WHAT IS… an Anabelian Scheme?

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Suppose we are given a set of polynomial equations that we wish to solve. A scheme is an object which records the solutions to these polynomials as the domain of the variables ranges over many rings. Allowing the domain to vary helps solve equations even over a fixed ring. Schemes were invented around 1960 by Alexander Grothendieck as an early step in his reworking of algebraic geometry, and they had a revolutionary impact on the subject. As he put it in his *Récoltes et Semailles*, schemes "represent a metamorphosis of the old notion of 'algebraic variety'."

In an Anabelian Scheme, the solutions are controlled not by the usual algebraic manipulations but rather by using the loops on the complex solutions together with a Galois group. This difference opens up new possibilities for understanding the solutions, which is one reason to care about Anabelian Schemes. Before giving precise definitions, let's look at an example.

Let $f(x)$ be a polynomial with coefficients in $\mathbb{Q}$, or to be even more specific, let's suppose $f(x) = \prod_{n=0}^{5}(x-n)$. The solutions

$$X(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 : y^2 = f(x)\}$$

are drawn in Figure 1 (see p. 286). Note that these solutions form a genus 2 surface with two punctures. Replacing $\mathbb{C}$ by any $\mathbb{Q}$-algebra R yields a corresponding set of solutions, also called R-points,

$$X(R) = \{(x,y) \in R^2 : y^2 - f(x) = 0\}.$$

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Editor's Note: *This month's installment of the "WHAT IS …?" column, providing as it does an unusually daring peek into some technical and abstract mathematics, seems a perfect accompaniment to this issue's tribute to Grothendieck.*

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For example, while $X(\mathbb{C})$ is a surface, $X(\mathbb{R})$ consists of three circles with two points removed from one of them, and $X(\mathbb{Q})$ is only a finite set of points. All of these solution sets together determine a scheme X.

Note that $(0,0)$, and $(1,0)$ are both in $X(\mathbb{R})$, and view $(0,0)$ as a designated base point. If we travel along a path γ from $(0,0)$ to $(1,0)$ in $X(\mathbb{C})$ and then travel backwards along the path $\overline{\gamma}$ given by taking the complex conjugates of the coordinates of points of γ, we get a loop $\gamma\overline{\gamma}^{-1}$ from $(0,0)$ to itself. The point $(1,0)$ in $X(\mathbb{R})$ is controlled by analogues of the loop $\gamma\overline{\gamma}^{-1}$. Conjecturally, all points of $X(k)$ are controlled by analogous loops formed from paths from $(0,0)$ to (x,y) when k is a finite extension of $\mathbb{Q}$. This is the way in which the solutions to the polynomials defining an Anabelian Scheme are controlled by the loops.

To be more precise, we need a generalization of the loops on $X(\mathbb{C})$ which also incorporates field automorphisms, such as complex conjugation. This generalization is called the étale fundamental group $\pi_1^{\text{ét}}$ and its definition uses the classification of covering spaces by the fundamental group to define a notion of fundamental group given a notion of covering space. This process was discovered by Grothendieck and $\pi_1^{\text{ét}}$ records information both about topological fundamental groups and Galois groups. For example, suppose a scheme X is such that all of its defining polynomials have coefficients in k and the only $X(R)$ considered are those where R is a k-algebra. Such a scheme is said to be *over* k. Under mild hypotheses, there is a short exact sequence

$$1 \to \pi_1(X(\mathbb{C}))^\wedge \to \pi_1^{\text{ét}}X \to G \to 1,$$

where $\pi_1(X(\mathbb{C}))^\wedge$ denotes the inverse limit of finite quotients of $\pi_1(X(\mathbb{C}))$, and G denotes the absolute Galois group of the number field k.

The procedure given two paragraphs above associating the point $(1,0)$ to the loop $\gamma\overline{\gamma}^{-1}$ on $X(\mathbb{C})$ generalizes to give a map

$$X(k) \to \mathrm{Map}_G^{\text{out}}(G, \pi_1^{\text{ét}}X), \tag{1}$$

where $\mathrm{Map}_G^{\text{out}}(G, \pi_1^{\text{ét}}X)$ denotes the outer continuous group homomorphisms from G to $\pi_1^{\text{ét}}X$ which respect

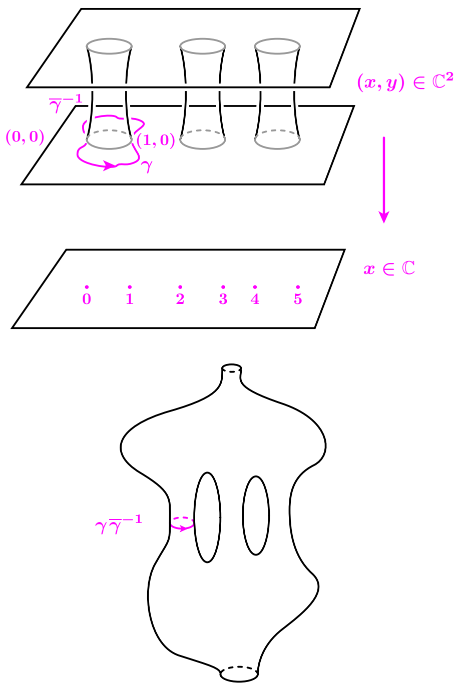

Figure 1. The map $f : X(\mathbb{C}) \to \mathbb{C}$ defined by $(x,y) \mapsto x$ is a branched covering map; over every point x of $\mathbb{C}$ such that $f(x) \neq 0$ there are two points of $X(\mathbb{C})$, and over every point x in $\mathbb{C}$ such that $f(x) = 0$ there is one point in $X(\mathbb{C})$. We partition the zeros of $f(x)$ into pairs and cut a slit running from each point of a pair to the other. The inverse image under f of the complex plane $\mathbb{C}$ minus the slits S is two disjoint copies of $\mathbb{C} - S$ because any loop in the base must wrap around an even number of zeros of f, which causes a lift of that loop to stay on the same sheet of the covering space. These two copies of $\mathbb{C} - S$ are attached along the inverse images of the slits. When a loop on the base passes through a slit, the lift of the loop must change sheets. Since the associated gluing would result in self-intersections, it is easier to see the shape of the solutions if we flip the bottom copy of $\mathbb{C} - S$ over the real axis. We then glue or add small cylinders and can see that the solutions form a genus 2 surface with two punctures.

the map $\pi_1^{\text{ét}} X \to G$. More precisely, $\mathrm{Map}_G^{\text{out}}(G, \pi_1^{\text{ét}} X)$ denotes the set of equivalence classes of continuous group homomorphisms $G \to \pi_1^{\text{ét}} X$ such that the diagram

$$\begin{tikzcd} G \arrow[rr] \arrow[dr, "1"'] & & \pi_1^{\text{ét}} X \arrow[dl] \\ & G & \end{tikzcd}$$

commutes and where two group homomorphisms $f_1, f_2 : G \to \pi_1^{\text{ét}} X$ are considered equivalent if there is γ in $\pi_1(X(\mathbb{C}))^\wedge$ such that $f_2(g) = \gamma f_1(g)\gamma^{-1}$. The purpose of considering outer homomorphisms instead of homomorphisms is to eliminate the dependency on the choice of base point. More generally still, there is a map

$$\mathrm{Map}(Y, X) \to \mathrm{Map}_G^{\text{out}}(\pi_1^{\text{ét}} Y, \pi_1^{\text{ét}} X) \tag{2}$$

for any scheme Y over k.

Roughly speaking, Anabelian schemes are a conjectural type of scheme for which maps similar to (2) and (1) are bijections, which is to say that the solutions to the polynomial equations underlying X correspond to maps of étale fundamental groups. Grothendieck gave specific examples of schemes he predicted to be anabelian in this way, and for definiteness, let's use that as a definition. Let k be a finitely generated field, and we'll also assume characteristic 0 as a precaution.

Definition. A finite type scheme[1] X over k is said to be *anabelian* if it can be constructed by successive smooth fibrations of curves with negative Euler characteristic.

Armed with this definition, let's give two theorems saying that Anabelian Schemes behave as Grothendieck predicted. The first is due to Neukirch and Uchida and says that any isomorphism of absolute Galois groups of number fields $\mathrm{Gal}(\overline{L}/L) \cong \mathrm{Gal}(\overline{k}/k)$ corresponds to an isomorphism of fields. For example, it follows that for any a,b in $\mathbb{Q}^*$, if $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{a}]) \cong \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{b}])$, we must have that the fields $\mathbb{Q}[\sqrt{a}]$ and $\mathbb{Q}[\sqrt{b}]$ are themselves equal, or equivalently, that $a = b$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, the rational numbers modulo their squares. Using the identification of Galois groups of fields with étale fundamental groups of the corresponding schemes, the Neukirch–Uchida theorem can be restated to say that an analogue of (2) where isomorphisms replace maps is a bijection.

To state our second theorem saying that Anabelian Schemes behave as Grothendieck predicted, we need the notion of a dominant map between schemes. A map between schemes whose image is dense is called *dominant*. One can refine the map (2) to a map

$$\mathrm{Map}^{\text{dom}}(Y, X) \to \mathrm{Map}_G^{\text{out,open}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X)), \tag{3}$$

from the set of dominant maps from Y to X to the subset

$$\mathrm{Map}_G^{\text{out,open}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X)) \subset \mathrm{Map}_G^{\text{out}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X))$$

consisting of triangles such that $\pi_1^{\text{ét}}(Y) \to \pi_1^{\text{ét}}(X)$ has open image. Grothendieck conjectured that for X and Y anabelian, the map (3) is bijective. Shinichi Mochizuki proved an impressive case of this conjecture.

Theorem. *(Mochizuki 1999) For Y any smooth scheme and X a smooth curve with negative Euler characteristic,* (3) *is bijective.*

The prediction that (2) is a bijection when $Y = \mathrm{Spec}\, k$ is called the Section Conjecture and is a major open problem in the field.

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[1]Finite type *is a mild technical assumption on a scheme, which can be thought of as the requirement that the scheme be a finite union of subschemes of affine space, and isn't terribly important here.*