Heisenberg Sculpture

This month's cover was suggested by Moon Duchin's article in this issue (p. 871) on word counting in groups. It was produced by Greg McShane, and illustrates the remarkable geometry on which word lengths in the integral three dimensional Heisenberg group $H(\mathbb{Z})$ are based. Duchin writes:

"McShane's image illustrates several key features of Heisenberg geometry, also known as *nilgeometry*—one of Thurston's eight three-dimensional geometries, the building blocks from which all three-manifolds are made. In brief, the image is trying to show you what it's like to walk around in the geometry: the pale planes show allowable directions of travel and the gold tubes show efficient paths.

"The Heisenberg group $H(\mathbb{R})$ is the three-dimensional Lie group of matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \quad (x, y, z \in \mathbb{R}) .$$

"This group has a class of left-invariant metrics called Carnot–Carathéodory (or CC) metrics. Identify the xy-plane with a subspace of the tangent space at the identity. You can push it around by left-multiplication in the group to get planes at every point, which recover the standard contact structure on $\mathbb{R}^3$. This is the plane field shown in the cover image. Now if you norm the initial plane, you get norms on every plane in this subbundle of the tangent bundle. Curves as in Figure 1 are called *admissible* (or horizontal, or Legendrian) if their tangent vectors lie in this subbundle, and then one gets the length of an admissible curve by integrating lengths of tangent vectors, and the distance between points by minimizing length of admissible paths. It's easy to see that any two points may be joined by admissible paths, and that minimizing paths are geodesics.

"This restriction to admissible curves produces a striking feature of the geometry. Consider the *shadow* of an admissible curve, i.e., its projection into the xy-plane. Then the shadow has an admissible lift (unique up to starting height) covering it. A simple application of Stokes' theorem shows that the amount of height gained by the curve is exactly equal to the signed area swept out by rays from the origin to the shadow path. For a closed path, this is equal to the signed area inside the path.

"So what you see in the cover image as gold tubes are actually local geodesics of the ambient geometry, if the norm on the base plane is the usual Euclidean norm. The shadows of these geodesics would be circles in the xy-plane, which lift to geodesics precisely because circles are isoperimetrically optimal in Euclidean geometry: If you follow them around once, they use as little length as possible while obtaining a certain area (i.e., while the lifts

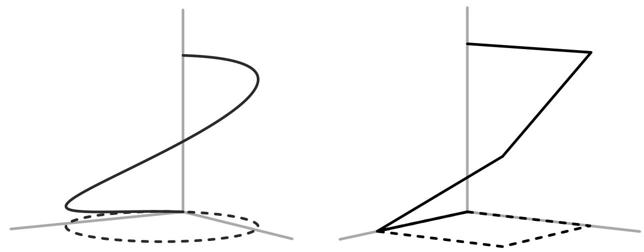

Figure 1. Two admissible paths and their shadows.

reach a certain height). That's why the ones with a larger circular radius rise faster in height.

"These CC metrics turn out to be a key tool for understanding the discrete Heisenberg group—the subgroup $H(\mathbb{Z})$ with integer entries—which features in my article. By work of Pierre Pansu from the 1980s, we know that the word metric on $H(\mathbb{Z})$ with any generating set is asymptotic to a CC metric on $H(\mathbb{R})$ endowed with an appropriate polygonal norm rather than a circle. For instance, standard generators induce the L^1 norm, which has a square shape. That means that the Cayley graphs for these discrete groups have some of the same coiled geodesics that you can see in McShane's art, but with polygonal coils [as] in the top path on the right in Figure 1. Counting problems in the discrete group can be approached by a good understanding of the CC geometry."

Greg McShane used **Blender**™ to produce his image. (This free rendering program has by now become both capable and convenient. It was also used by Thilo Rörig to produce the cover for the June/July *Notices*.) Other interesting graphics related to Heisenberg groups can be found in *Fine Asymptotic Geometry in the Heisenberg Group* by Duchin and Christopher Mooney (available on the arXiv). For analytical aspects of Heisenberg groups, Duchin recommends the book *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem* by Capogna, Danielli, Pauls, and Tyson.

All images mentioned here use coordinates on the group induced by the exponential map on the Lie algebra, in which structures are rotationally symmetric.

—Bill Casselman
Graphics Editor
notices-covers@ams.org