Shape in Two Dimensions

In her April feature article, Alice Barbara Tumpach explains how to relate two surfaces in 3D. For the cover, she put together something on the same theme involving plane curves. The playing card motif is mostly artifice, but with a mathematical rationale. She writes:

“Two parametrized smooth curves in 2D have the same shape if one can be obtained from the other by reparametrization (applying a smooth strictly increasing function). Any rectifiable planar curve admits a canonical parametrization, its arc-length parametrization, which draws the same shape, but at constant unit speed. The set of 2D-shapes can be therefore identified with the set of arc-length parametrized curves, which is an infinite-dimensional submanifold of the space of parametrized curves.

When working with realistic shapes (such as the outline of Elie Cartan below), one usually doesn’t have any parametrization at hand, but only a finite number of ordered points on the curve. The discrete version of an arc-length parametrized curve is an equilateral polygon. Given a finite set of ordered points in the plane, one can interpolate between them using cubic splines and then choose an ordered set of points on this interpolation forming an equilateral polygon. This process is called ‘resampling uniformly’.

In order to draw an equilateral polygon, one just needs to know the length of the edges, the position of the first edge, and the angles between two successive edges. In other words, the sequence of turning-angles defines an equilateral polygon modulo scaling, rotation, and translation.

Elie Cartan and the outline of his head.

The sequence of turning angles of an equilateral polygon is the discrete analogue of the signed curvature κ of an arc-length parametrized smooth curve, which is the rate of the turning of its unit tangent vector. As in the discrete case, one can reconstruct a curve (modulo rotation and translation) from its signed curvature.

Any deformation of a 2D-shape into another is equivalent to a deformation of the corresponding arc-length parametrized curves or, equivalently, of the corresponding curvature functions. One can easily interpolate between two shapes using the linear interpolation between the corresponding curvature functions.

In the figure above at the right, the exterior boundary is the outline of Elie Cartan’s head, as pictured below left, with the curvature function graphed above. The interior boundary is the profile of Emmy Noether, with its computed curvature function. In between are depicted the arc-length parametrized curves obtained by reconstructing a curve from the curvature function on the segment joining κCartan and κNoether. The length of the curves varies linearly just for appearance.

Unfortunately, this construction does not preserve the set of closed curves (look for example, at the bottom of the right image above). However, when the two curves we start with possess central symmetry, the sequence of curves constructed this way will all be closed. “This was the motivation to create playing cards from the outlines of Cartan and Noether. On the cover picture, the most external curve is a symmetric version of Noether’s outline, and the most internal curve is a symmetric version of Cartan’s profile. In between is depicted the sequence of closed curves obtained by reconstructing a curve from the curvature lying on the segment connecting the curvatures of Cartan’s and Noether’s playing card versions.”

The photographs of Cartan and Noether have been taken from their Wikipedia entries (The images are fair use and public domain, respectively.) One good place to find more information about shapes is David Mumford’s web page:

www.dam.brown.edu/people/mumford/vision/shape.html

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