WHAT IS…

a Complex Symmetric Operator?

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What do these matrices have in common:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \text{ and } \begin{bmatrix} 9 & 8 & 9 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}?$$

They each possess a well-hidden symmetry, for they are unitarily similar to the symmetric, but non-Hermitian, matrices

$$\begin{bmatrix} -\frac{1}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} \frac{5-\sqrt{34}}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{5+\sqrt{34}}{2} \end{bmatrix},$$

$$\begin{bmatrix} 2+\sqrt{\frac{57}{2}} & 0 & -\frac{1}{2}i\sqrt{37-73\sqrt{\frac{2}{57}}} \\ 0 & 2-\sqrt{\frac{57}{2}} & -\frac{1}{2}i\sqrt{37+73\sqrt{\frac{2}{57}}} \\ -\frac{1}{2}i\sqrt{37-73\sqrt{\frac{2}{57}}} & -\frac{1}{2}i\sqrt{37+73\sqrt{\frac{2}{57}}} & 3 \end{bmatrix},$$

and

$$\begin{bmatrix} 8-\frac{\sqrt{149}}{2} & \frac{9}{2}i\frac{\sqrt{16837+64\sqrt{149}}}{\sqrt{13093}} & i\frac{\sqrt{133672-1296\sqrt{149}}}{\sqrt{13093}} \\ \frac{9}{2}i\frac{\sqrt{16837+64\sqrt{149}}}{\sqrt{13093}} & \frac{207440+9477\sqrt{149}}{26186} & \frac{18\sqrt{3978002+82324\sqrt{149}}}{13093} \\ i\frac{\sqrt{133672-1296\sqrt{149}}}{\sqrt{13093}} & \frac{18\sqrt{3978002+82324\sqrt{149}}}{13093} & \frac{92675+1808\sqrt{149}}{13093} \end{bmatrix},$$

respectively. ($n \times n$ matrices A and B are *unitarily similar* if $A = U^*BU$, where U is unitary and U^* is its adjoint; operator theorists prefer the term *unitarily equivalent* instead.) The existence of these hidden symmetries is best explained in the framework of complex symmetric operators, a surprisingly large class of tractable and well-behaved operators.

Let $\mathcal{H}$ be a complex Hilbert space. Examples include $\mathbb{C}^n$, the Lebesgue spaces $L^2(X,\mu)$ of square-integrable functions on X with respect to a measure μ, the spaces $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ of square integrable sequences indexed by $\mathbb{N}$ and $\mathbb{Z}$, and the Hardy Hilbert space H^2 of holomorphic functions on the unit disk with square-summable Taylor coefficients at the origin. A conjugate-linear, isometric, involution $C : \mathcal{H} \to \mathcal{H}$ is a *conjugation* on $\mathcal{H}$; these are the Hilbert space analogues of complex conjugation. An example is $[Cf](x) = \overline{f(1-x)}$ on $L^2[0,1]$.

A linear operator $T : \mathcal{H} \to \mathcal{H}$ is *bounded* if $\|T\| := \sup\{\|T\mathbf{x}\| : \|\mathbf{x}\| \leq 1\}$ is finite. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is *C-symmetric* if $T = CT^*C$; it is *complex symmetric* if T is C-symmetric with respect to some C. Unbounded examples appear in the complex scaling theory for Schrödinger operators, certain non-self-adjoint boundary value problems, and $\mathcal{PT}$-symmetric quantum theory [1].

What is the relationship between complex symmetric operators and complex symmetric matrices? If C is a conjugation on $\mathcal{H}$, then there is an orthonormal basis $(\mathbf{e}_n)$ of $\mathcal{H}$ whose elements are fixed by C: $C\mathbf{e}_n = \mathbf{e}_n$ for all n. Since $\langle C\mathbf{x}, C\mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, the matrix of a C-symmetric operator T with respect to $(\mathbf{e}_n)$ is symmetric:

$$\begin{aligned}[T]_{i,j} &= \langle T\mathbf{e}_j, \mathbf{e}_i\rangle = \langle CT^*C\mathbf{e}_j, \mathbf{e}_i\rangle = \langle C\mathbf{e}_i, T^*C\mathbf{e}_j\rangle \\ &= \langle T\mathbf{e}_i, \mathbf{e}_j\rangle = [T]_{j,i}.\end{aligned}$$

For example, $T = CT^*C$ for

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \overline{z_3} \\ \overline{z_2} \\ \overline{z_1} \end{bmatrix}.$$

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Form a unitary

$$U = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{i}{\sqrt{2}} \end{bmatrix},$$

each of whose columns is fixed by C, and perform the corresponding change of basis:

$$U^*TU = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{i}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{2} \\ -\frac{i}{2} & \frac{i}{2} & 0 \end{bmatrix}.$$

Voilá! A hidden symmetry is revealed! There are now procedures to test for the existence of a compatible conjugation; this is how some of the matrices above were discovered.

This suggests a striking result: each square complex matrix is similar to a complex symmetric matrix. Here is the proof: every matrix is similar to its Jordan canonical form, and every Jordan block is unitarily similar to a complex symmetric matrix (mimic the example above). Thus, $A = A^{\mathsf{T}}$ reveals *nothing* about the Jordan structure of A. On the other hand, $A = A^*$ ensures that A has an orthonormal basis of eigenvectors and only real eigenvalues. How can this be? It takes $2(1+2+\cdots+n) = n^2+n$ real parameters to specify an $n \times n$ complex symmetric matrix but only $2(1+2+\cdots+(n-1))+n = n^2$ real parameters to specify an $n \times n$ Hermitian matrix, since its diagonal entries are real. These n real degrees of freedom make all the difference!

Although less prevalent than their Hermitian counterparts, complex symmetric matrices arise throughout mathematics and its applications. For instance, suppose f is holomorphic on $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. Then f is injective if and only if for any distinct $z_1, z_2, \ldots, z_n \in \mathbb{C}$, the Grunsky-Goluzin inequality

$$\left| \sum_{j,k=1}^{n} w_j w_k \log\left(\frac{z_j z_k}{f(z_j) f(z_k)} \cdot \frac{f(z_j) - f(z_k)}{z_j - z_k} \right) \right| \leq \sum_{j,k=1}^{n} w_j \overline{w_k} \log \frac{1}{1 - z_j \overline{z_k}}$$

holds for all $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$. This is a *Hermitian-symmetric inequality*:

$$|\langle A\mathbf{w}, \overline{\mathbf{w}} \rangle| \leq \langle B\mathbf{w}, \mathbf{w} \rangle,$$

in which $B = B^*$ is positive semidefinite and $A = A^{\mathsf{T}}$. In applications, complex symmetric matrices have appeared in the study of thermoelastic waves, quantum reaction dynamics, vertical cavity surface emitting lasers, electric power modeling, multicomponent transport, and the numerical simulation of high-voltage insulators.

The most familiar result about complex symmetric matrices is the Autonne-Takagi decomposition: if $A \in \mathsf{M}_n(\mathbb{C})$ and $A = A^{\mathsf{T}}$, then $A = U\Sigma U^{\mathsf{T}}$, in which U is unitary and Σ is the diagonal matrix of singular values of A (the square roots of the eigenvalues of the positive semidefinite matrix A^*A). It was discovered by Léon Autonne in 1915 and subsequently rediscovered throughout the early twentieth century in various contexts: T. Takagi (function theory, 1925), N. Jacobson (projective geometry, 1939), C.L. Siegel (symplectic geometry, 1943), L.-K. Hua (automorphic functions of matrices, 1944), and I. Schur (quadratic forms, 1945).

The innocent-looking *Volterra operator*

$$[Tf](x) = \int_0^x f(y)\,dy$$

on $L^2[0,1]$ is a familiar counterexample to many conjectures made by budding operator theorists. For instance, it has no eigenvalues and it is properly quasinilpotent: $\|T^n\|^{1/n} \to 0$ and $T^n \neq 0$ for $n = 0, 1, 2, \ldots$. It is a standard example of a complex symmetric operator: $T = CT^*C$, in which $[Cf](x) = \overline{f(1-x)}$. Each element of the orthonormal basis $(e^{2\pi i n x})_{n \in \mathbb{Z}}$ is fixed by C. With respect to this basis, the Volterra operator has the matrix

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \frac{i}{6\pi} & 0 & 0 & -\frac{i}{6\pi} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{i}{2\pi} & -\frac{i}{2\pi} & 0 & 0 & 0 & \cdots \\ \cdots & -\frac{i}{6\pi} & -\frac{i}{4\pi} & -\frac{i}{2\pi} & \boxed{\frac{1}{2}} & \frac{i}{2\pi} & \frac{i}{4\pi} & \frac{i}{6\pi} & \cdots \\ \cdots & 0 & 0 & 0 & \frac{i}{2\pi} & -\frac{i}{2\pi} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{i}{6\pi} & 0 & 0 & -\frac{i}{6\pi} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

in which the $(0,0)$ entry has been highlighted. This drives home the fact that T is a rank-one perturbation of a skew-Hermitian operator. One might jest that definite integration is the study of a sparse, infinite complex symmetric matrix!

Examples of complex symmetric operators abound. For instance, every idempotent operator, normal operator, truncated Toeplitz operator, and Hankel matrix is a complex symmetric operator. What sort of properties do they have?

An old result of Godič and Lucenko tells us that each unitary U acting on a Hilbert space factors as $U = CJ$, in which C and J are conjugations. This generalizes the fact that a planar rotation is the product of two reflections. A similar result holds for any complex symmetric operator: if T is C-symmetric, then $T = CJ|T|$, in which J is a conjugation that commutes with the positive operator $|T| = \sqrt{T^*T}$.

There are occasional parallels between the Hermitian and complex-symmetric worlds. This should be surprising since many "poorly behaved" operators, like Jordan blocks and the Volterra operator, are complex symmetric. The celebrated Courant minimax principle asserts that if A is an $n \times n$ Hermitian matrix, then the (necessarily real) eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ of A satisfy

$$\min_{\operatorname{codim} \mathcal{V} = k} \max_{\substack{\mathbf{x} \in \mathcal{V} \\ \|\mathbf{x}\| = 1}} \mathbf{x}^* A \mathbf{x} = \lambda_k.$$

On the other hand, Danciger's minimax principle ensures that if $A = A^{\mathsf{T}}$, then its singular values $s_0 \geq s_1 \geq \cdots \geq s_{n-1}$

satisfy

$$\min_{\operatorname{codim}\mathcal{V}=k}\ \max_{\substack{\mathbf{x}\in\mathcal{V}\\ \|\mathbf{x}\|=1}} \operatorname{Re}\mathbf{x}^{\mathsf{T}}A\mathbf{x} = \begin{cases} s_{2k} & \text{if } 0 \le k < \frac{n}{2}, \\ 0 & \text{if } \frac{n}{2} \le k \le n. \end{cases}$$

The peculiar singular value "skipping" phenomenon occurs because of significant cancellation in the complex-valued expression $\mathbf{x}^{\mathsf{T}}A\mathbf{x}$. Naturally, appropriate generalizations for compact operators exist.

We conclude with a complex-symmetric analogue of Weyl's criterion from spectral theory. Let $\sigma(A)$ denote the *spectrum* of a bounded linear operator T; that is, it is the set of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ does not have a bounded inverse. If $T = T^*$, then $\lambda \in \sigma(T)$ if and only if there exist unit vectors $\mathbf{x}_n$ such that

$$\lim_{n\to\infty} \|(T-\lambda I)\mathbf{x}_n\| = 0.$$

The familiar equation $T\mathbf{x} = \lambda\mathbf{x}$ characterizes the eigenvalues of T. A similar result holds in the complex-symmetric setting. If T is C-symmetric, then $|\lambda| \in \sigma(\sqrt{T^*T})$ if and only if there are unit vectors $\mathbf{x}_n$ so that

$$\lim_{n\to\infty} \|(T-\lambda C)\mathbf{x}_n\| = 0.$$

In particular, the "antilinear eigenvalue problem" $T\mathbf{x} = |\lambda|C\mathbf{x}$ characterizes the singular values of T. This can occasionally be used to obtain information about the spectrum of $|T|$ without computing T^*T itself.

References

[1] Stephan Ramon Garcia, Emil Prodan, and Mihai Putinar, Mathematical and physical aspects of complex symmetric operators, *J. Phys. A* **47** (2014), no. 35, 353001, 54 pp. MR 3254868

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Stephan Ramon Garcia got his PhD at UC Berkeley, worked at UC Santa Barbara, and now teaches at Pomona College. He is the author of two books and over seventy research articles in operator theory, complex analysis, matrix analysis, number theory, discrete geometry, and other fields. He has coauthored over two dozen articles with students.