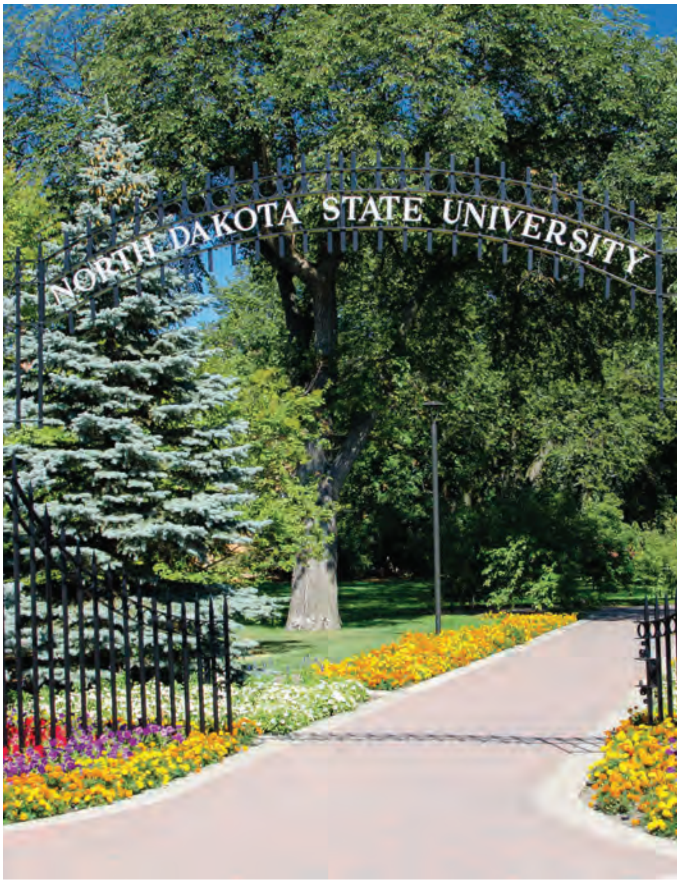
AMS SPRING SECTIONAL SAMPLER

LAURA FELICIA MATUSEVICH, associate professor of mathematics at Texas A&M University; **RODRIGO BAÑUELOS**, professor of mathematics at Purdue University and Fellow of the AMS and IMS, and **STEPH van WILLIGENBURG**, professor of mathematics at the University of British Columbia kindly provide introductions to their Invited Addresses for the upcoming AMS Spring Western Sectional (Utah, April 9–10) and the AMS Spring Central Sectional (North Dakota, April 16–17) Meetings.



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Laura Felicia Matusevich

Binomial Ideals

I am very excited to have been asked to deliver an invited address at the AMS Spring Central Sectional Meeting. I will talk about *binomial ideals*.

Binomial ideals are ideals in polynomial rings, in one or more commuting variables, with coefficients in a field. A binomial is a polynomial with at most two terms, and a binomial ideal is an ideal generated by binomials.

There is an exceptionally rich combinatorial theory of binomial ideals, which will be the subject of my address. The purpose of this piece, however, is to explain something that I won't have time to discuss in my talk: why I became interested in binomial ideals, and how my first results in this area came about.

When working with objects that can be described by equations with two terms, there is usually a binomial ideal lurking in the background, although sometimes it may be hidden. I came to binomial ideals from the study of *hypergeometric functions*, an area that can trace its origins to Euler and Gauss. Hypergeometric functions are represented by series

$$F(z_1,\dots,z_n) = \sum_{k_1,\dots,k_n=0}^{\infty} a_{k_1\cdots k_n} z_1^{k_1}\cdots z_n^{k_n},$$

whose coefficients satisfy first-order difference equations

$$q_j(k_1,\dots,k_n)a_{k_1\cdots k_j+1\cdots k_n} = p_j(k_1,\dots,k_n)a_{k_1\cdots k_n}, \tag{1}$$

where $p_1,\dots,p_n,q_1,\dots,q_n$ are polynomials. (Note that the above difference equations have two terms!)

It is well known that a system of recurrences for the coefficients of a series is equivalent to a system of differential equations for the series itself. Here, the shift operator $a_{k_1\cdots k_n} \mapsto a_{k_1\cdots k_j+1\cdots k_n}$ at the level of coefficients corresponds to multiplication by z_j at the level of series, while multiplication of a coefficient by its jth index k_j corresponds to the operator $z_j\partial/\partial z_j$ on the series. In this way, the system of first-order recurrences (1) is transformed into a linear system of n differential equations in n variables, which is highly structured but which has many more than two terms.

In the late 1980s, Gelfand, Graev, Kapranov, and Zelevinsky had an amazing idea. At the cost of introducing additional variables, the first-order recurrences could be translated into partial differential equations with constant coefficients and two terms! In effect, the recurrences produced a binomial ideal in a polynomial ring. Of course, in order to account for the additional variables, more than these differential equations were needed, and the new kind of hypergeometric system consisted of a binomial part with constant coefficients and a system of Euler-type equations. The different types of equations cannot be considered separately, as their interaction is at the root of many interesting behaviors. The situation is clearest when the binomial equations generate a prime ideal; this was the case that had been considered by Gelfand, Graev, Kapranov, and Zelevinsky.

On the other hand, the classical hypergeometric systems are given by n generators in n variables, and the corresponding binomial ideals (known as *lattice basis ideals*) are rarely prime. A natural question is then whether we can reduce from the lattice basis ideals we are interested in to the more tractable prime binomial ideals. It turns out that, under some genericity assumptions, the answer is "almost".

It was clear that we needed a new idea.

The key breakthrough comes from the influential article "Binomial ideals" by Eisenbud and Sturmfels, whose main result is that any binomial ideal (in a polynomial ring over an algebraically closed field) has a *primary decomposition* in terms of binomial ideals that are "almost" prime. Over ℂ (the natural field to work with when studying hypergeometric functions), Eisenbud and Sturmfels also gave a description of the binomials in each primary component. However, in order to use these results in the hypergeometric context, a combinatorial description of the monomials in each primary component was also necessary, and such descriptions were absent from the literature.

Which brings us to my kitchen table in 2005. Based on earlier joint work with Timur Sadykov, computational experiments, and a dose of inspiration, Alicia Dickenstein and I had come up with very satisfying conjectural combinatorial expressions for the solutions of a hypergeometric system that arise from the primary components of a lattice basis ideal. Our problem was that in order to prove that our solutions were *the* solutions, we needed to

Laura Felicia Matusevich at work.

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This article has benefitted from conversations with and suggestions from my colleagues Alicia Dickenstein and Ezra Miller.

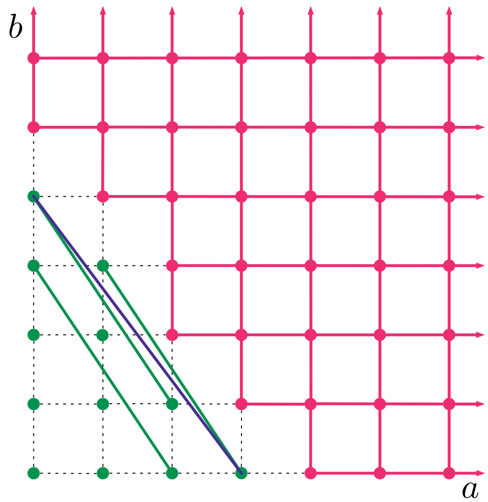

Figure 1. A combinatorial computation of binomial primary decomposition. The lattice points represent monomials $a^n b^m$, the segments represent binomials $a^p b^q - a^r b^s$, and the magenta points are the monomials in a primary component of a certain ideal.

know the monomials in those components. So we decided to write down what the primary components needed to be in order for our conjectures to be true and try to prove that those were indeed the components. And there we got stuck. It was clear that we needed a new idea.

A few months later I visited Ezra Miller in Minneapolis, and we spent three days discussing hypergeometric systems, lattice basis ideals, and their primary decompositions. By the end of my visit, Ezra had recognized that the combinatorial gadgets Alicia and I were using (see Figure 1 for a typical example) represented monoid congruences. This turned out to be the crucial new idea that was necessary to make everything work. In fact, this tool not only yielded combinatorial expressions for the primary components of lattice basis ideals but also applied to any binomial ideal over an algebraically closed field of characteristic zero. This enabled Alicia, Ezra, and me to prove all the hypergeometric results we wanted in much more generality than I had at first hoped for.

Since then the combinatorics of binomial primary decomposition has taken on a life of its own, independent of its hypergeometric origins. More general results have been found which do not depend on base field assumptions, but many open questions remain. For a survey on the recent developments on binomial primary decomposition, come to my talk!

Laura Felicia Matusevich

Rodrigo Bañuelos

Lévy Processes, Nonlocal Operators, and Spectral/Heat Asymptotics

In October 1910, the physicist Hendrik Antoon Lorentz delivered the Paul Wolfskehl lectures at the University of Göttingen on "Old and new problems in physics". During these lectures, with David Hilbert and Hermann Weyl in the audience, he formulated a problem: Prove that the number $N(\lambda)$ of eigenvalues of the Laplacian Δ on a planar region Ω (with zero boundary conditions), not exceeding the positive number λ, grows like the area of Ω times λ, as λ goes to infinity. The problem had been raised a month earlier by Arnold Sommerfeld at a lecture in Königsberg. Apparently Hilbert predicted that this would not be proved in his lifetime, yet the assertion was proved the following year by Weyl. More precisely, Weyl proved that $N(\lambda) = \frac{|\Omega|}{4\pi}\lambda + o(\lambda)$, as $\lambda \to \infty$, where $|\Omega|$ denotes the area of Ω. Weyl's celebrated theorem, commonly referred to as *Weyl's Law*, has been extended and refined in many directions, with connections to many areas of pure and applied mathematics and generating many interesting problems. Among these is Pólya's 1961 famous conjecture which asserts that the upper bound $N(\lambda) \leq \frac{|\Omega|}{4\pi}\lambda$ holds for all λ and which he proved for regions that tile the plane. The conjecture remains open even for the disc.

In subsequent papers in 1912, Weyl extended his result to regions in three space (the setting of the Sommerfeld-Lorentz problem) and to other boundary value problems. In 1913, he went one step further and conjectured a second-order asymptotic law: $N(\lambda) = \frac{|\Omega|}{4\pi}\lambda \pm \frac{|\partial\Omega|}{4\pi}\sqrt{\lambda} + o(\sqrt{\lambda})$, $\lambda \to \infty$, where $|\partial\Omega|$ is the length of the boundary of Ω, the $(-)$ corresponds to Dirichlet boundary conditions, and the $(+)$ to Neumann. Weyl's conjecture (*Weyl's second Law*) remained open until 1980, when it was proved by Ivrii (and shortly thereafter by Melrose) under certain conditions on Ω.

There has been intense activity in recent years in the study of problems where local operators, such as the Laplacian, are replaced by nonlocal operators, such as the fractional Laplacian, $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$. Many of the nonlocal operators of interest correspond to generators of Lévy processes, a rich class of stochastic processes introduced by Paul Lévy in the 1930s sharing some important properties with Brownian motion. They have independent and stationary increments, but their paths are only stochastically continuous, which allows for jumps. The Poisson and compound Poisson processes are classical examples of Lévy processes with jumps, as are the rotationally symmetric stable processes of order

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α, $0 < \alpha < 2$, whose generators are the fractional Laplacians. Problems in the area of nonlocal operators often branch in different directions involving probabilistic and/or analytic techniques and belonging to different areas of both pure and applied mathematics.

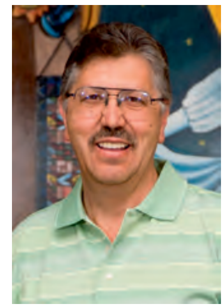
The problems in this talk belong to the area of spectral theory for nonlocal operators and particularly for fractional Laplacians and other closely related operators, such as the relativistic Laplacian and Schrödinger fractional Laplacians. They are motivated by Weyl's first and second laws, corresponding questions for traces of heat semigroups and the *heat content*, and its connections to the expected volume of the *Wiener sausage*. They are also motivated by probabilistic ideas and techniques that have greatly influenced the study of heat kernels, eigenfunctions, and eigenvalues for the Laplacian and operators corresponding to more general diffusions in different geometric settings, including manifolds and fractals. From the point of view of Lévy processes, the problems are quite natural, as Brownian motion, with the Laplacian as its generator, is "just" an example of such a process. On the other hand, the introduction of jumps requires the development of new techniques, both on the probability and on the analysis side, and especially when dealing with boundary value problems. In fact, even the formulation of the "proper" boundary conditions requires careful consideration.

This talk first gives a review of the elegant connections of spectral and heat asymptotics for the Laplacian to Brownian motion explored by Mark Kac in the 1950s and 1960s and connections to his celebrated 1966 paper "Can one hear the shape of a drum?" It then explains how, using the entire complement of Ω as its "boundary", the Dirichlet eigenvalues for many nonlocal operators form a sequence $\{\lambda_k\}$ satisfying $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \to \infty$, as in the case of the Laplacian. However, unlike the case of the Laplacian, where the eigenvalues (and eigenfunctions) can be explicitly given for various simple regions, the eigenvalues for nonlocal operators, such as the fractional Laplacian, are not explicitly known even in the simplest case of an interval in one dimension. Nevertheless, one can raise many questions on the dependence of the eigenvalues (and eigenfunctions) on the geometry of the region. In particular, are there analogues of Weyl's first and second laws, and versions of these (first- and second-order small time asymptotics) for heat traces? If so, what geometric quantities of the region are revealed by these asymptotics? As we shall see, most questions, beyond first-order asymptotics, remain quite open. But there is progress (by several researchers) to report on second-order heat trace asymptotics for regions $\Omega \subset \mathbb{R}^d$ and on heat trace and heat content asymptotics of any order for fractional Schrödinger operators with smooth potentials on $\mathbb{R}^d$. Curiously, the latter reveals quantities involving the potential that are quite different from those arising for the Laplacian.

If it is indeed the case, as has often being said, that "the geometry of the Laplacian (Brownian motion) does not reveal its secrets lightly," this is even more so for the geometry of nonlocal operators (Lévy processes)!

About the Author

Rodrigo Bañuelos is a Fellow of the AMS and IMS (Institute of Mathematical Statistics). His research is at the interface of probability and analysis.



Steph van Willigenburg

Quasisymmetric Schur Functions

Dating from an 1815 paper of Cauchy, Schur functions have long been a central object of study due to their multifaceted nature. Starting from their definition, they can be viewed as the determinant of a matrix, computed using divided differences, expressed using raising operators, or written as a sum of combinatorially computed monomials. Additionally, they arise in a number of different guises in mathematics, including as an orthonormal basis for the algebra of symmetric functions, Sym, and as characters of the irreducible polynomial representations of $GL(n, \mathbb{C})$, whence the name from Schur's seminal work in 1901. They also play a pivotal role in Hilbert's 15th problem on Schubert calculus and more recently in quantum physics.

Their pervasive nature has led to their generalization in a variety of ways. Perhaps the best known generalization is Macdonald polynomials, which encompass a plethora of functions and have additional parameters q and t that reduce to Schur functions when $q = t = 0$. Likewise, Sym has been generalized in a number of ways.

One significant nonsymmetric generalization is the algebra of quasisymmetric functions, $QSym$, which is itself of interest, since quasisymmetric functions can be seen as generating functions for P-partitions and flags in graded posets in discrete geometry and enumerative

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combinatorics. In representation theory, quasisymmetric functions arise in the study of Hecke algebras, Lie representations, and crystal graphs for general linear Lie superalgebras, while in probability they arise in the investigation of random walks and riffle shuffles.

Since *Sym* is a subalgebra of *QSym*, a natural question to ask is whether there exists a basis of *QSym* that reflects the algebraic and combinatorial properties of Schur functions. Recently Haglund, Luoto, Mason, and van Willigenburg discovered such a basis of quasisymmetric Schur functions, whose genesis lies in the combinatorics of Macdonald polynomials. This basis is already having an impact, being key to the resolution by Lauve and Mason of a long-standing conjecture that *QSym* over *Sym* has a stable basis and initiating a new avenue of research for other Schur-like bases.

About the Author

Courtesy of Niall Christie.

Steph van Willigenburg's awards for research include fellowships from the Leverhulme Trust and the Alexander von Humboldt Foundation, and she has won a Killam Award for her teaching. Professor van Willigenburg is also one of the co-founders and organizers of the Algebraic Combinatorixx workshops at the Banff International Research Station to foster mentoring, collaborations, and networking for women in algebraic combinatorics and related areas.