, how would you define the curvature of a surface at a point p? Slice the surface by a plane normal to the surface at p, as in Figure 1.

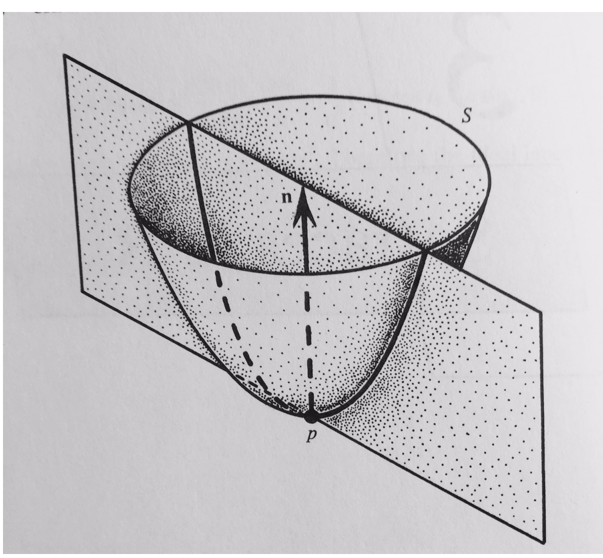

Image by J. Bredt; ©Frank Morgan, *Riemannian Geometry*, Figure 3.1.

Figure 1. To define the Gauss curvature, consider the curvature of slice curves. The Gauss curvature is the product of the largest (positive) and the smallest or most negative such curvatures.

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Take the curvature of the slice-curve. Thus you get a curvature in every direction. The largest (positive, upward) and smallest (most negative) such curvatures are called the principal curvatures, and they occur in orthogonal directions. Their average is called the *mean curvature*, and their product is called the *Gauss curvature*. For the unit sphere, both principal curvatures are 1 and hence the Gauss curvature is 1. For a unit cylinder, the principal curvatures are 1 and 0 and hence the Gauss curvature is 0. For a suitable hyperbolic paraboloid as in Figure 2, the principal curvatures are 1 and -1, and the Gauss curvature is -1.

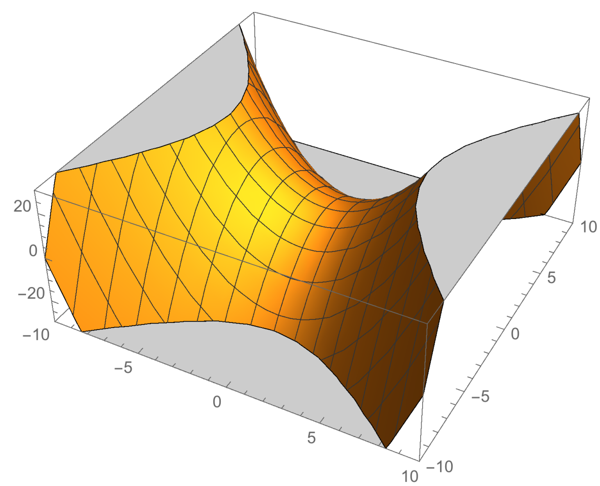

Image courtesy of Cesar E. Silva.

Figure 2. For this hyperbolic paraboloid, the Gauss curvature is $(+1)(-1) = -1$.

Gauss's Theorema Egregium says that this Gauss curvature is intrinsic; in other words, it can be measured by a bug within the surface. That means that if you bend the surface without stretching it, the Gauss curvature cannot change. If you roll the plane up into a cylinder, as in Figure 3, the principal curvatures change from 0, 0 to 0, 1, but the Gauss curvature remains 0.

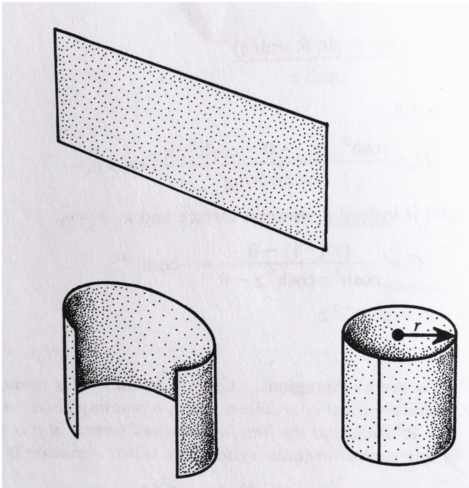

Image by J. Bredt; ©Frank Morgan, *Riemannian Geometry*, Figure 3.7.

Figure 3. Rolling a piece of plane into a cylinder of radius r changes the principal curvatures from 0, 0 to $1/r$, 0 but the Gauss curvature–the product–remains 0.

One way a bug can measure the Gauss curvature of a surface at a point p is by looking at a tiny "geodesic" circle of tiny radius r about p:

$$\{x : \mathrm{dist}(p, x) = r\}.$$

If the Gauss curvature is 0, the circumference is $2\pi r$ to high order. If the Gauss curvature is positive as on the sphere, the circumference is smaller. If the Gauss curvature is negative as on the hyperbolic paraboloid, the circumference is greater. In general, for Gauss curvature G, the circumference is given by

$$2\pi r - G\pi\frac{r^3}{3} + \dots.$$

For a closed surface, there is the amazing Gauss-Bonnet formula relating the Gauss curvature G and the Euler characteristic χ. The Euler characteristic is a purely combinatorial quantity, easily computed from any triangulation as the number of vertices minus the number of edges plus the number of regions. The Gauss-Bonnet Theorem says that

$$\int G = 2\pi\chi.$$

Since the Euler characteristic of the sphere is 2, this says that the integral of the Gaussian curvature is 4π. This is easy to check for the round unit sphere: since the Gauss curvature is 1, it just says that the area is 4π. The amazing thing is that it remains true for a deformed sphere of any size and shape.

The Gauss-Bonnet Theorem is an amazing equality of the geometric and the combinatorial. It immediately implies that the Euler characteristic is independent of triangulation, because the integral of the Gauss curvature is. Similarly it implies that the integral of the Gauss curvature remains constant under deformations, because the Euler characteristic does.

As you can read in the article on Nirenberg (page 126), in 1970 Nirenberg[1] asked which functions on the sphere can be the Gauss curvature for some embedding of the sphere in space. By the Gauss-Bonnet Theorem, the integral must be 4π. Are there any other conditions?

[1] *Technically Nirenberg considered not only spheres embedded in space but more abstract Riemannian surfaces given by a Riemannian metric on the sphere. But since Nash (who'll be featured in the May* Notices*) proved that any such sphere can be embedded in high-dimensional space, it really doesn't matter.*