

WHAT IS...

# A Multiple Orthogonal Polynomial?

Andrei Martínez-Finkelshtein and Walter Van Assche

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*Multiple orthogonal polynomials* are polynomials of one variable that satisfy orthogonality conditions with respect to *several* measures. They are a very useful extension of orthogonal polynomials and recently received renewed interest because tools have become available to investigate their asymptotic behavior. They appear in rational approximation, number theory, random matrices, integrable systems, and geometric function theory. Various families of special multiple orthogonal polynomials have been found, extending the classical orthogonal polynomials but also giving completely new special functions [1, Ch. 23].

**Definition**

The classical Legendre polynomials  $P_n(x)$ ,

$$1, x, (1/2)(3x^2 - 1), (1/2)(5x^3 - 3x), \dots,$$

are orthogonal on  $[-1, 1]$  with respect to the Lebesgue measure:

$$\int_{-1}^1 P_m(x)P_n(x)dx = 0, \text{ for } m \neq n.$$

Equivalently,

$$\int_{-1}^1 x^m P_n(x)dx = 0, \text{ for } m < n.$$

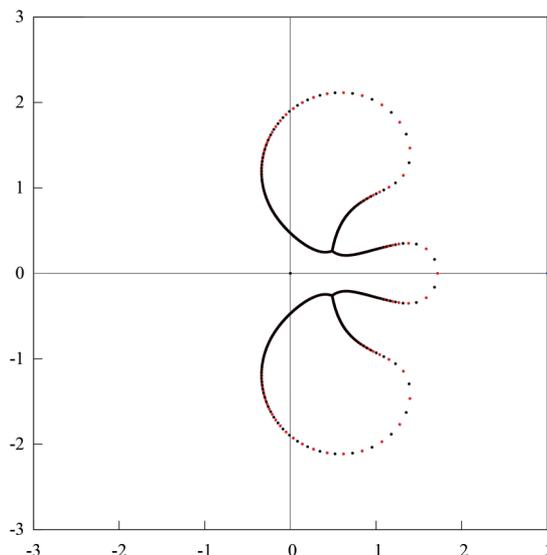
Their direct generalization comprises the Jacobi polynomials, orthogonal on the same interval with respect to the beta density  $(1 - x)^\alpha(1 + x)^\beta$ ,  $\alpha, \beta > -1$ . Similarly, the classical Hermite polynomials are orthogonal with respect to normal density  $e^{-x^2}$  on the whole real line, and

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**Figure 1. Zeros of  $A_{\vec{n},1}$ ,  $A_{\vec{n},2}$  and  $B_{\vec{n},1}$ ,  $\vec{n} = (350, 350)$ , corresponding to Type I Hermite-Padé approximation to the functions  $f_1 = w, f_2 = w^2$ , where  $w(z)$  is a solution of the algebraic equation  $w^3 + 3(z - 3)^2w + 2i(3z - 1)^3 = 0$ .**

the classical Laguerre polynomials are orthogonal with respect to the gamma density  $x^\alpha e^{-x}$ ,  $\alpha > -1$ , on  $[0, \infty)$ .

Multiple orthogonal polynomials satisfy orthogonality relations with respect to *several* measures,  $\mu_1, \mu_2, \dots, \mu_r$ , on the real line. Given a multi-index  $\vec{n} = (n_1, n_2, \dots, n_r)$  of positive integers of length  $|\vec{n}| = \sum_{i=1}^r n_i$ , *type I multiple orthogonal polynomials* are a vector  $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$  of polynomials such that  $A_{\vec{n},j}$  has degree at most  $n_j - 1$  and

$$(1) \quad \sum_{j=1}^r \int x^k A_{\vec{n},j}(x) d\mu_j(x) = 0, \quad 0 \leq k \leq |\vec{n}| - 2.$$

The *type II multiple orthogonal polynomial*  $P_{\vec{n}}$  is the monic polynomial of degree  $|\vec{n}|$  that satisfies the orthogonality conditions

$$(2) \quad \int P_{\vec{n}}(x)x^k d\mu_j(x) = 0, \quad 0 \leq k \leq n_j - 1,$$

for  $1 \leq j \leq r$ . Both relations (1), endowed with an additional normalization condition, and (2) yield a corresponding linear system of  $|\vec{n}|$  equations in the  $|\vec{n}|$  unknowns: either the coefficients of the polynomials  $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$  or the coefficients of the monic polynomial  $P_{\vec{n}}$ . The matrix of each of these linear systems is the transpose of the other and contains moments of the  $r$  measures  $(\mu_1, \dots, \mu_r)$ . A solution of these linear systems may not exist or may not be unique. In order for a solution to exist and be unique, one needs extra assumptions on the measures  $(\mu_1, \dots, \mu_r)$ . If a unique solution exists for a multi-index  $\vec{n}$ , then the multi-index is said to be *normal*. If all multi-indices are normal, then the system  $(\mu_1, \dots, \mu_r)$  is said to be a *perfect system*.

## Hermite-Padé Approximation

Multiple orthogonal polynomials originate from *Hermite-Padé approximation*, a simultaneous rational approximation to several functions  $f_1, f_2, \dots, f_r$  given by their analytic germs at infinity,

$$f_j(z) = \sum_{k=0}^{\infty} \frac{C_{k,j}}{z^{k+1}}, \quad 1 \leq j \leq r.$$

A type I Hermite-Padé approximation consists of finding polynomials  $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$  and  $B_{\vec{n}}$ , with  $\deg A_{\vec{n},j} \leq n_j - 1$ , such that

$$\sum_{j=1}^r A_{\vec{n},j}(z)f_j(z) - B_{\vec{n}}(z) = \mathcal{O}\left(\frac{1}{z^{|\vec{n}|}}\right), \quad z \rightarrow \infty;$$

see Figure 1. A type II Hermite-Padé approximation consists of finding rational approximants  $Q_{\vec{n},j}/P_{\vec{n}}$  with the common denominator  $P_{\vec{n}}$  in the sense that for  $1 \leq j \leq r$ ,

$$P_{\vec{n}}(z)f_j(z) - Q_{\vec{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right), \quad z \rightarrow \infty.$$

If the function  $f_j$  is the Cauchy transform of the measure  $\mu_j$ ,

$$f_j(z) = \int \frac{d\mu_j(x)}{z-x},$$

then  $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$  are the type I multiple orthogonal polynomials and  $P_{\vec{n}}$  is the type II multiple orthogonal polynomial for the measures  $(\mu_1, \dots, \mu_r)$ .

## Number Theory

The construction of the previous section goes back to the end of the nineteenth century, especially to Hermite and his student Padé, as well as to Klein and his scientific descendants Lindemann and Perron. Hermite's proof that  $e$  is a transcendental number is based on Hermite-Padé approximation. Using these ideas, Lindemann generalized this result, proving that  $e^{\alpha_1 z}, \dots, e^{\alpha_r z}$  are algebraically independent over  $\mathbb{Q}$  as long as the algebraic numbers  $\alpha_1, \dots, \alpha_r$  are linearly independent over  $\mathbb{Q}$  (which, in turn, yields transcendence of  $e$  and  $\pi$ ).

More recently, Apéry proved in 1979 that  $\zeta(3)$  is irrational. His proof can be seen as a problem of Hermite-Padé approximation to three functions with a mixture of type I and type II multiple orthogonal polynomials. Similar Hermite-Padé constructions were used by Ball and Rivoal in 2001 to show that infinitely many  $\zeta(2n+1)$  are irrational, and somewhat later Zudilin was able to prove that at least one of the numbers  $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$  is irrational.

## Random Matrices and Nonintersecting Random Paths

A research field where multiple orthogonal polynomials have appeared more recently and turned out to be very useful is random matrix theory. It was well known that orthogonal polynomials play an important role in the so-called Gaussian Unitary Ensemble of random matrices. Another random matrix model is one with an external source, i.e., a fixed (nonrandom) hermitian matrix  $A$  and probability density

$$\frac{1}{Z_n} e^{-n\text{Tr}(M^2 - AM)} dM.$$

If  $A$  has  $n_j$  eigenvalues  $a_j$  ( $1 \leq j \leq r$ ) and  $n = |\vec{n}|$ , then the average characteristic polynomial

$$P_{\vec{n}}(z) = \mathbb{E} \det(zI_n - M)$$

is a type II multiple orthogonal polynomial with orthogonality conditions

$$\int_{-\infty}^{\infty} P_{\vec{n}}(x)x^k e^{-n(x^2 - a_j x)} dx = 0, \quad 0 \leq k \leq n_j - 1,$$

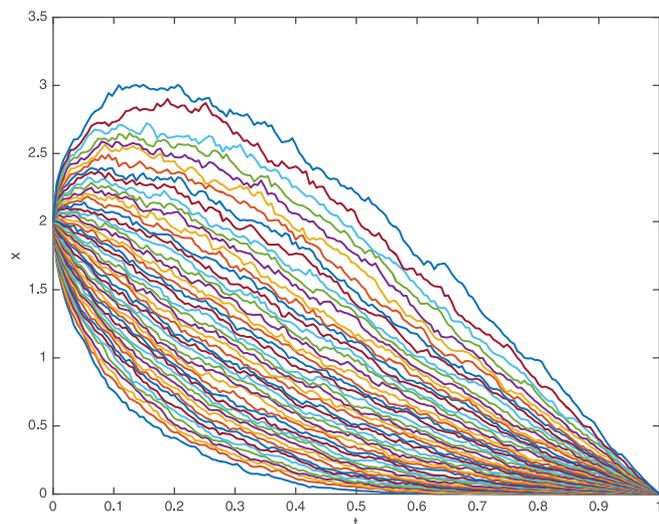
for  $1 \leq j \leq r$ . This is called a *multiple Hermite polynomial*. Instead of Hermitian matrices, one can also use positive definite matrices from the Wishart ensemble and find multiple Laguerre polynomials. More recently, products of Ginibre random matrices were investigated by Akemann, Ipsen, and Kieburg in 2013. The singular values of such matrices are described in terms of multiple orthogonal polynomials for which the weight functions are Meijer G-functions.

Eigenvalues and singular values of random matrices are special cases of determinantal point processes. These are point processes for which the correlation function can be written as a determinant of a kernel function  $K$ :

$$\rho_n(x_1, \dots, x_n) = \det(K(x_i, x_j))_{1 \leq i, j \leq n}$$

for every  $n \geq 1$ . For random matrices with an external source the kernel is in terms of type I and type II multiple orthogonal polynomials. This kernel extends the well-known Christoffel-Darboux kernel for orthogonal polynomials.

Other determinantal point processes are related to non-intersecting random paths. As Kuijlaars and collaborators have shown, nonintersecting Brownian motions leaving at  $r$  points and arriving at 1 point can be described in terms of multiple Hermite polynomials, and the analysis uses the same tools as random matrices with an external source. If the Brownian motions leave from  $r$  points and



**Figure 2. Numerical simulation of 50 rescaled nonintersecting squared Bessel paths, starting at  $a = 2$  and ending at the origin. The statistics of the position of the paths at each time  $t$  is described in terms of multiple orthogonal polynomials with respect to weights related to modified Bessel functions.**

arrive at  $r$  points, then one needs a mixture of type I and type II multiple orthogonal polynomials.

Brownian motions in  $d$ -dimensional space, in which one tracks the square of the distance of the particle from the origin, give rise to nonintersecting squared Bessel paths, whose determinantal structure is written now in terms of multiple orthogonal polynomials related to modified Bessel functions  $I_\nu$  and  $I_{\nu-1}$ , where  $\nu = d/2$ ; see Figure 2.

## Analytic Theory of MOPs

As the pioneering work of Nikishin and others showed, various tools from complex analysis and geometric function theory illuminate the study of multiple orthogonal polynomials. For example, one needs to consider extremal problems for the logarithmic energy for a vector of measures. In addition, algebraic functions solving a polynomial equation of order  $r + 1$  often come into play. The related Riemann surfaces typically have  $r + 1$  sheets, and matrices of order  $r + 1$  arise in a matrix Riemann-Hilbert characterization.

One sees here not mere technical complications but rather the richness of the asymptotic behavior of multiple orthogonal polynomials. At least for the near future, it is difficult to envision a general theory.

## Reference

- [1] M. E. H. ISMAIL, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, 2005 (paperback edition 2009). MR2191786 (2007f:33001)

## Credits

Figure 1 courtesy of S. P. Suetin.

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