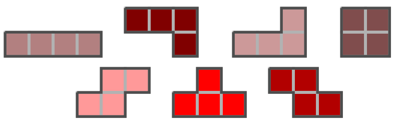
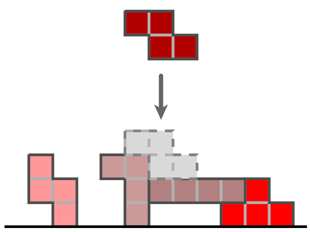
About the Cover

TETRIS® Statistics

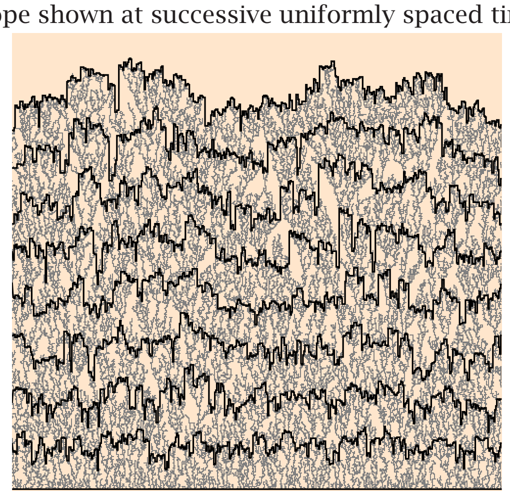
The theme of the cover is a mathematical version of the familiar game TETRIS®. The basic objects are the 7 oriented polyominoes of size 4, which are called tetrominoes.



At each moment a random tetromino, oriented randomly, descends to a random horizontal location and sticks on to the lowest point at which it will not overlap with the current configuration.



As time goes on these pile up, albeit rather loosely. The following figure illustrates a typical run, with the top envelope shown at successive uniformly spaced times:



Mathematical questions arise immediately. For example, *how does the interface at the top of the configuration grow?* Conjecturally, the answer to this question is related to the Kardar-Parisi-Zhang (KPZ) statistics discussed in Ivan Corwin's article in this issue. Corwin writes,

"The first natural conjecture is that the envelope proceeds to first order with a constant velocity, whose value is the same for all runs. To make a precise formulation, assume the boundary to be periodic, say of width W. Drop W tetrominoes in each unit of time, at a constant rate. With this normalization, does the envelope grow linearly with the same velocity for all widths and almost all runs? If so, this would be an analogue of the law of large numbers. In fact, if one drops unit squares instead of tetrominoes then each stack grows independently of other stacks, and linear growth at a unit rate follows exactly from the law of large numbers. I call this the random deposition model. Such a prediction has been verified for at least one of the models referred to in my article. (Finding exactly what that velocity is, however, seems impossible.)

Despite the overall linear growth of the envelope height, it fluctuates significantly, and the size of fluctuations seems to grow over time. What order of magnitude is this growth as a function of time? For example, if we were dropping unit squares the fluctuation would be $O(\sqrt{t})$. But this is where the KPZ universality class predictions come into play. If $\sigma(t)$ is the standard deviation of the envelope from its mean, then it is expected that

$$\sigma(t) \approx \begin{cases} t^{1/3} & t \ll W^{3/2} \\ W^{1/2} & t \gg W^{3/2} \end{cases}$$

with a crossover occurring on the time scale $W^{3/2}$. Here, we assume also that W is very large, and the $\approx$ symbol means up to constants (whose values are not known). This prediction could be checked numerically by running random TETRIS many times and for many different choices of time so as to estimate the curve $\sigma(t)$. A log-log plot would then be best suited then to reveal whether the prediction holds.

There are two very striking features of this prediction. The first is that the fluctuations of the envelope grow like $t^{1/3}$, unlike in the case of the random deposition model where they would grow like $t^{1/2}$. The intuitive reason for this is that in TETRIS attachment is not independent, and there is some evident correlation in the envelope. In TETRIS, if one section gets high, then nearby sections tend to catch up quickly because pieces latch on and spread outwardly. In fact, it is exactly this spatial correlation which leads to the second striking feature of this prediction—the saturation of the standard deviation on the time scale $O(W^{3/2})$. The reason for this is that the KPZ prediction holds that spatial correlations grow over time like $t^{2/3}$. This means that in time $t = W^{3/2}$, the correlations in space will be of order $t^{2/3} = O(W)$. Therefore, the system will feel its finiteness and will approach a steady state after which the height function fluctuations cannot continue to grow.

There is more to the KPZ prediction. In particular, using connections to integrable systems, the exact statistical distributions describing the fluctuations of the height function have been predicted (and proved in some very simple analogous models). Confirming this finer scale distributional behavior is more subtle."

—Bill Casselman
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