

# Notices

of the American Mathematical Society

March 2017

Volume 64, Number 3

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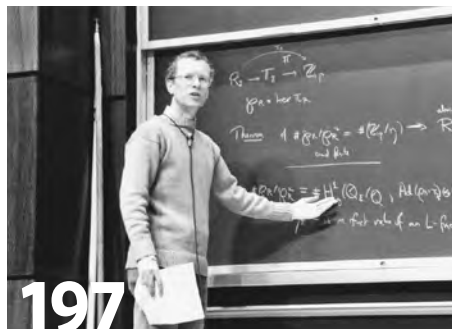
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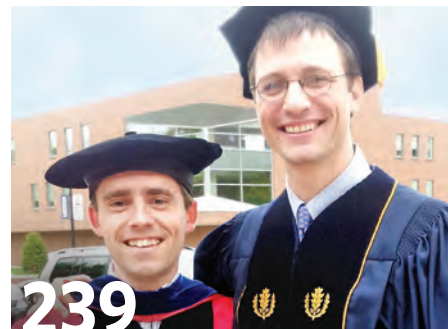
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*To fend off spring fever, read the Notices, submit an article or an item for the BackPage, participate in the commentary on the webpage [www.ams.org/notices](http://www.ams.org/notices), and send in some Letters to the Editor. —Frank Morgan, Editor-in-Chief*

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Yuval Peres, Microsoft Research, Redmond, WA

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# Leroy P. Steele Prizes

## Call for Nominations

The selection committee for these prizes requests nominations for consideration for the 2018 awards. Further information about the prizes can be found in the November 2015 Notices pp.1249–1255 (also available at [www.ams.org/profession/prizes-awards/ams-prizes/steele-prize](http://www.ams.org/profession/prizes-awards/ams-prizes/steele-prize)).

Three Leroy P. Steele Prizes are awarded each year in the following categories: (1) the Steele Prize for Lifetime Achievement: for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students; (2) the Steele Prize for Mathematical Exposition: for a book or substantial survey or expository-research paper; and (3) the Steele Prize for Seminal Contribution to Research: for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research. In 2018 the prize for Seminal Contribution to Research will be awarded for a paper in **Discrete Mathematics/Logic**.

Further information and instructions for submitting a nomination can be found at the Leroy P. Steele Prizes website:

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Nominations for the Steele Prizes for Lifetime Achievement and for Mathematical Exposition will remain active and receive consideration for three consecutive years.

For questions contact the AMS Secretary at [secretary@ams.org](mailto:secretary@ams.org).

The nomination period is February 1, 2017 through March 31, 2017.



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Wiles received the Abel Prize from Crown Prince Haakon of Norway.

**Editor's Note:** *Christopher Skinner kindly accepted our invitation to put together this feature in honor of Andrew Wiles on the occasion of his receiving the 2016 Abel Prize.*

On May 24, 2016, Sir Andrew J. Wiles received the Abel Prize in a ceremony held in the Aula of the University of Oslo in Oslo, Norway. Wiles, who received the prize from H.R.H. Crown Prince Haakon at the award ceremony, was the fourteenth recipient of the 6 million NOK (about 750,000 USD) prize. A prize honoring the Norwegian mathematician Niels Henrik Abel was first proposed by the world-renowned mathematician Sophus Lie, also from Norway, and initially planned for the one-hundredth anniversary of Abel's birth in 1902, but the establishment of the Abel Prize had to wait another hundred years. The Abel Prize is administered by the Norwegian Academy of Science and Letters.

The Abel Prize was awarded to Wiles for "his stunning proof of Fermat's Last Theorem," which opened a new era in number theory. The citation of the Abel Committee, read by committee chair John Rognes on the occasion of the announcement of the 2016 Abel Prize, recounts the early history of Fermat's Last Theorem—the assertion that for any given integer  $n \geq 3$ , there are no integer solutions to  $x^n + y^n = z^n$  with  $xyz \neq 0$ —and how it was eventually linked to the then-conjectural modularity of semistable elliptic curves and that it was this modularity that Andrew

Wiles ultimately established in a proof both surprising and profound. It is especially appropriate that Wiles's groundbreaking work on elliptic curves was recognized by the awarding of the Abel Prize, elliptic curves being the natural domains of the elliptic functions introduced by Abel. As the citation for the 2016 Abel Prize concludes: "Few results have as rich a mathematical history and as dramatic a proof as Fermat's Last Theorem."

The awarding of the Abel Prize was followed by the Abel Lectures on the next day, May 25. In his lecture, Wiles explained how his proof of Fermat's Last Theorem exemplified the movement of number theory from the abelian to the nonabelian. Henri Darmon, in his lecture "Andrew Wiles's Marvelous Proof," described Wiles's work as "a centerpiece of the Langlands program" and explained its transformative impact, and Manjul Bhargava spoke about how Wiles's work has implications for the Birch–Swinnerton-Dyer Conjecture.

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The full text of the citation of the Abel Committee can be found at [www.abelprize.no/c54154/binfil/download.php?tid=67039](http://www.abelprize.no/c54154/binfil/download.php?tid=67039). An expanded version of Henri Darmon's Abel Lecture is included in this issue.

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# Interview with Abel Laureate Sir Andrew J. Wiles

*Martin Raussen and Christian Skau*

Andrew J. Wiles is the recipient of the 2016 Abel Prize of the Norwegian Academy of Science and Letters. The following interview was conducted by Martin Raussen and Christian Skau in Oslo on May 23, 2016, in conjunction with the Abel Prize celebration. This article originally appeared in the September 2016 issue of the *Newsletter of the European Mathematical Society*—see [www.ems-ph.org/journals/newsletter/pdf/2016-09-101.pdf#page=31](http://www.ems-ph.org/journals/newsletter/pdf/2016-09-101.pdf#page=31), pp. 29–38—and is reprinted here with permission of the EMS.

**Raussen and Skau:** Professor Wiles, please accept our congratulations for having been selected as the Abel Prize Laureate for 2016. To be honest, the two of us expected this interview to take place several years ago!

You are famed not only among mathematicians but also among the public at large for (and now we cite the Abel Committee): “the stunning proof of Fermat’s Last Theorem, by way of the Modularity Conjecture for elliptic curves, opening a new era in number theory.” This proof goes back to 1994, which means that you had to wait for more than twenty years before it earned you the Abel Prize. Nevertheless, you are the youngest Abel Prize Laureate so far. After you finished your proof of Fermat’s Last Theorem you had to undergo a deluge of interviews, which makes our task difficult. How on earth are we to come up with questions that you have not answered several times before? Well, we will try our best.

## Fermat’s Last Theorem: A Historical Account

We have to start at the very beginning, with a citation in Latin: “...nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere,” which means: “it is impossible to separate any power higher than the second into two like powers.” In modern mathematical jargon, this can be written: “The equation  $x^n + y^n = z^n$  has no solution in natural numbers for  $n$  greater than two.” And then it continues: “cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet,” which means: “I have discovered a truly marvellous proof of this, which



Sir Andrew J. Wiles received the Abel Prize from Crown Prince Haakon of Norway.

this margin is too narrow to contain.” This remark was written in the year 1637 by the French lawyer and amateur mathematician Pierre de Fermat [1601–1665] in the margin of his copy of Diophantus’ *Arithmetica*. He certainly did not expect that it would keep mathematicians, professionals, and amateurs alike busy for centuries trying to unearth the proof.

Could you please give us a short account of some of the attempts towards proving Fermat’s Last Theorem up until the time you embarked on your successful journey? Furthermore, why was such a simple-minded question so attractive, and why were attempts to prove it so productive in the development of number theory?

**Wiles:** The first serious attempt to solve it was presumably by Fermat himself. But, unfortunately, we know nothing about it except for what he explained about his proofs in the specific cases of  $n = 3$  and  $n = 4$ .<sup>1</sup> That is, he showed that you can’t have the sum of two cubes be another cube or the sum of two fourth powers being a

<sup>1</sup>Strictly speaking, Euler was the first to spell out a complete proof in the case  $n = 3$ .

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fourth power. He did this by a beautiful method, which we call infinite descent. It was a new method of proof, or at least a new way of presenting proofs in arithmetic. He explained this method to his colleagues in letters, and he also wrote about it in his famous margin, which was big enough for some of it at least. After the marginal notes were published by Fermat's son after his father's death, it lay dormant for a while. Then it was picked up by Euler [1707–1783] and others who tried to find this truly marvellous proof. And they failed. It became quite dramatic in the mid-nineteenth century—various people thought they could solve it. There was a discussion concerning this in the French Academy: Lamé [1795–1870] claiming he was just about to prove it, and Cauchy [1789–1857] saying he thought he could too, and so on.

In fact, it transpired that the German mathematician Kummer [1810–1893] had already written a paper where he explained that the fundamental problem was what is known now as the fundamental theorem of arithmetic. In our normal number system, any number can be factorized in essentially one way into prime factors. Take a number like 12; it is 2 times 2 times 3. There is no other way of breaking it up. But in trying to solve the Fermat problem, you actually want to use systems of numbers where this uniqueness does not hold. Every attempt that was made to solve the Fermat problem had stalled because of this failure of unique factorization. Kummer analyzed this in incredible detail. He came up with the most beautiful results, and the end product was that he could solve it for many, many cases. For example, for  $n \leq 100$  he solved it for all primes except for 37, 59, and 67. But he did not finally solve it. His method was based on the idea that Fermat had introduced—the method of infinite descent—but in these new number systems.

The new number systems he was using spawned algebraic number theory as we see it today. One tries to solve equations in these new systems of numbers instead of solving them with ordinary integers and rational numbers. Attempts in the style of Fermat carried on for a while but somewhat petered out in the twentieth century. No one came up with a fundamentally new idea. In the second half of the twentieth century, number theory moved on and considered other questions. Fermat's problem was all but forgotten by the professionals.

Then, in 1985, Ger-

hard Frey, a German mathematician, came up with a stunning new idea where he took a hypothetical solution to the Fermat problem and rewrote it so that it made what is called an elliptic curve. And he showed, or suggested, that this elliptic curve had very

*It was a  
roadblock right in  
the middle of  
modern  
mathematics.*

peculiar properties. He conjectured that you can't really have such an elliptic curve. Building on this a year later, an American mathematician, Kenneth Ribet, demonstrated, using this Frey curve, that any solution of Fermat would contradict another well-known conjecture called the Modularity Conjecture. This conjecture had been proposed in a weak form by Taniyama [1927–1958] and was refined by Shimura, but the first real evidence for it came from André Weil [1906–1998], who made it possible to check this precise form of the Modularity Conjecture in some detail. And a lot of evidence was amassed showing that this should certainly be true. So, at that point, mathematicians could see that: “Yes, Fermat is going to be true. Moreover, there has to be a proof of it.” What happened was that the Modularity Conjecture was a problem that mathematics could not just put to one side and go on for another five hundred years. It was a roadblock right in the middle of modern mathematics. It was a very, very central problem. As for Fermat, you could just leave it aside and forget it almost forever. This Modularity Conjecture you could not forget. So, at the point when I heard that Ribet had done this, I knew that this problem could be solved and I was going to try.

**Raussen and Skau:** *Concerning speculations about Fermat's claimed proof, do you think he had the same idea as Lamé, assuming, wrongly as it turned out, that the cyclotomic integers have unique factorization?*

**Wiles:** No, I don't think so, though the idea might be in there somewhere. It is very hard to understand. André Weil wrote about this. All the other problems Fermat considered had to do with curves that were of genus zero or genus one. And suddenly he is writing down a curve that has higher genus. How is he going to think about it? When I was trying this myself as a teenager, I put myself in Fermat's frame of mind because there was hardly anything else I could do. I was capable of understanding his mathematics from the seventeenth century but probably not much beyond that. It seemed to me that everything he did came down to something about quadratic forms, and I thought that might be a way of trying to think about it. Of course, I never succeeded, but there is nothing else that suggests Fermat fell into this trap with unique factorization. In fact, from the point of view of quadratic forms, he understood that sometimes there was unique factorization and sometimes there was not. So he understood that difference in his own context. I think it is unlikely that that was the mistake.

**Raussen and Skau:** *In the same book by André Weil that you referred to, titled Number Theory: An approach through History from Hammurapi to Legendre, it is mentioned that Fermat looked at the equation of a cube minus a square equal to 2 [ $x^3 - y^2 = 2$ ] and he showed that it has essentially only one solution, namely  $x = 3$  and  $y = \pm 5$ . André Weil speculates that Fermat at the time looked at the ring  $\mathbb{Z}[\sqrt{-2}]$ , which does have unique factorization.*

**Wiles:** Yes, he used unique factorization, but the way he did it was in terms of quadratic forms. And I think he also looked at quadratic forms corresponding to  $\mathbb{Z}[\sqrt{-6}]$  where there is not unique factorization. So I think he understood. It was my impression when I thought about it that he understood the difference.

## A Mathematical Education

**Raussen and Skau:** *You were apparently already interested in mathematical puzzles as quite a young boy. Have you any thoughts about where this interest came from? Were you influenced by anyone in particular?*

**Wiles:** I just enjoyed mathematics when I was very young. At the age of ten, I was looking through library shelves devoted to mathematics. I would pull out books and at one point I pulled out a book of E. T. Bell [1883–1960] titled *The Last Problem*, which on its cover describes the Fermat equation, the Wolfskehl Prize, and the romantic history of the problem. I was completely captivated by it.

**Raussen and Skau:** *Were there other things that fascinated you in this book by Eric Temple Bell?*

**Wiles:** It is entirely about that one equation, really. And it is actually quite wordy. So there is less mathematics in some sense than you might think. I think it was more the equation. Then, when I found this equation, I looked for other elementary books on number theory and learned about congruences and solved congruences and so on, and looked at other things that Fermat did.

**Raussen and Skau:** *You did this work besides your ordinary schoolwork?*

**Wiles:** Yes, I don't think my schoolwork was too taxing from that point of view.

**Raussen and Skau:** *Was it already clear to you at that time that you had an extraordinary mathematical talent?*

**Wiles:** I certainly had a mathematical aptitude and obviously loved to do mathematics, but I don't think I felt that I was unique. In fact, I don't believe I was unique in the school I attended. There were others who had just as strong a claim to be future mathematicians, and some of them have become mathematicians, too.

**Raussen and Skau:** *Did you already plan to study mathematics and to embark on a mathematical career at that age?*

**Wiles:** No, I don't think I really understood you could spend your life doing mathematics. I think that only came later. But I certainly wanted to study it as long as I could. I'm sure that as far as my horizon extended, it involved mathematics.

**Raussen and Skau:** *You started to study mathematics as a student at Oxford in 1971. Can you tell us a little bit about how that worked out? Were there any particular teachers or any particular areas that were particularly important for you?*

**Wiles:** Before I went to college (actually in high school), one of my teachers had a PhD in number theory. He gave me a copy of Hardy and Wright's *An Introduction to the Theory of Numbers*, and I also found a copy of Davenport's *The Higher Arithmetic*. And these two books I found very, very inspiring in terms of number theory.

**Raussen and Skau:** *So you were on track before you started studying?*

**Wiles:** Yes, I was on track before. In fact, to some extent, I felt college was a distraction because I had to do all these other things: applied maths, logic, and so on, and I just wanted to do number theory. You were not

allowed to do number theory in your first year. And you could not really get down to it before your third year.

**Raussen and Skau:** *But you were not interested in geometry, not as much as in algebra and number theory, anyway?*

**Wiles:** No, I was primarily interested in algebra and number theory. I was happy to learn these other things, but I really was most excited about number theory. My teachers arranged for me to take extra classes in number theory, but there was not that much to offer.

At one point, I decided that I should put all the years of Latin I had done at school to good use and try to read some of Fermat in the original, but I found that was actually too hard. Even if you translated the Latin, the way they wrote in those days wasn't in the algebraic symbols I was used to, so it was quite difficult.

**Raussen and Skau:** *It must have been a relief when you were done and came to Cambridge to start studying number theory for real, with John Coates as your supervisor.*

**Wiles:** That's right. I had a year, a preliminary year, in which I just studied a range of subjects and then I could do a special paper. John Coates was not yet at Cambridge, but I think he helped me, maybe over the summer. Anyway, that summer I met him and started working with him right away, and that was just wonderful. The transition from undergraduate work, where you were just reading and studying, to research—that was the real break for me. It was just wonderful.

## Elliptic Curves

**Raussen and Skau:** *We assume it was John Coates who initiated your work on elliptic curves and Iwasawa theory?*

**Wiles:** Absolutely. He had some wonderful ideas and was generous to share them with me.

**Raussen and Skau:** *Did you tell John Coates that you were interested in the Fermat problem?*

**Wiles:** Perhaps I did. I don't remember. It is really true that there hadn't been any new ideas since the nineteenth century. People were trying to refine the old methods and, yes, there were refinements. But it didn't look like these refinements and the solution were going to converge. It was just too hard that way.

**Raussen and Skau:** *At the time you started to work with John Coates, you had no idea that these elliptic curves were going to be crucial for the solution of Fermat's Last Theorem?*

**Wiles:** No, it's a wonderful coincidence. The strange thing is that, in a way, the two things that are most prominent in Fermat that we remember today are his work on elliptic curves and his famous last theorem. For example, this equation you mentioned,  $y^2 + 2 = x^3$ , is an elliptic curve. And the two strands came together in the proof.

**Raussen and Skau:** *Could you explain what an elliptic curve is and why elliptic curves are of interest in number theory?*

**Wiles:** For a number theorist, the life of elliptic curves started with Fermat as equations of the form  $y^2$  equals a cubic polynomial in  $x$  with rational coefficients. Then,

the problem is to find the rational solutions to such an equation. What Fermat noticed was the following. Sometimes you can start with one or even two rational solutions and use them to generate infinitely many others. And yet sometimes there are no solutions. This latter situation occurs, for example, in the case  $n = 3$  of Fermat's Last Theorem, the equation being, in fact, an elliptic curve in disguise. Sometimes you can show there are no rational solutions. You could have infinitely many and you could have none. This was already apparent to Fermat.

In the early nineteenth century, one studied these equations in complex numbers. Abel [1802–1829] himself came in at this point and studied elliptic functions and related these to elliptic curves, implying that elliptic curves have a group structure. They were very well understood in terms of doubly periodic functions in the early nineteenth century. But that is what underlies the complex solutions, solutions to the equation in complex numbers.

The solutions to the equation in rational numbers were studied by Poincaré [1854–1912]. What's now known as the Mordell–Weil theorem was proved by Mordell [1888–1972] and then Weil in the 1920s, answering a question of Poincaré. In our setting, it says that the  $K$ -rational points on an elliptic curve over a number field  $K$ , in particular for  $K$  equal to the rationals, form a finitely generated abelian group. That is, from Fermat's language, you can start with a finite number of solutions and, using those, generate all the solutions by what he called the chord-and-tangent process.

By now you know the structure; it is a very beautiful algebraic structure, the structure of a group, but that does not actually help you find the solutions. So, no one really had any general methods for finding the solutions until the conjectures of the 1960s, which emerged from the Birch and Swinnerton-Dyer Conjecture. There are two aspects to it; one is somewhat analytic, and one is in terms of what is called the Tate–Shafarevich group. Basically, the Tate–Shafarevich group gives you the obstruction to an algorithm for finding the solutions. And the Birch and Swinnerton-Dyer Conjecture tells you that there is actually an analytic method for analyzing this so-called Tate–Shafarevich group. If you combine all this together, ultimately it should give you an algorithm for finding the solutions.

### Birch and Swinnerton-Dyer, Tate-Shafarevich, Selmer...

**Raussen and Skau:** *You worked on the Birch and Swinnerton-Dyer Conjecture when you were a graduate student together with John Coates?*

**Wiles:** Yes, that is exactly what he proposed working on. We got the first result in certain special families of elliptic curves on this analytic link between the solutions and what is called the  $L$ -function of the elliptic curve.

**Raussen and Skau:** *These were curves admitting complex multiplication?*

**Wiles:** Exactly; these were the elliptic curves with complex multiplication.



**Interviewed in Oslo in May, Wiles told Martin Raussen and Christian Skau that he set out to prove the Modularity Conjecture with no idea from what branch of mathematics the answer would come.**

**Raussen and Skau:** *Was this the first general result concerning the Birch and Swinnerton-Dyer Conjecture?*

**Wiles:** It was the first one that treated a family of cases rather than individual cases. There was a lot of numerical data for individual cases, but this was the first infinite family of cases.

**Raussen and Skau:** *This was over the rational numbers?*

**Wiles:** Yes.

**Raussen and Skau:** *We should mention that the Birch and Swinnerton-Dyer Conjecture is one of the Clay Millennium Prize Problems, which would earn a person who solves it one million dollars.*

**Wiles:** That's right. I think it's appealing, partly because it has its roots in Fermat's work, just like the Fermat problem. It is another "elementary-to-state" problem concerned with equations—in this case of very low degree—which we can't master and which Fermat initiated. I think it is a very appealing problem.

**Raussen and Skau:** *Do you think it is within reach? In other words, do we have the necessary tools for somebody daring enough to attack it and succeed? Or do we have to wait for another three hundred years to see it solved?*

**Wiles:** I don't suppose it will take three hundred years, but I don't think it is the easiest of the Millennium Problems. I think we are still lacking something. Whether the tools are all here now, I am not sure. They may be. There are always these speculations with these really difficult problems; it may be that the tools simply aren't there. I don't believe that anyone in the nineteenth century could have solved Fermat's Last Theorem, certainly not in the way it was eventually solved. There was just too big a gap in mathematical history. You had to wait another hundred years for the right pieces to be in place. You can never be quite sure about these problems, whether they are accessible to your time. That is really what makes them so challenging; if you had the intuition for what can be done now and what can't be done now, you would be a long way towards a solution!

**Raussen and Skau:** You mentioned the Tate-Shafarevich group and in that connection the Selmer group appears. Selmer [1920–2006] was a Norwegian mathematician, and it was Cassels [1922–2015] who was responsible for naming this group the Selmer group. Could you say a few words about the Selmer group and how it is related to the Tate-Shafarevich group, even if it's a little technical?

**Wiles:** It is technical, but I can probably explain the basic idea of what the Selmer group is. What you are trying to do is to find the rational solutions on an elliptic curve. The method is to take the rational points on the elliptic curve—suppose you have got some—and you generate field extensions from these. So when I say generate extensions, I mean that you can take roots of those points on the elliptic curve. Just like taking the  $n$ th root of 5 or the cube root of 2. You can do the same thing on an elliptic curve; you can take the  $n$ th root of a point. These are all points which *added to themselves  $n$  times* give you the point you started with. They generate certain extensions of the number field you started with, in our case the rational number field  $\mathbf{Q}$ .

You can put a lot of restrictions on those extensions. And the Selmer group is basically the smallest set of extensions you can get putting on all the obvious restrictions.

Let me summarize this. You've got the group of points. They generate some extensions; that's too big—you don't want all extensions. You cut that down as much as you can using local criteria, using  $p$ -adic numbers; that's called the Selmer group. And the essential difference between the group generated by the points and the Selmer group is the Tate-Shafarevich group. So the Tate-Shafarevich group gives you the error term, if you like, in trying to get at the points via the Selmer group.

**Raussen and Skau:** Selmer's paper, which Cassels refers to, studied the Diophantine equation  $3x^3 + 4y^3 + 5z^3 = 0$  and similar ones. Selmer showed that it has just a trivial solution in the integers, while modulo  $n$  it has nontrivial solutions for all  $n$ . In particular, this curve has no rational points. Why did Cassels invoke Selmer's name in naming the group?

**Wiles:** Yes, there are quite subtle relationships between these. What happens is you are actually looking at one elliptic curve, which in this case would be  $x^3 + y^3 + 60z^3 = 0$ . That is an elliptic curve, in disguise, if you like, and the Tate-Shafarevich group involves looking at other ones like it, for example,  $3x^3 + 4y^3 + 5z^3 = 0$ , which is a genus one curve but which has no rational points. Its Jacobian is the original elliptic curve  $x^3 + y^3 + 60z^3 = 0$ . One way of describing the Tate-Shafarevich group is in terms of these curves that have genus one but don't have rational points. And by assembling these together you can make the Tate-Shafarevich group, and that is reflected in the Selmer group. It is too intricate to explain in words, but it is another point of view. I gave it in more arithmetic terminology in terms of extensions. The more geometric terminology was in terms of these twisted forms.

## The Modularity Conjecture

**Raussen and Skau:** What you proved in the end was a special case of what is now called the Modularity Conjecture. In order to explain this, one has to start with modular forms and how modular forms can be put in relation with elliptic curves. Could you give us some explanations?

**Wiles:** Yes; we have described an elliptic curve (over the rationals) as an equation  $y^2 = x^3 + ax + b$ , where the  $a$  and  $b$  are assumed to be rational numbers. (There is also a condition that the discriminant should not vanish.) As I said, at the beginning of the nineteenth century you could describe the complex solutions to this equation. You could describe these very nicely in terms of the Weierstrass  $\wp$ -function, in terms of a special elliptic function. But what we want is actually a completely different uniformization of these elliptic curves which captures the fact that the  $a$  and  $b$  are rational numbers. It is a parametrization just for the rational elliptic curves. And because it captures the fact that it is defined over the rationals, it gives you a much better hold on solutions over the rationals than the elliptic functions do. The latter really only sees the complex structure.

And the place it comes from are modular forms or modular curves. To describe modular functions first: we are used to functions which satisfy the relation of being invariant under translation. Every time we write down a Fourier series, we have a function which is invariant under translation. Modular functions are ones which are invariant under the action of a much bigger group, usually a subgroup of  $SL_2(\mathbf{Z})$ . So, you would ask for a function  $f(z)$  in one complex variable, usually on the upper-half-plane, which satisfies  $f(z)$  is the same as  $f((az + b)/(cz + d))$  (or, more generally, is that times a power of  $cz + d$ ).

## Modular functions hold the key to the arithmetic of elliptic curves.

These are called modular functions, and they were extensively studied in the nineteenth century. Surprisingly, they hold the key to the arithmetic of elliptic curves. Perhaps the simplest way to describe it is that because we have an action of  $SL_2(\mathbf{Z})$  on the upper-half-plane  $H$ —by the action  $z$  goes to  $(az + b)/(cz + d)$ —we can look at the quotient  $H$  modulo this action. You can then give the quotient the structure of a curve. In fact, it naturally gets the structure of a curve over the rational numbers. If you take a subgroup of  $SL_2(\mathbf{Z})$ , or more precisely what is called a congruence subgroup, defined by the  $c$  value being divisible by  $N$ , then you call the curve a modular curve of level  $N$ . The Modularity Conjecture asserts that every elliptic curve over the rationals is actually a quotient of one of these modular curves for some integer  $N$ . It gives you a uniformization of elliptic curves by these other entities, these modular curves. On the face of it, it might seem we are losing because this is a high genus curve—it is more complicated. But it actually has a lot more structure because it is a moduli space.

**Raussen and Skau:** And that is a very powerful tool?

**Wiles:** That is a very powerful tool, yes. You have function theory, you have deformation theory, geometric methods, etc. You have a lot of tools to study it.

**Raussen and Skau:** Taniyama, the young Japanese mathematician who first conjectured or suggested these connections; his conjecture was more vague, right?

**Wiles:** His conjecture was more vague. He didn't pin it down to a function invariant under the modular group. I've forgotten exactly what he conjectured; it was invariant under some kind of group, but I forget exactly which group he was predicting. But it was not as precise as the congruence subgroups of the modular group. I think it was originally written in Japanese, so it was not circulated as widely as it might have been. I believe it was part of notes compiled after a conference in Japan.

**Raussen and Skau:** It was an incredibly audacious conjecture at that time, wasn't it?

**Wiles:** Apparently, yes.

**Raussen and Skau:** But then it gradually caught the attention of other mathematicians. You told us already about Gerhard Frey, who came up with a conjecture relating Fermat's Last Theorem with the Modularity Conjecture.

**Wiles:** That's right. Gerhard Frey showed that if you take a solution to the Fermat problem, say  $a^p + b^p = c^p$ , and you create the elliptic curve  $y^2 = x(x - a^p)(x + b^p)$ , then the discriminant of that curve would end up being a perfect  $p$ th power. And if you think about what that means assuming the Modularity Conjecture—you have to assume something a bit stronger as well (the so-called epsilon conjecture of Serre)—then it forces this elliptic curve to have the level  $N$  that I spoke about to be equal to one, and hence the associated congruence subgroup is equal to  $SL_2(\mathbb{Z})$ . But  $H$  modulo  $SL_2(\mathbb{Z})$  is a curve of genus zero. It has no elliptic curve quotient, so it wasn't there after all, and hence there can't be a solution to the Fermat problem.

## The Quest for a Proof

**Raussen and Skau:** That was the point of departure for your own work, with crucial further ingredients due to Serre and Ribet making this connection clear. May we briefly summarize the story that then followed? It has been told by you many times, and it is the focus of a BBC documentary.

You had moved to the United States, first to Harvard, then to Princeton University, becoming a professor there. When you heard of Ribet's result, you devoted all your research time to proving the Modularity Conjecture for semistable elliptic curves over the rationals. This work went on for seven years of really hard work in isolation. At the same time you were working as a professor in Princeton and you were raising small children.

A proof seems to be accomplished in 1993, and the development culminates in a series of three talks at the Isaac Newton Institute in Cambridge back in England, announcing your proof of Fermat's Last Theorem. You are celebrated by your mathematical peers. Even the world press takes an interest in your results, which happens very rarely for mathematical results.

But then, when your result is scrutinized by six referees for a highly prestigious journal, it turns out that there is a subtle gap in one of your arguments and you are sent back to the drawing board. After a while, you send for your former student, Richard Taylor, to come to Princeton to help you in your efforts. It takes a further ten months of hard and frustrating work; we think we do not exaggerate by calling it a heroic effort under enormous pressure. Then, in a sudden flash of insight, you realize that you can combine some of your previous attempts with new results to circumvent the problem that had caused the gap. This turns out to be what you need in order to get the part of the Modularity Conjecture that implies Fermat's Last Theorem. What a relief that must have been! Would you like to give a few comments on this dramatic story?

**Wiles:** With regard to my own work, when I became a professional mathematician working with Coates, I realized I really had to stop working on Fermat because it was time-consuming and I could see that in the last hundred years almost nothing had been

done. And I saw others, even very distinguished mathematicians, had come to grief on it. When Frey came out with this result, I was a bit skeptical that the Serre part of the conjecture was going to be true, but when Ribet proved it, then, okay, this was it!

And it was a long, hard struggle. In some sense, it is irresponsible to work on one problem to the exclusion of everything else, but this is the way I tend to do things. Whereas Fermat is very narrow (I mean, it is just this one equation, whose solution may or may not help with anything else), the setting of the Modular Conjecture was one of the big problems in number theory. It was a great thing to work on anyway, so it was just a tremendous opportunity.

When you are working on something like this, it takes many years to really build up the intuition to see the kinds of things you need and the kinds of things a solution will depend on. It's something like discarding everything you can't use and won't work till your mind is so focused that even making a mistake, you've seen enough that you'll find another way to the end.

Funnily enough, concerning the mistake in the argument that I originally gave, people have worked on that aspect of the argument and quite recently they have actually shown that you can produce arguments very like that. In fact, in every neighboring case, arguments similar to the original method seem to work, but there is this unique case that it doesn't seem to work for, and there is not yet any real explanation for it. So the same kind of argument I was trying to use, using Euler systems and so on, has been made to work in every surrounding case, but not the one I needed for Fermat. It's really extraordinary.

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**Raussen and Skau:** You once likened this quest for the proof of the Modularity Theorem to a journey through a dark, unexplored mansion. Could you elaborate?

**Wiles:** I started off really in the dark. I had no prior insights of how the Modularity Conjecture might work or how you might approach it. One of the troubles with this problem—it's a little like the Riemann Hypothesis but perhaps even more so—was that you didn't even know what branch of mathematics the answer would be coming from.

To start with, there are three ways of formulating the problem: one is geometric, one is arithmetic, and one is analytic. And there were analysts—I would not understand their techniques at all well—who were trying to make headway on this problem.

I think I was a little lucky because my natural instinct was with the arithmetic approach and I went straight for the arithmetic route, but I could have been wrong. The only previously known cases where the Modularity Conjecture was known to hold were the cases of complex multiplication, and that proof is analytic, completely analytic.

Partly out of necessity, I suppose, and partly because that's what I knew, I went straight for an arithmetic approach. I found it very useful to think about it in a way that I had been studying in Iwasawa theory. With John Coates, I had applied Iwasawa theory to elliptic curves. When I went to Harvard, I learned about Barry Mazur's work, where he had been studying the geometry of modular curves using a lot of the modern machinery. There were certain ideas and techniques I could draw on from that. I realized after a while, I could actually use that to make a beginning—to find some kind of entry into the problem.

**Raussen and Skau:** Before you started on the Modularity Conjecture, you published a joint paper with Barry Mazur, proving the main theorem of Iwasawa theory over the rationals. Can you please tell us what Iwasawa theory is all about?

**Wiles:** Iwasawa theory grew out of the work of Kummer on cyclotomic fields and his approach to Fermat's Last Theorem. He studied the arithmetic, and in particular the ideal class groups, of prime cyclotomic fields. Iwasawa's idea was to consider the tower of cyclotomic fields obtained by taking all  $p$ -power roots of unity at once. The main theorem of Iwasawa theory proves a relation between the action of a generator of the Galois group on the  $p$ -primary class groups and the  $p$ -adic  $L$ -functions. It is analogous to the construction used in the study of curves over finite fields where the characteristic polynomial of Frobenius is related to the zeta function.

**Raussen and Skau:** And these tools turned out to be useful when you started to work on the Modularity Conjecture?

**Wiles:** They did; they gave me a starting point. It wasn't obvious at the time, but when I thought about it for a while, I realized that there might be a way to start from there.

## Parallels to Abel's Work

**Raussen and Skau:** We want to read you a quotation: "The ramparts are raised all around but, enclosed in its last redoubt, the problem defends itself desperately. Who will be the fortunate genius who will lead the assault upon it or force it to capitulate?"

**Wiles:** It must have been E. T. Bell, I suppose. Is it?

**Raussen and Skau:** No, it's not. It is actually a quote from the book *Histoire des Mathématiques* by Jean-Étienne Montucla [1725–1799], written in the late eighteenth century. It is really the first book ever written on the history of mathematics. The quotation refers to the solvability or unsolvability of the quintic equation by radicals. As you know, Abel [1802–1829] proved the unsolvability of the general quintic equation when he was twenty-one years old. He worked in complete isolation, mathematically speaking, here in Oslo. Abel was obsessed, or at least extremely attracted, to this problem. He also got a false start. He thought he could prove that one could actually solve the quintic by radicals. Then he discovered his mistake and he finally found the unsolvability proof. Well, this problem was, at that time, almost three hundred years old and very famous. If we move fast-forward two hundred years, the same quotation could be used about the Fermat problem, which was around three hundred fifty years old when you solved it. It is a very parallel story in many ways. Do you have any comments?

**Wiles:** Yes. In some sense, I do feel that Abel, and then Galois [1811–1832], were marking a transition in algebra from these equations, which were solvable in some very simple way, to equations which can't be solved by radicals. But this is an algebraic break that came with the quintic. In some ways, the whole trend in number theory now is the transition from basically abelian and possibly solvable extensions to insoluble extensions. How do we do the arithmetic of insoluble extensions?

I believe the Modularity Conjecture was solved because we had moved on from this original abelian situation to a nonabelian situation, and we were developing tools, modularity and so on which are fundamentally nonabelian tools. (I should say, though, that the proof got away mostly with using the solvable situation, not because it was more natural but because we have not solved the relevant problems in the general nonsolvable case.)

It is the same transition in number theory that he was making in algebra, which provides the tools for solving this equation. So I think it is very parallel.

**Raussen and Skau:** There is an ironic twist with Abel and the Fermat problem. When he was twenty-one years old, Abel came to Copenhagen to visit Professor Degen [1766–1825], who was the leading mathematician in Scandinavia at that time. Abel wrote a letter to his mentor in Oslo, Holmboe [1795–1850], stating three results about the Fermat equation without giving any proofs—one of them is not easy to prove, actually. This, of course, is just a curiosity today.

But in the same letter, he gives vent to his frustration, intimating that he can't understand why he gets an equation of degree  $n^2$  and not  $n$  when dividing the lemniscate arc in  $n$  equal pieces. It was only after returning to Oslo

that he discovered the double periodicity of the lemniscate integral and also of the general elliptic integral of the first kind.

If one thinks about it, what he did on the Fermat equation turned out to be just a curiosity. But what he achieved on elliptic functions, and implicitly on elliptic curves, turned out later to be a relevant tool for solving it. Of course, Abel had no idea that this would have anything to do with arithmetic. So this story tells us that mathematics sometimes develops in mysterious ways.

**Wiles:** It certainly does, yes.

## Work Styles

**Raussen and Skau:** May we ask for some comments about work styles of mathematicians in general and also about your own? Freeman Dyson, a famous physicist and mathematician at IAS in Princeton, said in his Einstein Lecture in 2008: “Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects and they solve problems one at a time.”

Freeman Dyson didn't say that birds were better than frogs or the other way around. He considered himself a frog rather than a bird.

When we are looking at your work, for us it seems rather difficult to decide where to place you in his classification scheme: among the birds (those who create theories) or among the frogs (those who solve problems). What is your own perception?

**Wiles:** Well, I don't feel like either. I'm certainly not a bird—unifying different fields. I think of frogs as jumping a lot. I think I'm very, very focused. I don't know what the animal analogy is, but I think I'm not a frog in the sense of enjoying the nearby landscape. I'm very, very concentrated on the problem I happen to work on and I am very selective. And I find it very hard to even take my mind off it enough to look at any of the flowers around, so I don't think that either of the descriptions quite fit.

**Raussen and Skau:** Based on your own experience, could you describe the interplay between hard, concentrated, and persevering work on the one side and, on the other side, these sudden flashes of insight that seemingly come out of nowhere, often appearing in a more relaxed setting. Your mind must have worked unconsciously on the problem at hand, right?

**Wiles:** I think what you do is that you get to a situation where you know a theory so well, and maybe even more than one theory, so that you have seen every angle and tried a lot of different routes.

There is this tremendous amount of work in the preparatory stage where you have to understand all the details and maybe some examples—that is your essential launch pad. When you have developed all this, you let the mind relax and then at some point—maybe when you go away and do something else for a little bit—you come back and suddenly it is all clear. Why did you not think

of that? This is something the mind does for you. It is the flash of insight.

I remember (this is a trivial example in a non-mathematical setting) someone once showed me some script—it was some Gothic script—and I couldn't make head nor tail of it. I was trying to understand a few letters, and I gave up. Then I came back half an hour later and I could read the whole thing. The mind somehow does this for you and we don't quite know how, but we do know what we have to do to set up the conditions where it will happen.

**Raussen and Skau:** This is reminiscent of a story about Abel. While in Berlin, he shared an apartment with some Norwegian friends who were not mathematicians. One of his friends said that Abel typically woke up during the night, lit a candle, and wrote down ideas that he woke up with. Apparently his mind was working while asleep.

**Wiles:** Yes, I do that, except I don't feel the need to write them down when I wake up with it because I know I will not forget it. But if I have an idea when I am about to go to sleep, I am terrified that I will not wake up with that idea, so then I have to write it down.

**Raussen and Skau:** Are you thinking in terms of formulas or in terms of geometric pictures or what?

**Wiles:** It is not really geometric. I think it is patterns and I think it is just parallels between situations I have seen elsewhere and the one I am facing now. In a perfect world, what is it all pointing to? What are the ingredients that ought to go into this proof? What am I not using that I still have in my pocket? Sometimes it is just desperation. I assemble every piece of evidence I have and that's all I've got. I have got to work with that and there is nothing else.

I often feel that doing mathematics is like being a squirrel and there are some nuts at the top of a very tall tree. But there are several trees and you don't know which one. What you do is that you run up one and you think, no, it does not look good on this one, and you go down and up another one, and you spend your whole life just going up and down these trees. But you've only got up to thirty feet. Now, if someone told you the rest of the trees—it's not in them, you have only one tree left—then you would just keep going until you found it. In some sense, it is ruling out the wrong things—that is really crucial. And if you just believe in your intuition and your intuition is correct and you stick with your one tree, then you will find it.

## Problems in Mathematics

**Raussen and Skau:** Felix Klein [1849–1925] once said: “Mathematics develops as old results are being understood and illuminated by new methods and insights. Proportionally with a better and deeper understanding new problems

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naturally arise.” And David Hilbert [1862–1943] stressed that “problems are the lifeblood of mathematics.” Do you agree?

**Wiles:** I certainly agree with Hilbert, yes. Good problems are the lifeblood of mathematics. I think you can see this clearly in number theory in the second half of the last century. For me personally, there is obviously the Modularity Conjecture but also the whole Langlands program and the Birch and Swinnerton-Dyer Conjecture. These problems give you a very clear focus on what we should try to achieve. We also have the Weil Conjectures on curves and varieties over finite fields and the Mordell Conjecture and so on.

These problems somehow concentrate the mind and also simplify our goals in mathematics. Otherwise, we can get very, very spread out and not sure what’s of value and what’s not of value.

**Raussen and Skau:** Do we have as good problems today as when Hilbert formulated his twenty-three problems in 1900?

**Wiles:** I think so, yes.

**Raussen and Skau:** Which one do you think is the most important problem today? And how does the Langlands program fit in?

**Wiles:** Well, I think the Langlands program is the broadest spectrum of problems related to my field. I think that the Riemann Hypothesis is the single greatest problem from the areas I understand. It is sometimes hard to say exactly why that is, but I do believe that solving it would actually help solve some of these other problems. And then, of course, I have a very personal attachment to the Birch and Swinnerton-Dyer Conjecture.

**Raussen and Skau:** Intuition can lead us astray sometimes. For example, Hilbert thought that the Riemann Hypothesis would be solved in his lifetime. There was another problem on his list, the seventh, that he never thought would be solved in his lifetime, but which was solved by Gelfond [1906–1968] in 1934. So our intuition can be wrong.

**Wiles:** That is right. I’m not surprised that Hilbert felt that way. The Riemann Hypothesis has such a clear statement, and we have the analogue in the function field setting. We understand why it is true there and we feel we ought to be able to translate it. Of course, many people have tried and failed. But I would still expect it to be solved before the Birch and Swinnerton-Dyer Conjecture.

### Investing in Mathematics

**Raussen and Skau:** Let’s hope we’ll find out in our lifetimes!

Classical mathematics has, roughly speaking, two sources: one of them coming from the physical sciences and the other one from—let’s for simplicity call it number theoretical speculations, with number theory not associated to applications.

That has changed. For example, your own field of elliptic curves has been applied to cryptography and security. People are making money with elliptic curves nowadays! On the other hand, many sciences apart from physics really take advantage and profit from mathematical thinking and mathematical results. Progress in industry

nowadays often depends on mathematical modelling and optimization methods. Science and industry propose challenges to the mathematical world.

In a sense, mathematics has become more applied than it ever was. One may ask whether this is a problem for pure mathematics. It appears that pure mathematics sometimes is put to the sidelines, at least from the point of view of the funding agencies. Do you perceive this as a serious problem?

**Wiles:** Well, I think in comparison with the past, one feels that mathematicians two, three hundred years ago were able to handle a much broader spectrum of mathematics, and a lot more of it touched applied mathematics than a typical pure mathematician would do nowadays. On the other hand, that might be because we only remember the very best and most versatile mathematicians from the past.

I think it is always going to be a problem if funding agencies are short-sighted. If they want to see a result in three years, then it is not going to work. It is hard to imagine a pure development and then the application all happening within three to five years. It is probably not going to happen.

On the other hand, I don’t believe you can have a happily functioning applied maths world without the pure maths to back it up, providing the future and keeping them on the straight and narrow. So it would be very foolish not to invest in pure mathematics. It is a bit like only investing in energy resources that you can see now. You have to invest in the future; you have to invest in fusion power or solar power or these other things. You don’t just use up what is there and then start worrying about it when it is gone. It is the same with mathematics; you can’t just use up the pure mathematics we have now and then start worrying about it when you need a pure result to generate your applications.

### Mathematical Awards

**Raussen and Skau:** You have already won a lot of prizes as a result of your achievements, culminating in proving Fermat’s Last Theorem. You have won the Rolf Schock Prize, given by the Swedish Academy; the Ostrowski Prize, which was given to you in Denmark; the Fermat Prize in France; the Wolf Prize in Israel; the Shaw Prize in Hong Kong (the prize that has been named the Nobel Prize of the East), and the list goes on, culminating with the Abel Prize tomorrow. May we ask you whether you enjoy these awards and the accompanying celebrations?

**Wiles:** I certainly love them, I have to say. I think they are a celebration of mathematics. I think with something like Fermat, it is something people are happy to see in their lifetime. I would obviously be very happy to see the Riemann Hypothesis solved. It is just exciting to see how it finally gets resolved and just to understand the end of the story—because a lot of these stories we won’t live to see the end of. Each time we do see the end of such a story, it is something we will naturally celebrate. For me, I learned about the Fermat problem from this book of E. T. Bell and about the Wolfskehl Prize attached to it. The Wolfskehl Prize was still there—only just, I may

say; I only had a few years left before the deadline for it expired.

**Raussen and Skau:** *This gives us the lead to talk a little about that prize. The Wolfskehl Prize was founded in 1906 by Paul Wolfskehl [1856–1906], who was a German physician with an interest in mathematics. He bequeathed one hundred thousand Reichmarks (equivalent to more than one million dollars in today's money) to the first person to prove Fermat's Last Theorem. The prize was, according to the testament, valid until 13 September 2007 and you received it in 1997. By then, due in part to the hyperinflation Germany suffered after World War I, the prize money had dwindled a lot.*

**Wiles:** For me, the amount of money was unimportant. It was the sentimental feeling attached to the Wolfskehl Prize that was important for me.

### Graduate Students

**Raussen and Skau:** *You have had altogether twenty-one PhD students and you have attracted very gifted students. Some of them are really outstanding. One of them, Manjul Bhargava, won the Fields Medal in 2014. It must be a pleasure to be advisor to such students.*

**Wiles:** Yes, I don't want to take too much credit for it. In the case of Manjul, I suggested a problem to him, but after that I had nothing much more to do. He was coming up with these absolutely marvelous discoveries. In some sense, you get more credit if you have very gifted students, but the truth is that very gifted students don't really require that much help.

**Raussen and Skau:** *What is the typical way for you of interacting with graduate students?*

**Wiles:** Well, I think the hardest thing to learn as a graduate student is that afterwards you need to carry on with the rest of your professional life; it's hard to pick problems. And if you just assign a problem and they do it, in some sense that hasn't given them terribly much. Okay, they solved that problem, but the hard thing is then to have to go off and find other problems! So I prefer it if we come to a decision on the problem together.

I give them some initial idea and which area of mathematics to look at, having not quite focused on the problem. Then, as they start working and become experts, they can see a better way of pinning down what the right question is. And then they are part of the process of choosing the problem. I think that is a much better investment for their future. It doesn't always work out that way, and sometimes the problem you give them turns out to be the right thing. But usually it is not that way and usually it's a process to find the right problem.

### Hobbies and Interests

**Raussen and Skau:** *We always end the Abel interviews by asking the laureate what they enjoy doing when they are not working with mathematics. What are your hobbies and interests outside mathematics?*

**Wiles:** Well, it varies at different times. When I was doing Fermat and being a father with young children, that combination was all-consuming.

I like to read and I like various kinds of literature: novels, some biographies—it is fairly balanced. I don't have any other focused obsessions. When I was in school, I played on chess teams and bridge teams, but when I started to do serious mathematics, I completely lost interest in those.

**Raussen and Skau:** *What about music; are you fond of music?*

**Wiles:** I go and listen to concerts, but I am not myself actively playing anything. I enjoy listening to music—classical, preferably.

**Raussen and Skau:** *Are you interested in other sciences apart from mathematics?*

**Wiles:** I would say somewhat. These are things I do to relax, so I don't like them to be too close to mathematics. If it is something like animal behavior or astrophysics or something from a qualitative point of view—I certainly enjoy learning about those—likewise about what machines are capable of, and many other kinds of popular science, but I'm not going to spend my time learning the details of string theory. I'm too focused to be willing to do that. Not that I would not be interested, but this is my choice.

**Raussen and Skau:** *We would like to thank you very much for this wonderful interview, first of all on behalf of the two of us but also on behalf of the Norwegian, the Danish, and the European Mathematical Societies. Thank you so much!*

**Wiles:** Thank you very much!

### Photo Credits

Photo of Sir Andrew J. Wiles with Crown Prince Haakon is courtesy of Audun Braastad.

Photo of Sir Andrew J. Wiles with Martin Raussen, and Christian Skau is courtesy of Erik F. Baardsen, DNVA.

### Sidebar 1. Abel Prize Winners

2016: Andrew Wiles  
2015: John Forbes Nash Jr. and Louis Nirenberg  
2014: Yakov Sinai  
2013: Pierre Deligne  
2012: Endre Szemerédi  
2011: John Milnor  
2010: John Tate  
2009: Mikhail Leonidovich Gromov  
2008: John G. Thompson and Jacques Tits  
2007: S. R. Srinivasa Varadhan  
2006: Lennart Carleson  
2005: Peter Lax  
2004: Michael Atiyah and Isadore Singer  
2003: Jean-Pierre Serre

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## Sidebar 2. *Notices* articles on Wiles

**July/August 1993:** *Wiles Proves Taniyama's Conjecture; Fermat's Last Theorem Follows*, by Kenneth A. Ribet, [math.berkeley.edu/~ribet/Articles/notices.pdf](http://math.berkeley.edu/~ribet/Articles/notices.pdf)

**October 1993:** *Fermat Fest Draws a Crowd*, by Allyn Jackson

**October 1994:** *Another Step Toward Fermat*, by Allyn Jackson, [www.ams.org/notices/199501/rubin.pdf](http://www.ams.org/notices/199501/rubin.pdf)

**July 1995:** *The Proof of Fermat's Last Theorem by R. Taylor and A. Wiles*, by Gerd Faltings, [www.ams.org/notices/199507/faltings.pdf](http://www.ams.org/notices/199507/faltings.pdf)

**July 1996:** *Wiles Receives NAS Award in Mathematics*, by John Coates, [www.ams.org/notices/199607/comm-wiles.pdf](http://www.ams.org/notices/199607/comm-wiles.pdf)

**January 1997:** *Review of BBC's Horizon Program, "Fermat's Last Theorem,"* reviewed by Andrew Granville, [www.ams.org/notices/199701/comm-granville.pdf](http://www.ams.org/notices/199701/comm-granville.pdf)

**March 1997:** *Announcement: 1997 Cole Prize*, [www.ams.org/notices/199703/comm-cole.pdf](http://www.ams.org/notices/199703/comm-cole.pdf)

**November 1997:** *Paul Wolfskehl and the Wolfskehl Prize*, by Klaus Barner, [www.ams.org/notices/199710/barner.pdf](http://www.ams.org/notices/199710/barner.pdf)

**November 1997:** *Book Review: Fermat's Enigma by Simon Singh*, reviewed by Allyn Jackson, [www.ams.org/notices/199710/comm-fermat.pdf](http://www.ams.org/notices/199710/comm-fermat.pdf)

**December 1999:** *Research News: A Proof of the Full Shimura-Taniyama-Weil Conjecture Is Announced*, by Henri Darmon, [www.ams.org/notices/199911/comm-darmon.pdf](http://www.ams.org/notices/199911/comm-darmon.pdf)

**December 2001:** *Theater Review: Fermat's Last Tango*, reviewed by Robert Osserman, [www.ams.org/notices/200111/rev-osserman.pdf](http://www.ams.org/notices/200111/rev-osserman.pdf)

**September 2005:** *Wiles Receives 2005 Shaw Prize*, by Allyn Jackson, [www.ams.org/notices/200508/comm-shaw.pdf](http://www.ams.org/notices/200508/comm-shaw.pdf)

**June/July 2016:** *Sir Andrew J. Wiles Awarded Abel Prize*, [www.ams.org/publications/journals/notices/201606/rnoti-p608.pdf](http://www.ams.org/publications/journals/notices/201606/rnoti-p608.pdf)

# Andrew Wiles's Marvelous Proof

Henri Darmon

Fermat famously claimed to have discovered “a truly marvelous proof” of his Last Theorem, which the margin of his copy of Diophantus’s *Arithmetica* was too narrow to contain. While this proof (if it ever existed) is lost to posterity, Andrew Wiles’s marvelous proof has been public for over two decades and has now earned him the Abel Prize. According to the prize citation, Wiles merits this recognition “for his stunning proof of Fermat’s Last Theorem by way of the modularity conjecture for semistable elliptic curves, opening a new era in number theory.”

Few can remain insensitive to the allure of Fermat’s Last Theorem, a riddle with roots in the mathematics of ancient Greece, simple enough to be understood and appreciated by a novice (like the ten-year-old Andrew Wiles browsing the shelves of his local public library), yet eluding the concerted efforts of the most brilliant minds for well over three centuries, becoming over its long history the object of lucrative awards

*It is also a centerpiece of the “Langlands program,” the imposing, ambitious edifice of results and conjectures which has come to dominate the number theorist’s view of the world.*

like the Wolfskehl Prize and, more importantly, motivating a cascade of fundamental discoveries: Fermat’s method of infinite descent, Kummer’s theory of ideals, the ABC conjecture, Frey’s approach to ternary diophantine equations, Serre’s conjecture on mod  $p$  Galois representations,....

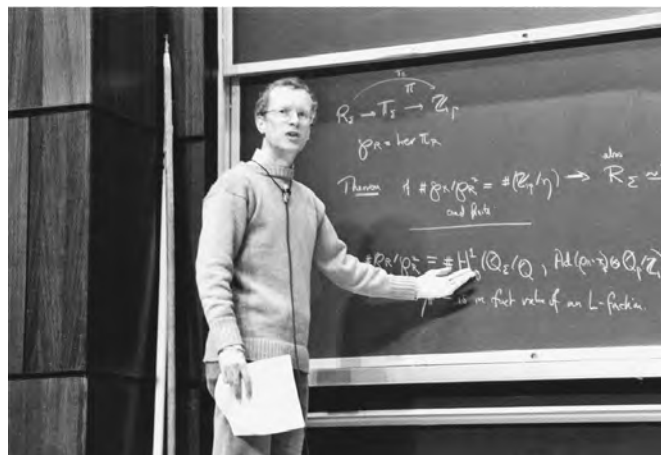
Even without its seemingly serendipitous connection to Fermat’s Last Theorem, Wiles’s modularity theorem is a fundamental statement about elliptic curves (as evidenced, for instance, by the key role it plays in the proof

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Wiles giving his first lecture in Princeton about his approach to proving the Modularity Conjecture in early 1994.

of Theorem 2 of Karl Rubin’s contribution in this volume). It is also a centerpiece of the “Langlands program,” the imposing, ambitious edifice of results and conjectures which has come to dominate the number theorist’s view of the world. This program has been described as a “grand unified theory” of mathematics. Taking a Norwegian perspective, it connects the objects that occur in the works of Niels Hendrik Abel, such as elliptic curves and their associated Abelian integrals and Galois representations, with (frequently infinite-dimensional) linear representations of the continuous transformation groups whose study was pioneered by Sophus Lie. This report focuses on the role of Wiles’s theorem and its “marvelous proof” in the Langlands program in order to justify the closing phrase in the prize citation: how Wiles’s proof has opened “a new era in number theory” and continues to have a profound and lasting impact on mathematics.

Our “beginner’s tour” of the Langlands program will only give a partial and undoubtedly biased glimpse of the full panorama, reflecting the author’s shortcomings as well as the inherent limitations of a treatment aimed at a general readership. We will motivate the Langlands program by starting with a discussion of *diophantine equations*: for the purposes of this exposition, they are equations of the form

$$(1) \quad X: P(x_1, \dots, x_{n+1}) = 0,$$

where  $P$  is a polynomial in the variables  $x_1, \dots, x_{n+1}$  with integer (or sometimes rational) coefficients. One can examine the set, denoted  $X(F)$ , of solutions of (1) with coordinates in any ring  $F$ . As we shall see, the subject draws much of its fascination from the deep and subtle ways in which the behaviours of different solution sets

can resonate with each other, even if the sets  $\mathcal{X}(\mathbb{Z})$  or  $\mathcal{X}(\mathbb{Q})$  of integer and rational solutions are foremost in our minds. Examples of diophantine equations include Fermat's equation  $x^d + y^d = z^d$ , the Brahmagupta-Pell equation  $x^2 - Dy^2 = 1$  with  $D > 0$ , as well as elliptic curve equations of the form  $y^2 = x^3 + ax + b$ , in which  $a$  and  $b$  are rational parameters, the solutions  $(x, y)$  with rational coordinates being the object of interest in the latter case.

It can be instructive to approach a diophantine equation by first studying its solutions over *simpler* rings, such as the complete fields of real or complex numbers. The set

$$(2) \quad \mathbb{Z}/n\mathbb{Z} := \{0, 1, \dots, n-1\}$$

of remainders after division by an integer  $n \geq 2$ , equipped with its natural laws of addition, subtraction, and multiplication, is another particularly simple collection of numbers of *finite cardinality*. If  $n = p$  is *prime*, this ring is even a *field*: it comes equipped with an operation of division by nonzero elements, just like the more familiar collections of rational, real, or complex numbers. The fact that  $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$  is a field is an algebraic characterisation of the primes that forms the basis for most known efficient primality tests and factorisation algorithms. One of the great contributions of Evariste Galois, in addition to the eponymous theory which plays such a crucial role in Wiles's work, is his discovery of a field of cardinality  $p^r$  for any prime power  $p^r$ . This field, denoted  $\mathbb{F}_{p^r}$  and sometimes referred to as the Galois field with  $p^r$  elements, is even *unique* up to isomorphism.

For a diophantine equation  $X$  as in (1), the most basic invariant of the set

$$(3) \quad \mathcal{X}(\mathbb{F}_{p^r}) := \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{F}_{p^r}^{n+1} \text{ such that } P(x_1, \dots, x_{n+1}) = 0 \right\}$$

of solutions over  $\mathbb{F}_{p^r}$  is of course its *cardinality*

$$(4) \quad N_{p^r} := \#\mathcal{X}(\mathbb{F}_{p^r}).$$

What patterns (if any) are satisfied by the sequence

$$(5) \quad N_p, N_{p^2}, N_{p^3}, \dots, N_{p^r}, \dots?$$

This sequence can be packaged into a generating series like

$$(6) \quad \sum_{r=1}^{\infty} N_{p^r} T^r \quad \text{or} \quad \sum_{r=1}^{\infty} \frac{N_{p^r}}{r} T^r.$$

For technical reasons it is best to consider the exponential of the latter:

$$(7) \quad \zeta_p(X; T) := \exp \left( \sum_{r=1}^{\infty} \frac{N_{p^r}}{r} T^r \right).$$

This power series in  $T$  is known as the *zeta function* of  $X$  over  $\mathbb{F}_p$ . It has integer coefficients and enjoys the following remarkable properties:

(1) It is a *rational function* in  $T$ :

$$(8) \quad \zeta_p(X; T) = \frac{Q(T)}{R(T)},$$

where  $Q(T)$  and  $R(T)$  are polynomials in  $T$  whose degrees (for all but finitely many  $p$ ) are *independent of  $p$*  and determined by the shape—the complex topology—of the set  $\mathcal{X}(\mathbb{C})$  of complex solutions;

(2) the reciprocal roots of  $Q(T)$  and  $R(T)$  are complex numbers of absolute value  $p^{i/2}$  with  $i$  an integer in the interval  $0 \leq i \leq 2n$ .

The first statement—the rationality of the zeta function, which was proved by Bernard Dwork in the early 1960s—is a key part of the Weil conjectures, whose formulation in the 1940s unleashed a revolution in arithmetic geometry, driving the development of étale cohomology by Grothendieck and his school. The second statement, which asserts that the complex function  $\zeta_p(X; p^{-s})$  has its roots on the real lines  $\Re(s) = i/2$  with  $i$  as above, is known as the Riemann hypothesis for the zeta functions of diophantine equations over finite fields. It was proved by Pierre Deligne in 1974 and is one of the major achievements for which he was awarded the Abel Prize in 2013.

That the asymptotic behaviour of  $N_p$  can lead to deep insights into the behaviour of the associated diophantine equations is one of the key ideas behind the Birch and Swinnerton-Dyer conjecture. Understanding the patterns satisfied by the function

$$(9) \quad p \mapsto N_p \quad \text{or} \quad p \mapsto \zeta_p(X; T)$$

as the prime  $p$  varies will also serve as our motivating question for the Langlands program.

It turns out to be fruitful to package the zeta functions over all the finite fields into a single function of a complex variable  $s$  by taking the infinite product

$$(10) \quad \zeta(X; s) = \prod_p \zeta_p(X; p^{-s})$$

as  $p$  ranges over all the prime numbers. In the case of the simplest nontrivial diophantine equation  $x = 0$ , whose solution set over  $\mathbb{F}_{p^r}$  consists of a single point, one has  $N_{p^r} = 1$  for all  $p$ , and therefore

$$(11) \quad \zeta_p(x = 0; T) = \exp \left( \sum_{r=1}^{\infty} \frac{T^r}{r} \right) = (1 - T)^{-1}.$$

It follows that

$$(12) \quad \zeta(x = 0; s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}$$

$$(13) \quad = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots \right)$$

$$(14) \quad = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

The zeta function of even the humblest diophantine equation is thus a central object of mathematics: the celebrated Riemann zeta function, which is tied to some of the deepest questions concerning the distribution of prime numbers. In his great memoir of 1860, Riemann proved that, even though (13) and (14) only converge absolutely on the right half-plane  $\Re(s) > 1$ , the function  $\zeta(s)$  extends to a meromorphic function of  $s \in \mathbb{C}$  (with a single pole at  $s = 1$ ) and possesses an elegant functional equation relating its values at  $s$  and  $1 - s$ . The zeta functions of linear equations  $X$  in  $n + 1$  variables are just shifts of the Riemann zeta function, since  $N_{p^r}$  is equal to  $p^{nr}$ , and therefore  $\zeta(X; s) = \zeta(s - n)$ .

Moving on to equations of degree two, the general quadratic equation in one variable is of the form  $ax^2 + bx + c = 0$ , and its behaviour is governed by its *discriminant*

$$(15) \quad \Delta := b^2 - 4ac.$$

This purely algebraic fact remains true over the finite fields, and for primes  $p \nmid 2a\Delta$  one has

$$(16) \quad N_p = \begin{cases} 0 & \text{if } \Delta \text{ is a nonsquare modulo } p, \\ 2 & \text{if } \Delta \text{ is a square modulo } p. \end{cases}$$

A priori, the criterion for whether  $N_p = 2$  or 0—whether the integer  $\Delta$  is or is not a quadratic residue modulo  $p$ —seems like a subtle condition on the prime  $p$ . To get a better feeling for this condition, consider the example of the equation  $x^2 - x - 1$ , for which  $\Delta = 5$ . Calculating whether 5 is a square or not modulo  $p$  for the first few primes  $p \leq 101$  leads to the following list:

$$(17) \quad N_p = \begin{cases} 2 & \text{for } p = 11, 19, 29, 31, 41, 59, 61, 71, 79, 89, 101, \dots \\ 0 & \text{for } p = 2, 3, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, \dots \end{cases}$$

A clear pattern emerges from this experiment: whether  $N_p = 0$  or 2 seems to depend only on the rightmost digit of  $p$ , i.e., on what the remainder of  $p$  is modulo 10. One is led to surmise that

$$(18) \quad N_p = \begin{cases} 2 & \text{if } p \equiv 1, 4 \pmod{5}, \\ 0 & \text{if } p \equiv 2, 3 \pmod{5}, \end{cases}$$

a formula that represents a dramatic improvement over (16), allowing a much more efficient calculation of  $N_p$  for example. The guess in (18) is in fact a consequence of Gauss's celebrated law of quadratic reciprocity:

**Theorem (Quadratic reciprocity).** *For any equation  $ax^2 + bx + c$ , with  $\Delta := b^2 - 4ac$ , the value of the function  $p \mapsto N_p$  (for  $p \nmid a\Delta$ ) depends only on the residue class of  $p$  modulo  $4\Delta$  and hence is periodic with period length dividing  $4|\Delta|$ .*

The repeating pattern satisfied by the  $N_p$ 's as  $p$  varies greatly facilitates the manipulation of the zeta functions of quadratic equations. For example, the zeta function of  $X : x^2 - x - 1 = 0$  is equal to

$$(19) \quad \zeta(X; s) = \zeta(s) \times \left\{ \left(1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s}\right) + \left(\frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s}\right) + \left(1 - \frac{1}{11^s} - \frac{1}{12^s} - \frac{1}{13^s} + \frac{1}{14^s}\right) + \dots \right\}.$$

The series that occurs on the right-hand side is a prototypical example of a *Dirichlet L-series*. These  $L$ -series, which are the key actors in the proof of Dirichlet's theorem on the infinitude of primes in arithmetic progressions, enjoy many of the same analytic properties as the Riemann zeta function: an analytic continuation to the entire complex plane and a functional equation relating their values at  $s$  and  $1 - s$ . They are also expected to satisfy a Riemann hypothesis which generalises Riemann's original statement and is just as deep and elusive.

It is a (not completely trivial) fact that the zeta function of the general quadratic equation in  $n$  variables

$$(20) \quad \sum_{i,j=1}^n a_{ij}x_i x_j + \sum_{i=1}^n b_i x_i + c = 0$$

involves the same basic constituents, Dirichlet series, as in the one-variable case. This means that quadratic diophantine equations in any number of variables are well understood, at least as far as their zeta functions are concerned.

The plot thickens when equations of higher degree are considered. Consider for instance the cubic equation  $x^3 - x - 1$  of discriminant  $\Delta = -23$ . For all  $p \neq 23$ , this cubic equation has no multiple roots over  $\mathbb{F}_p$ , and therefore  $N_p = 0, 1$ , or 3. A simple expression for  $N_p$  in this case is given by the following theorem of Hecke:

**Theorem (Hecke).** *The following hold for all primes  $p \neq 23$ :*

(1) *If  $p$  is not a square modulo 23, then  $N_p = 1$ .*

(2) *If  $p$  is a square modulo 23, then*

$$(21) \quad N_p = \begin{cases} 0 & \text{if } p = 2a^2 + ab + 3b^2, \\ 3 & \text{if } p = a^2 + ab + 6b^2, \end{cases}$$

for some  $a, b \in \mathbb{Z}$ .

Hecke's theorem implies that

$$(22) \quad \zeta(x^3 - x - 1; s) = \zeta(s) \times \sum_{n=1}^{\infty} a_n n^{-s},$$

where the generating series

$$(23) \quad F(q) := \sum a_n q^n = q - q^2 - q^3 + q^6 + q^8 - q^{13} - q^{16} + q^{23} + \dots$$

is given by the explicit formula

$$(24) \quad F(q) = \frac{1}{2} \left( \sum_{a,b \in \mathbb{Z}} q^{a^2+ab+6b^2} - q^{2a^2+ab+3b^2} \right).$$

The function  $f(z) = F(e^{2\pi iz})$  that arises by setting  $q = e^{2\pi iz}$  in (24) is a prototypical example of a *modular form*: namely, it satisfies the transformation rule

$$(25) \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)f(z), \quad \left\{ \begin{array}{l} a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \\ 23|c, \quad \left(\frac{a}{23}\right) = 1. \end{array} \right.$$

under so-called *modular substitutions* of the form  $z \mapsto \frac{az+b}{cz+d}$ . This property follows from the *Poisson summation formula* applied to the expression in (24). Thanks to (25), the zeta function of  $X$  can be manipulated with the same ease as the zeta functions of Riemann and Dirichlet. Indeed, Hecke showed that the  $L$ -series  $\sum_{n=1}^{\infty} a_n n^{-s}$  attached to a modular form  $\sum_{n=1}^{\infty} a_n e^{2\pi inz}$  possesses very similar analytic properties, notably an analytic continuation and a Riemann-style functional equation.

The generating series  $F(q)$  can also be expressed as an infinite product:

$$(26) \quad \frac{1}{2} \left( \sum_{a,b \in \mathbb{Z}} q^{a^2+ab+6b^2} - q^{2a^2+ab+3b^2} \right) = q \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{23n}).$$

The first few terms of this power series identity can readily be verified numerically, but its proof is highly nonobvious and indirect. It exploits the circumstance that the space of holomorphic functions of  $z$  satisfying the transformation rules (25) together with suitable growth properties is a one-dimensional complex vector space which also contains

the infinite product above. Indeed, the latter is equal to  $\eta(q)\eta(q^{23})$ , where

$$(27) \quad \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind eta function whose logarithmic derivative (after viewing  $\eta$  as a function of  $z$  through the change of variables  $q = e^{2\pi iz}$ ) is given by

$$(28) \quad \frac{\eta'(z)}{\eta(z)} = -\pi i \left( \frac{-1}{12} + 2 \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) e^{2\pi inz} \right)$$

$$(29) \quad = \frac{i}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz + n)^2},$$

where the term attached to  $(m, n) = (0, 0)$  is excluded from the last sum. The Dedekind  $\eta$ -function is also connected to the generating series for the partition function  $p(n)$  describing the number of ways in which  $n$  can be expressed as a sum of positive integers via the identity

$$(30) \quad \eta^{-1}(q) = q^{-1/24} \sum_{n=0}^{\infty} p(n)q^n,$$

which plays a starring role alongside Jeremy Irons and Dev Patel in a recent film about the life of Srinivasa Ramanujan.

Commenting on the “unreasonable effectiveness and ubiquity of modular forms,” Martin Eichler once wrote, “There are five elementary arithmetical operations: addition, subtraction, multiplication, division,...and modular forms.” Equations (26), (29), and (30) are just a few of the many won-

drous identities which abound, like exotic strains of fragrant wild orchids, in what Roger Godement has called the “garden of modular delights.”

The example above and many others of a similar type are described in Jean-Pierre Serre’s delightful monograph [Se], touching on themes that were also covered in Serre’s lecture at the inaugural Abel Prize ceremony in 2003.

Hecke was able to establish that all cubic polynomials in one variable are *modular*; i.e., the coefficients of their zeta functions obey patterns just like those of (24) and (25). Wiles’s achievement was to extend this result to a large class of cubic diophantine equations in two variables over the rational numbers: the *elliptic curve* equations which can be brought to the form

$$(31) \quad y^2 = x^3 + ax + b$$

after a suitable change of variables and which are non-singular, a condition equivalent to the assertion that the *discriminant*  $\Delta := -16(4a^3 + 27b^2)$  is nonzero.

To illustrate Wiles’s theorem with a concrete example, consider the equation

$$(32) \quad E : y^2 = x^3 - x,$$

of discriminant  $\Delta = 64$ . After setting

$$(33) \quad \zeta(E; s) = \zeta(s-1) \times (a_1 + a_2 2^{-s} + a_3 3^{-s} + a_4 4^{-s} + \dots)^{-1},$$

the associated generating series satisfies the following identities reminiscent of (24) and (26):

$$(34) \quad F(q) = \sum a_n q^n = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} + \dots$$

$$(35) \quad = \sum_{a,b} a \cdot q^{(a^2+b^2)}$$

$$(36) \quad = q \prod_{n=1}^{\infty} (1 - q^{4n})^2 (1 - q^{8n})^2,$$

where the sum in (35) runs over the  $(a, b) \in \mathbb{Z}^2$  for which the Gaussian integer  $a + bi$  is congruent to 1 modulo  $(1+i)^3$ . (This identity follows from Deuring’s study of zeta functions of elliptic curves *with complex multiplication* and may even have been known earlier.) Once again, the holomorphic function  $f(z) := F(e^{2\pi iz})$  is a modular form satisfying the slightly different transformation rule

$$(37) \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z), \quad \begin{cases} a, b, c, d \in \mathbb{Z}, \\ ad - bc = 1, \\ 32|c. \end{cases}$$

Note the exponent 2 that appears in this formula. Because of it, the function  $f(z)$  is said to be a *modular form of weight 2* and level 32. The modular forms of (25) attached to cubic equations in one variable are of weight 1, but otherwise the parallel of (35) and (36) with (24) and (26) is striking. The original conjecture of Shimura-Taniyama, and Weil asserts the same pattern for all elliptic curves:

**Conjecture (Shimura-Taniyama-Weil).** *Let  $E$  be any elliptic curve. Then*

$$(38) \quad \zeta(E; s) = \zeta(s-1) \times \left( \sum_{n=1}^{\infty} a_n n^{-s} \right)^{-1},$$

where  $f_E(z) := \sum a_n e^{2\pi inz}$  is a modular form of weight 2.

The conjecture was actually more precise and predicted that the level of  $f_E$ —i.e., the integer that appears in the transformation property for  $f_E$  as the integers 23 and 32 in (25) and (37) respectively—is equal to the *arithmetic conductor* of  $E$ . This conductor, which is divisible only by primes for which the equation defining  $E$  becomes singular modulo  $p$ , is a measure of the arithmetic complexity of  $E$  and can be calculated explicitly from an equation for  $E$  by an algorithm of Tate. An elliptic curve is said to be *semistable* if its arithmetic conductor is squarefree. This class of elliptic curves includes those of the form

$$(39) \quad y^2 = x(x-a)(x-b)$$

with  $\gcd(a, b) = 1$  and  $16|b$ . The most famous elliptic curves in this class are those that ultimately do not exist:

“There are five elementary arithmetical operations: addition, subtraction, multiplication, division,...and modular forms.”



**Andrew Wiles, Henri Darmon, and Mirela Çiperiani in June 2016 at Harvard University during a conference in honor of Karl Rubin's sixtieth birthday.**

the “Frey curves”  $y^2 = x(x - a^p)(x + b^p)$  arising from putative solutions to Fermat’s equation  $a^p + b^p = c^p$ , whose nonexistence had previously been established in a landmark article of Kenneth Ribet<sup>1</sup> under the assumption of their modularity. It is the proof of the Shimura–Taniyama–Weil conjecture for semistable elliptic curves that earned Andrew Wiles the Abel Prize:

**Theorem (Wiles).** *Let  $E$  be a semistable elliptic curve. Then  $E$  satisfies the Shimura–Taniyama–Weil conjecture.*

The semistability assumption in Wiles’s theorem was later removed by Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor around 1999. (See, for instance, the account [Da] that appeared in the *Notices* at the time.)

As a prelude to describing some of the important ideas in its proof, one must first try to explain why Wiles’s theorem occupies such a central position in mathematics. The Langlands program places it in a larger context by offering a vast generalisation of what it means for a diophantine equation to be “associated to a modular form.” The key is to view modular forms attached to cubic equations or to elliptic curves as in (24) or (34) as vectors in certain irreducible infinite-dimensional representations of the locally compact topological group

$$(40) \quad \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) = \prod_p' \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{R})$$

(where  $\prod_p'$  denotes a restricted direct product over all the prime numbers consisting of elements  $(\gamma_p)_p$  for which the  $p$ th component  $\gamma_p$  belongs to the maximal

compact subgroup  $\mathrm{GL}_2(\mathbb{Z}_p)$  for all but finitely many  $p$ ). The shift in emphasis from modular forms to the so-called *automorphic representations* which they span is decisive. Langlands showed how to attach an  $L$ -function to any irreducible automorphic representation of  $G(\mathbb{A}_{\mathbb{Q}})$  for an arbitrary reductive algebraic group  $G$ , of which the matrix groups  $\mathrm{GL}_n$  and more general algebraic groups of Lie type are prototypical examples. This greatly enlarges the notion of what it means to be “modular”: a diophantine equation is now said to have this property if its zeta function can be expressed in terms of the Langlands  $L$ -functions attached to automorphic representations. One of the fundamental goals in the Langlands program is to establish further cases of the following conjecture:

**Conjecture.** *All diophantine equations are modular in the above sense.*

This conjecture can be viewed as a far-reaching generalisation of quadratic reciprocity and underlies the non-Abelian reciprocity laws that are at the heart of Andrew Wiles’s achievement.

Before Wiles’s proof, the following general classes of diophantine equations were known to be modular:

- Quadratic equations, by Gauss’s law of quadratic reciprocity;
- Cubic equations in one variable, by the work of Hecke and Maass;
- Quartic equations in one variable.

This last case deserves further comment, since it has not been discussed previously and plays a primordial role in Wiles’s proof. The modularity of quartic equations follows from the seminal work of Langlands and Tunnell. While it is beyond the scope of this survey to describe their methods, it must be emphasised that Langlands and Tunnell make essential use of the *solubility by radicals* of the general quartic equation, whose underlying symmetry group is contained in the permutation group  $S_4$  on 4 letters. Solvable extensions are obtained from a succession of Abelian extensions, which fall within the purview of the class field theory developed in the nineteenth and first half of the twentieth century. On the other hand, the modularity of the general equation of degree  $> 4$  in one variable, which cannot be solved by radicals, seemed to lie well beyond the scope of the techniques that were available in the “pre-Wiles era.” The reader who perseveres to the end of this essay will be given a glimpse of how our knowledge of the modularity of the general quintic equation has progressed dramatically in the wake of Wiles’s breakthrough.

Prior to Wiles’s proof, modularity was also not known for any interesting general class of equations (of degree  $> 2$ , say) in more than one variable; in particular it had only been verified for finitely many elliptic curves over  $\mathbb{Q}$  up to isomorphism over  $\mathbb{Q}$  (including the elliptic curves over  $\mathbb{Q}$  with complex multiplication, of which the elliptic curve of (31) is an instance.) Wiles’s modularity theorem confirmed the Langlands conjectures in the important test case of elliptic curves, which may seem like (and, in fact, are) very special diophantine equations, but have provided a fertile terrain for arithmetic investigations,

<sup>1</sup>See the interview with Ribet as the new AMS president in this issue, page 229.

both in theory and in applications (cryptography, coding theory...).

*Wiles's proof is also important for having introduced a revolutionary new approach which has opened the floodgates for many further breakthroughs in the Langlands program.*

Returning to the main theme of this report, Wiles's proof is also important for having introduced a revolutionary new approach which has opened the floodgates for many further breakthroughs in the Langlands program.

To expand on this point, we need to present another of the *dramatis personae* in Wiles's proof: *Galois representations*. Let  $G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  be the absolute Galois group of  $\mathbb{Q}$ , namely, the automorphism group of the field of all algebraic numbers. It is a profinite group, endowed with a natural topology for which the subgroups  $\text{Gal}(\bar{\mathbb{Q}}/L)$  with  $L$  ranging over the finite extensions of  $\mathbb{Q}$  form a basis of open subgroups. Following the

original point of view taken by Galois himself, the group  $G_{\mathbb{Q}}$  acts naturally as permutations on the roots of polynomials with rational coefficients. Given a finite set  $S$  of primes, one may consider only the monic polynomials with integer coefficients whose discriminant is divisible only by primes  $\ell \in S$  (eventually after a change of variables). The topological group  $G_{\mathbb{Q}}$  operates on the roots of such polynomials through a quotient, denoted  $G_{\mathbb{Q},S}$ —the automorphism group of the maximal algebraic extension *unramified* outside  $S$ , which can be regarded as the symmetry group of all the zero-dimensional varieties over  $\mathbb{Q}$  having “nonsingular reduction outside  $S$ .”

In addition to the permutation representations of  $G_{\mathbb{Q}}$  that were so essential in Galois's original formulation of his theory, it has become important to study the (continuous) *linear* representations

$$(41) \quad \varrho : G_{\mathbb{Q},S} \rightarrow GL_n(L)$$

of this Galois group, where  $L$  is a complete field, such as the fields  $\mathbb{R}$  or  $\mathbb{C}$  of real or complex numbers, the finite field  $\mathbb{F}_{\ell^r}$  equipped with the discrete topology, or a finite extension  $L \subset \bar{\mathbb{Q}}_{\ell}$  of the field  $\mathbb{Q}_{\ell}$  of  $\ell$ -adic numbers.

Galois representations were an important theme in the work of Abel and remain central in modern times. Many illustrious mathematicians in the twentieth century have contributed to their study, including three former Abel Prize winners: Jean-Pierre Serre, John Tate, and Pierre Deligne. Working on Galois representations might seem to be a prerequisite for an algebraic number theorist to receive the Abel Prize!

Like diophantine equations, Galois representations also give rise to analogous zeta functions. More precisely, the group  $G_{\mathbb{Q},S}$  contains, for each prime  $p \notin S$ , a distinguished element called the *Frobenius element* at  $p$ , denoted  $\sigma_p$ . Strictly speaking, this element is well defined only up to conjugacy in  $G_{\mathbb{Q},S}$ , but this is enough to make the arithmetic sequence

$$(42) \quad N_{p^r}(\varrho) := \text{Trace}(\varrho(\sigma_p^r))$$

well defined. The zeta function  $\zeta(\varrho; s)$  packages the information from this sequence in exactly the same way as in the definition of  $\zeta(X; s)$ .

For example, if  $X$  is attached to a polynomial  $P$  of degree  $d$  in one variable, the action of  $G_{\mathbb{Q},S}$  on the roots of  $P$  gives rise to a  $d$ -dimensional permutation representation

$$(43) \quad \varrho_X : G_{\mathbb{Q},S} \rightarrow \text{GL}_d(\mathbb{Q}),$$

and  $\zeta(X, s) = \zeta(\varrho_X, s)$ . This connection goes far deeper, extending to diophantine equations in  $n+1$  variables for general  $n \geq 0$ . The glorious insight at the origin of the Weil conjectures is that  $\zeta(X; s)$  can be expressed in terms of the zeta functions of Galois representations arising in the *étale cohomology* of  $X$ , a cohomology theory with  $\ell$ -adic coefficients which associates to  $X$  a collection

$$\{H_{\text{et}}^i(X/\bar{\mathbb{Q}}, \mathbb{Q}_{\ell})\}_{0 \leq i \leq 2n}$$

of finite-dimensional  $\mathbb{Q}_{\ell}$ -vector spaces endowed with a continuous linear action of  $G_{\mathbb{Q},S}$ . (Here  $S$  is the set of primes  $q$  consisting of  $\ell$  and the primes for which the equation of  $X$  becomes singular after being reduced modulo  $q$ .) These representations generalise the representation  $\varrho_X$  of (43), insofar as the latter is realised by the action of  $G_{\mathbb{Q},S}$  on  $H_{\text{et}}^0(X/\bar{\mathbb{Q}}, \mathbb{Q}_{\ell})$  after extending the coefficients from  $\mathbb{Q}$  to  $\mathbb{Q}_{\ell}$ .

**Theorem (Weil, Grothendieck,...).** *If  $X$  is a diophantine equation having good reduction outside  $S$ , there exist Galois representations  $\varrho_1$  and  $\varrho_2$  of  $G_{\mathbb{Q},S}$  for which*

$$(44) \quad \zeta(X; s) = \zeta(\varrho_1; s) / \zeta(\varrho_2; s).$$

The representations  $\varrho_1$  and  $\varrho_2$  occur in  $\oplus H_{\text{et}}^i(X/\bar{\mathbb{Q}}, \mathbb{Q}_{\ell})$ , where the direct sum ranges over the odd and even values of  $0 \leq i \leq 2n$  for  $\varrho_1$  and  $\varrho_2$  respectively. More canonically, there are always *irreducible* representations  $\varrho_1, \dots, \varrho_t$  of  $G_{\mathbb{Q},S}$  and integers  $d_1, \dots, d_t$  such that

$$(45) \quad \zeta(X; s) = \prod_{i=1}^t \zeta(\varrho_i; s)^{d_i},$$

arising from the decompositions of the (semisimplification of the)  $H_{\text{et}}^i(X/\bar{\mathbb{Q}}, \mathbb{Q}_{\ell})$  into a sum of irreducible representations. The  $\zeta(\varrho_i, s)$  can be viewed as the “atomic constituents” of  $\zeta(X, s)$  and reveal much of the “hidden structure” in the underlying equation. The decomposition of  $\zeta(X; s)$  into a product of different  $\zeta(\varrho_i; s)$  is not unlike the decomposition of a wave function into its simple harmonics.

A Galois representation is said to be *modular* if its zeta function can be expressed in terms of generating series attached to modular forms and automorphic representations and is said to be *geometric* if it can be realised in

an étale cohomology group of a diophantine equation as above. The “main conjecture of the Langlands program” can now be amended as follows:

**Conjecture.** *All geometric Galois representations of  $G_{\mathbb{Q},S}$  are modular.*

Given a Galois representation

$$(46) \quad \varrho : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_n(\mathbb{Z}_\ell)$$

with  $\ell$ -adic coefficients, one may consider the resulting mod  $\ell$  representation

$$(47) \quad \bar{\varrho} : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_n(\mathbb{F}_\ell).$$

The passage from  $\varrho$  to  $\bar{\varrho}$  amounts to replacing the quantities  $N_{p^r}(\varrho) \in \mathbb{Z}_\ell$  as  $p^r$  ranges over all the prime powers with their mod  $\ell$  reduction. Such a passage would seem rather contrived for the sequences  $N_{p^r}(X)$ —why study the solution counts of a diophantine equation over different finite fields, taken modulo  $\ell$ ?—if one did not know a priori that these counts arise from  $\ell$ -adic Galois representations with coefficients in  $\mathbb{Z}_\ell$ . There is a corresponding notion of what it means for  $\bar{\varrho}$  to be modular, namely, that the data of  $N_{p^r}(\bar{\varrho})$  agrees, very loosely speaking, with the mod  $\ell$  reduction of similar data arising from an automorphic representation. We can now state Wiles’s celebrated *modularity lifting theorem*, which lies at the heart of his strategy:

**Wiles’s Modularity Lifting Theorem.** *Let*

$$(48) \quad \varrho : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_2(\mathbb{Z}_\ell)$$

*be an irreducible geometric Galois representation satisfying a few technical conditions (involving, for the most part, the restriction of  $\varrho$  to the subgroup  $G_{\mathbb{Q}_\ell} = \text{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$  of  $G_{\mathbb{Q},S}$ ). If  $\bar{\varrho}$  is modular and irreducible, then so is  $\varrho$ .*

This stunning result was completely new at the time: nothing remotely like it had ever been proved before! Since then, “modularity lifting theorems” have proliferated, and their study, in ever more general and delicate settings, has spawned an industry and led to a plethora of fundamental advances in the Langlands program.

Let us first explain how Wiles himself parlays his original modularity lifting theorem into a proof of the Shimura–Taniyama–Weil conjecture for semistable elliptic curves. Given such an elliptic curve  $E$ , consider the groups

$$(49) \quad E[3^n] := \{P \in E(\bar{\mathbb{Q}}) : 3^n P = 0\}, \quad T_3(E) := \varprojlim E[3^n],$$

the inverse limit being taken relative to the multiplication-by-3 maps. The groups  $E[3^n]$  and  $T_3(E)$  are free modules of rank 2 over  $(\mathbb{Z}/3^n\mathbb{Z})$  and  $\mathbb{Z}_3$  respectively and are endowed with continuous linear actions of  $G_{\mathbb{Q},S}$ , where  $S$  is a set of primes containing 3 and the primes that divide the conductor of  $E$ . One obtains the associated Galois representations:

$$(50) \quad \begin{aligned} \bar{\varrho}_{E,3} : G_{\mathbb{Q},S} &\rightarrow \text{Aut}(E[3]) \simeq \mathbf{GL}_2(\mathbb{F}_3), \\ \varrho_{E,3} : G_{\mathbb{Q},S} &\rightarrow \mathbf{GL}_2(\mathbb{Z}_3). \end{aligned}$$

The theorem of Langlands and Tunnell about the modularity of the general quartic equation leads to the conclusion that  $\bar{\varrho}_{E,3}$  is modular. This rests on the happy circumstance that

$$(51) \quad \mathbf{GL}_2(\mathbb{F}_3)/\langle \pm 1 \rangle \simeq S_4,$$

and hence that  $E[3]$  has essentially the same symmetry group as the general quartic equation! The isomorphism in (51) can be realised by considering the action of  $\mathbf{GL}_2(\mathbb{F}_3)$  on the set  $\{0, 1, 2, \infty\}$  of points on the projective line over  $\mathbb{F}_3$ .

If  $E$  is semistable, Wiles is able to check that both  $\varrho_{E,3}$  and  $\bar{\varrho}_{E,3}$  satisfy the conditions necessary to apply the Modularity Lifting Theorem, at least when  $\bar{\varrho}_{E,3}$  is irreducible. It then follows that  $\varrho_{E,3}$  is modular, and therefore so is  $E$ , since  $\zeta(E; s)$  and  $\zeta(\varrho_{E,3}; s)$  are the same.

Note the key role played by the result of Langlands–Tunnell in the above strategy. It is a dramatic illustration of the unity and historical continuity of mathematics that the solution in radicals of the general quartic equation, one of the great feats of the algebraists of the Italian Renaissance, is precisely what allowed Langlands, Tunnell, and Wiles to prove their modularity results more than five centuries later.

Having established the modularity of all semistable elliptic curves  $E$  for which  $\bar{\varrho}_{E,3}$  is irreducible, Wiles disposes of the others by applying his lifting theorem to the prime  $\ell = 5$  instead of  $\ell = 3$ . The Galois representation  $\bar{\varrho}_{E,5}$  is always irreducible in this setting, because no elliptic curve over  $\mathbb{Q}$  can have a rational subgroup of order 15. Nonetheless, the approach of exploiting  $\ell = 5$  seems hopeless at first glance, because the Galois representation  $E[5]$  is not known to be modular a priori, for much the same reason that the general quintic equation cannot be solved by radicals. (Indeed, the symmetry group  $\mathbf{SL}_2(\mathbb{F}_5)$  is a double cover of the alternating group  $A_5$  on 5 letters and thus is closely related to the symmetry group underlying the general quintic.) To establish the modularity of  $E[5]$ , Wiles constructs an auxiliary semistable elliptic curve  $E'$  satisfying

$$(52) \quad \bar{\varrho}_{E',5} = \bar{\varrho}_{E,5}, \quad \bar{\varrho}_{E',3} \text{ is irreducible.}$$

It then follows from the argument in the previous paragraph that  $E'$  is modular, hence that  $E'[5] = E[5]$  is modular as well, putting  $E$  within striking range of the modularity lifting theorem with  $\ell = 5$ . This lovely epilogue of Wiles’s proof, which came to be known as the “3-5 switch,” may have been viewed as an expedient trick at the time. But since then the prime switching argument has become firmly embedded in the subject, and many

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variants of it have been exploited to spectacular effect in deriving new modularity results.

## *The modularity of elliptic curves was only the first in a series of spectacular applications.*

Wiles's modularity lifting theorem reveals that "modularity is contagious" and can often be passed on to an  $\ell$ -adic Galois representation from its mod  $\ell$  reduction. It is this simple principle that accounts for the tremendous impact that the Modularity Lifting Theorem, and the many variants

proved since then, continue to have on the subject. Indeed, the modularity of elliptic curves was only the first in a series of spectacular applications of the ideas introduced by Wiles, and since 1994 the subject has witnessed a real golden age, in which open problems that previously seemed completely out of reach have succumbed one by one.

Among these developments, let us mention:

- The two-dimensional Artin conjecture, first formulated in 1923, concerns the modularity of all odd, two-dimensional Galois representations

$$(53) \quad \varrho : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_2(\mathbb{C}).$$

The image of such a  $\varrho$  modulo the scalar matrices is isomorphic either to a dihedral group, to  $A_4$ , to  $S_4$ , or to  $A_5$ . Thanks to the earlier work of Hecke, Langlands, and Tunnell, only the case of projective image  $A_5$  remained to be disposed of. Many new cases of the two-dimensional Artin conjecture were proved in this setting by Kevin Buzzard, Mark Dickinson, Nick Shepherd-Barron, and Richard Taylor around 2003 using the modularity of all mod 5 Galois representations arising from elliptic curves as a starting point.

- Serre's conjecture, which was formulated in 1987, asserts the modularity of all odd, two-dimensional Galois representations

$$(54) \quad \varrho : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_2(\mathbb{F}_{p^r}),$$

with coefficients in a finite field. This result was proved by Chandrasekhar Khare and Jean-Pierre Wintenberger in 2008 by a glorious extension of the "3-5 switching technique" in which essentially all the primes are used. (See Khare's report in this volume.) This result also implies the two-dimensional Artin conjecture in the general case.

- The two-dimensional Fontaine-Mazur conjecture concerning the modularity of odd, two-dimensional  $p$ -adic Galois representations

$$(55) \quad \varrho : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_2(\tilde{\mathbb{Q}}_p)$$

satisfying certain technical conditions with respect to their restrictions to the Galois group of  $\mathbb{Q}_p$ . This theorem was proved in many cases as a consequence of work of Pierre Colmez, Matthew Emerton, and Mark Kisin.

- The Sato-Tate conjecture concerning the distribution of the numbers  $N_p(E)$  for an elliptic curve  $E$  as the prime  $p$  varies, whose proof was known to follow from the modularity of all the symmetric power Galois representations attached to  $E$ , was proved in large part by Laurent Clozel, Michael Harris, Nick Shepherd-Barron, and Richard Taylor around 2006.
- One can also make sense of what it should mean for diophantine equations over more general number fields to be modular. The modularity of elliptic curves over all real quadratic fields has been proved very recently by Nuno Freitas, Bao Le Hung, and Samir Siksek by combining the ever more general and powerful modularity lifting theorems currently available with a careful diophantine study of the elliptic curves which could a priori fall outside the scope of these lifting theorems.
- Among the spectacular recent developments building on Wiles's ideas is the proof, by Laurent Clozel and Jack Thorne, of the modularity of certain symmetric powers of the Galois representations attached to holomorphic modular forms, which is described in Thorne's contribution to this volume.

These results are just a sampling of the transformative impact of modularity lifting theorems. The Langlands program remains a lively area, with many alluring mysteries yet to be explored. It is hard to predict where the next breakthroughs will come, but surely they will continue to capitalise on the rich legacy of Andrew Wiles's marvelous proof.

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Photo of the conference in honor of Karl Rubin's sixtieth birthday is courtesy of Kartik Prasanna.

## ABOUT THE AUTHOR

Henri Darmon received the 2017 AMS Cole Prize in Number Theory and the 2017 CRM-Fields-PIMS Prize for his contributions to the arithmetic of elliptic curves and modular forms.



# The Mathematical Works of Andrew Wiles

*Christopher Skinner, with contributions from Karl Rubin, Barry Mazur, Mirela Çiperiani, Chandrashekhar Khare, and Jack Thorne*

**S**ir Andrew J. Wiles was awarded the Abel Prize for 2016 for “his stunning proof of Fermat’s Last Theorem by way of the modularity conjecture for semistable elliptic curves, opening a new era in number theory.”<sup>1</sup> Andrew Wiles announced his proof of Fermat’s Last Theorem in June of 1993 in a series of three lectures at a conference at the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge. Overnight Wiles and his proof became an international media sensation, making headlines in papers around the world. The story of this proof—the subsequent discovery of a gap and its ultimate and beautiful completion in September of 1994—has entered into popular legend.<sup>2</sup> The surprising drama of the proof is told in the 1996 BBC Horizon documentary *Fermat’s Last Theorem*, directed by Simon Singh, which ably conveys the human side of what is often seen as the distant and rarefied world of mathematical research.<sup>3</sup>

All this is well known. What is less well known, possibly even among number theorists, is that before his proof of Fermat’s Last Theorem, Wiles had made significant contributions to two of the most important problems for late-twentieth-century number theory:

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<sup>1</sup>Citation by the Abel Prize Committee of the Norwegian Academy of Science and Letters for the 2016 Abel Prize Laureate: [www.abelprize.no/c67107/binfil/download.php?tid=67059](http://www.abelprize.no/c67107/binfil/download.php?tid=67059)

<sup>2</sup>No doubt many number theorists share my own experiences of striking up conversations with strangers, who, upon discovering that I am a mathematician and even a number theorist, ask about “that guy who solved that famous problem—the one who worked in his attic for seven years.”

<sup>3</sup>I have watched this documentary many times with groups of mathematically talented high school students from around the world. Twenty years later it still inspires questions and conversation about what it means to do mathematical research or what a life spent doing mathematics can be.

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**Andrew Wiles with his PhD advisor, John Coates (left); his first PhD student Karl Rubin (right); and Ken Ribet (second from right), whose work linking the Modularity Conjecture to Fermat’s Last Theorem inspired Wiles to reengage with the problem that had fascinated him since childhood. This photo was taken at the Newton Institute in Cambridge, UK, during the conference at which Wiles first announced his proof of the Modularity Conjecture.**

- Wiles proved, together with John Coates, the first theoretical evidence for the celebrated Birch–Swinnerton-Dyer Conjecture; this is now known as the Coates–Wiles Theorem.
- Wiles proved Iwasawa’s Main Conjecture for  $\mathbb{Q}$ , in joint work with Barry Mazur, and for all totally real fields.

Each of these is a landmark result on its own and would be considered the highlight of a distinguished career. Two of the following contributions describe these works and their proofs. Karl Rubin, Wiles’s first PhD student, writes about the Coates–Wiles Theorem. Barry Mazur, Wiles’s collaborator on his first proof of Iwasawa’s Main Conjecture for  $\mathbb{Q}$ , writes about Wiles’s work on the Main Conjectures. Anyone seeking to learn about the context, significance, and ideas of Wiles’s proof of Fermat’s Last Theorem can do no better than to read Henri Darmon’s Abel Prize lecture in this same issue of the *Notices*.

*What are you working on now?* This is a question that eminent mathematicians are frequently asked (or so I am reliably informed) and certainly one that Andrew Wiles has repeatedly faced in the years following his proof of Fermat's Last Theorem. In her contribution below, Mirela Çiperiani writes about her collaboration with Andrew Wiles, from the mid-2000s, on a very natural Diophantine question: does a genus one curve over  $\mathbb{Q}$  have a rational point over a solvable extension of  $\mathbb{Q}$ ? As Çiperiani explains, results and techniques arising from the proof of Fermat's Last Theorem also play a role in this work.

*What distinguishes a great mathematical proof?* There is, of course, no definitive answer to this question. Certainly proofs of famous or important open conjectures can lay claim to being great. By this measure, Andrew Wiles's proof of Fermat's Last Theorem is a truly great proof. But proofs that introduce new ideas or open doors to progress on problems that were previously viewed as out of reach also have their claim to greatness. As noted in the Abel Prize citation, Andrew Wiles's proofs also achieve greatness by this second measure.

The new techniques and ideas that led to a successful proof of the modularity of semistable elliptic curves have been remarkably robust, also leading to the resolution of a host of problems within the circle of the Langlands Program: proofs of Serre's conjecture, the Artin conjecture for odd two-dimensional representations, the Sato-Tate conjecture, meromorphic continuation of Hasse-Weil  $L$ -functions of elliptic curves over totally real fields, modularity of all elliptic curves over real quadratic fields, and automorphy of small symmetric powers of modular forms, to list just a few.

Much of this progress has been achieved by the efforts of Richard Taylor, who was a PhD student of Wiles and later a coauthor of the paper that introduced one of the key ingredients<sup>4</sup> of the proof of modularity; by Wiles's PhD students; and by Taylor's collaborators and PhD students. But Wiles's work also inspired many others to push his ideas further. Among these is Chandrashekhara Khare, who writes below about some of the progress on modularity of two-dimensional Galois representations that followed Wiles's proof and of the use of this in Khare's own proof, with Jean-Pierre Wintenberger, of Serre's conjecture. In a related contribution, Jack Thorne describes how Wiles's original techniques evolved and were adapted to proving the automorphy of higher-dimensional Galois representations, leading to his proof, with Laurent Clozel, of the automorphy of some small symmetric powers of a holomorphic modular form.

But we should not lose sight of Wiles's earlier contributions in the glow of the successes arising from the proof of Fermat's Last Theorem. The proof of the Coates-Wiles Theorem and Wiles's proof of Iwasawa's Main Conjecture for totally real fields have inspired similar progress. For example, Kazuya Kato's (2004) spectacular success in constructing an Euler system for elliptic curves and then relating it to the special values of the Hasse-Weil  $L$ -function of the curve via an explicit reciprocity law can

be seen as a vast generalization of the ideas in the proof of the Coates-Wiles theorem. My own work with Eric Urban (2014), which together with Kato's result proves much of the Main Conjecture in the Iwasawa theory of elliptic curves, is in large part the natural generalization to some unitary groups of Wiles's methods for proving the Iwasawa Main Conjecture for totally real fields. Andrew Wiles's ideas continue to inform and shape progress on some of the fundamental problems in algebraic number theory.

## Karl Rubin

### Wiles's Work on Elliptic Curves with Complex Multiplication

During my senior year at Princeton, 1975–76, there was a lot of buzz in the common room about something everyone referred to as “Coates-Wiles.” I had a vague idea of what an elliptic curve was, but I doubt that I had any clear idea of what John Coates and Andrew Wiles had done. But I could tell that it was important, and I had a mental image of two very senior mathematicians who had made a great breakthrough.

I arrived at Harvard as a graduate student in 1976. Wiles arrived a year later, and I discovered that this “very senior” mathematician was scarcely older than I was and had been a graduate student at the time of this spectacular result. I became his student and had the unusual experience of attending my advisor's thesis defense.

Here's what the excitement was about. An elliptic curve  $E$  over the field  $\mathbb{Q}$  of rational numbers is a curve defined by an equation  $y^2 = x^3 + ax + b$ , with  $a, b \in \mathbb{Q}$  satisfying  $4a^3 + 27b^2 \neq 0$ . It is classical that the rational points  $E(\mathbb{Q})$  on  $E$  over any field  $F$  containing  $\mathbb{Q}$  form an abelian group, and Mordell proved that  $E(\mathbb{Q})$  is finitely generated. The *rank* of  $E$  is the dimension of the  $\mathbb{Q}$ -vector space  $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

In the late 1950s, after extensive computations, Birch and Swinnerton-Dyer made the following conjecture.

**Conjecture** (Birch & Swinnerton-Dyer). *For every elliptic curve  $E$  over  $\mathbb{Q}$ , we have*

$$(\text{BSD}) \quad \text{rank}(E(\mathbb{Q})) = \text{ord}_{s=1} L(E, s).$$

Here  $L(E, s)$  is the Hasse-Weil  $L$ -function of  $E$ , defined by an Euler product that converges on the complex half-plane  $\Re(s) > 3/2$ , and  $\text{ord}_{s=1} L(E, s)$  is its order of vanishing at  $s = 1$ . This conjecture is still unproved and is one of the Clay Millennium Problems.

Clearly, to make progress on this conjecture, or even for the statement to make sense, one needs to know that  $L(E, s)$  has an analytic continuation at least to  $s = 1$ . This was already known, thanks to a theorem of Deuring, for elliptic curves with *complex multiplication*: we say that  $E$  has complex multiplication if the ring of endomorphisms  $\text{End}(E)$  is larger than  $\mathbb{Z}$ , in which case  $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$  is an

<sup>4</sup>Now generally known as the Taylor-Wiles method.

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imaginary quadratic field. (The case  $\text{End}(E) = \mathbb{Z}$  is much more common.) For example, the elliptic curve  $y^2 = x^3 + ax$  has complex multiplication because  $(x, y) \mapsto (-x, iy)$  is an automorphism of  $E$  of order 4. Deuring proved that if  $E$  has complex multiplication, then  $L(E, s)$  can be identified with the  $L$ -function attached to a Hecke character and hence has an analytic continuation to the entire complex plane. Further, in this case Damerell proved that there is an explicit  $\Omega \in \mathbb{R}^\times$  such that  $L(E, 1)/\Omega \in \mathbb{Z}$ .

By the mid-1970s little was known about (BSD) beyond computational examples. That was how things stood when Coates and Wiles made their breakthrough:

**Theorem 1** (Coates and Wiles, 1977 [2]). *Suppose  $E$  has complex multiplication. If  $E(\mathbb{Q})$  is infinite, then  $L(E, 1) = 0$ .*

In other words, if  $\text{rank}(E(\mathbb{Q})) > 0$ , then  $\text{ord}_{s=1} L(E, s) > 0$ .

The key to the proof of Theorem 1 is the use of Robert's elliptic units to provide the crucial link between the algebraic and analytic sides of (BSD). Elliptic units are global units in abelian extensions of imaginary quadratic fields, defined by analytic functions, so they live in both the algebraic and analytic worlds.

Suppose  $E$  has complex multiplication, and let  $K$  be the imaginary quadratic field  $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Suppose  $p \geq 5$  is a prime where  $E$  has good reduction, and  $p$  factors as  $p = \pi \bar{\pi} \in \text{End}(E) \subset K$ . For  $n \geq 1$  let  $E[\pi^n] \subset E(\bar{K})$  denote the kernel of the endomorphism  $\pi^n$ , let  $K_n := K(E[\pi^n])$ , and let

$$\Gamma_n := \text{Gal}(K_n/K) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times.$$

Since  $K_n$  is an abelian extension of  $K$ , there is a subgroup  $C_n \subset K_n^\times$  of elliptic units. We also let  $U_n$  denote the units in the ring of integers of the completion of  $K_n$  at the unique prime above  $\pi$ .

Coates and Wiles constructed a “logarithmic derivative” homomorphism  $\psi_n : U_n \rightarrow E[\pi^n]$  and computed that

$$\psi_n(C_n) = \frac{L(E, 1)}{\Omega} E[\pi^n].$$

Under a mild additional assumption on  $p$ , they showed that the group of  $\Gamma_n$ -equivariant homomorphisms  $\text{Hom}_{\Gamma_n}(U_n, E[\pi^n])$  is cyclic of order  $p^n$ , generated by  $\psi_n$ , and therefore

$$(1) \quad \text{Hom}_{\Gamma_n}(U_n/C_n, E[\pi^n]) \cong \mathbb{Z}/(p^n, \frac{L(E, 1)}{\Omega})\mathbb{Z}.$$

Now suppose  $E(\mathbb{Q})$  is infinite. Fix a point  $P \in E(\mathbb{Q})$  of infinite order, and for every positive integer  $n$  choose a point  $Q_n \in E(\bar{K})$  such that  $\pi^n(Q_n) = P$ . The “Kummer map” that sends  $\sigma \in \text{Gal}(\bar{K}/K_n)$  to  $\sigma(Q_n) - Q_n$  defines a homomorphism

$$\kappa_n \in \text{Hom}_{\Gamma_n}(\text{Gal}(\bar{K}/K_n), E[\pi^n]).$$

Using class field theory,  $\kappa_n$  induces a homomorphism

$$\tilde{\kappa}_n \in \text{Hom}_{\Gamma_n}(U_n/C_n, E[\pi^n]),$$

and Coates and Wiles showed that there is an integer  $k$ , independent of  $n$ , such that  $\tilde{\kappa}_n$  has order  $p^{n-k}$  for all  $n \geq k$ . Comparing this with (1) as  $n$  grows proves Theorem 1.

The ideas in the proof of Theorem 1, along with methods Wiles developed for his work on Iwasawa's Main

Conjecture and on the modularity of elliptic curves (see the contributions by Barry Mazur and Henri Darmon, respectively, in this issue), have continued to play an important role in progress on the Birch and Swinnerton-Dyer conjecture.

- Kolyvagin (1990) recognized that elliptic units form what he calls an *Euler system*. (In fact, as one of very few known examples, elliptic units helped him to formulate the concept of an Euler system.) Combining the methods of Coates and Wiles with Kolyvagin's Euler system machinery led to my 1991 proof of Iwasawa's Main Conjecture for imaginary quadratic fields.
- Using a quite different Euler system of Heegner points, the combined results in the 1980s of Kolyvagin, Gross-Zagier, Bump-Friedberg-Hoffstein, and Murty-Murty showed that (BSD) holds if  $E$  is modular and  $\text{ord}_{s=1} L(E, s) \leq 1$ .
- The work of Wiles on modularity [6], completed by Taylor-Wiles [4] and Breuil-Conrad-Diamond-Taylor [1], showed in the 1990s that every elliptic curve over  $\mathbb{Q}$  is modular. Hence we have the following result for all elliptic curves over  $\mathbb{Q}$ , with or without complex multiplication.

**Theorem 2.** *If  $\text{ord}_{s=1} L(E, s) \leq 1$ , then  $\text{rank}(E(\mathbb{Q})) = \text{ord}_{s=1} L(E, s)$ .*

Along with some Iwasawa-theoretic results due to Kato (2004) and Skinner and Urban (2014), this is currently the best result in the direction of the Birch and Swinnerton-Dyer conjecture for elliptic curves over  $\mathbb{Q}$ .

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## Barry Mazur

### Andrew Wiles's Work on the Main Conjecture of Iwasawa Theory

What a joy it was to work with Andrew on the Main Conjecture of Iwasawa theory over  $\mathbb{Q}$ , and how great that Andrew went on to establish it, more generally, for totally real fields.

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The basic idea behind the Main Conjecture might be thought of as having, as a starting place, the classical analytic formulas of number theory, such as Dirichlet's Class Number Theorem for quadratic imaginary fields:

$$\frac{L(\chi, 1)}{2\pi} = \frac{h(-d)}{w\sqrt{d}},$$

where  $\chi$  is the quadratic Dirichlet character cutting out the quadratic imaginary field  $\mathbb{Q}(\sqrt{-d})$  of (integer) conductor  $-d < 0$ ;  $h(-d)$  is the order of the ideal class group of that field;  $w$  is the number of roots of unity in it; and  $L(\chi, s)$  is the Dirichlet  $L$ -function. One of the striking aspects of this formula is that the left-hand side is *analytic*; the right-hand side is *arithmetic*.

The "Main Conjecture" is not a misnomer: for any prime number  $p$  the conjecture asserts a fundamental relationship that establishes a close tie between

- the  $p$ -adic  $L$ -functions of a number field  $k$ —these being  $p$ -adic analytic objects closely related to the classical (complex)  $L$ -functions of  $k$ ,
- the deep arithmetic of that number field—namely, the  $p$ -primary parts of ideal class groups of certain abelian extensions of  $k$ .

Slightly more specifically, the Main Conjecture identifies the zeroes of  $p$ -adic  $L$ -functions of a number field  $k$  with the eigenvalues of an operator on a  $p$ -adic vector space constructed from the  $p$ -primary parts of ideal class groups of the abelian extensions of  $k$  alluded to above or constructed from some closely related arithmetic objects.<sup>1</sup>

This puts the conjecture somewhat in the spirit of a suggestion attributed to Hilbert and Pólya that a complex  $L$ -function of a number field might arise naturally as the characteristic series attached to a certain unbounded operator on a (naturally defined) Hilbert space. (Their suggestion, though, goes further by noting that if these operators were self-adjoint, this would relate well to the Riemann Hypothesis.) It also is in the spirit of the classical theory of  $L$ -functions attached to varieties over finite fields, for such an  $L$ -function can also be thought of as the characteristic polynomial of the Frobenius operator on an appropriate étale cohomology group.

To outline an example of the Main Conjecture, we'll discuss the relevant *group of operators*, *vector space*,  *$p$ -adic  $L$ -functions*, and *the method for the construction of the relevant arithmetic objects*.

### The $p$ -Cyclotomic Tower and the Fundamental Operator in the Main Conjecture

Let  $n \geq 1$ . Consider the finite field extension  $\mathbb{Q}[\mu_n]/\mathbb{Q}$  obtained by adjoining the group of  $n$ th roots of unity,

$$\mu_n := \{e^{2\pi ia/n} \mid a = 0, 1, \dots, n-1\} \subset \mathbb{C}^*,$$

to the field  $\mathbb{Q}$  of rational numbers. The group  $(\mathbb{Z}/n\mathbb{Z})^*$  is canonically isomorphic to the group  $\text{Aut}(\mu_n)$  of automorphisms of the cyclic group  $\mu_n$ , this isomorphism being defined by sending  $a \in (\mathbb{Z}/n\mathbb{Z})^*$  to the automorphism

$\zeta \mapsto \zeta^a$  for any  $\zeta \in \mu_n$ . Any automorphism of  $\mu_n$  extends uniquely to an automorphism of the field  $\mathbb{Q}[\mu_n]$ , giving us canonical isomorphisms:

$$(\mathbb{Z}/n\mathbb{Z})^* \xrightarrow{\cong} \text{Aut}(\mu_n) \xrightarrow{\cong} \text{Gal}(\mathbb{Q}[\mu_n]/\mathbb{Q}).$$

Fixing a prime  $p$  (for simplicity, suppose  $p > 2$ ) and letting  $n$  run through powers of  $p$ , form the  **$p$ -cyclotomic tower of (abelian Galois) extensions**

$$\mathbb{Q} \subset \mathbb{Q}[\mu_p] \subset \mathbb{Q}[\mu_{p^2}] \subset \mathbb{Q}[\mu_{p^3}] \subset \dots$$

and put  $\mathbb{Q}[\mu_{p^\infty}] := \bigcup_{v=1}^{\infty} \mathbb{Q}[\mu_{p^v}]$ . We have that

$$\text{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) = \lim_{v \rightarrow \infty} (\mathbb{Z}/p^v\mathbb{Z})^* = \mathbb{Z}_p^*,$$

the latter,  $\mathbb{Z}_p^*$ , being the profinite topological group of  $p$ -adic units, which decomposes as a product,  $\mathbb{Z}_p^* = \mathbb{F}_p^* \times \{1 + p\mathbb{Z}_p\}$ , where  $\mathbb{F}_p^* \simeq \text{Gal}(\mathbb{Q}[\mu_p]/\mathbb{Q})$  is a cyclic group of order  $p-1$ , and the subgroup  $\Gamma := \{1 + p\mathbb{Z}_p\} \subset \mathbb{Z}_p^*$  is an infinite cyclic pro- $p$ -group. A neat topological generator to choose for  $\Gamma$  is the  $p$ -adic unit  $\gamma := (1+p) \in \Gamma \subset \mathbb{Z}_p^*$ .

The field  $\mathbb{Q}[\mu_{p^\infty}]$  is generated by two linearly disjoint subfields:  $\mathbb{Q}[\mu_p]$  and a field, call it  $\mathbb{Q}_\infty$ , Galois over  $\mathbb{Q}$  with Galois group

$$\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) = \Gamma := 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^*.$$

The topological group  $\Gamma$  is our "group of operators."

### The Vector Space Containing Basic Arithmetic Data Related to Number Fields

Let  $k \subset K$  be number fields, contained in  $\mathbb{C}$ , linearly disjoint from  $\mathbb{Q}_\infty$ , with  $K$  totally real, and  $K/k$  a cyclic Galois extension. Put  $K_\infty = K \cdot \mathbb{Q}_\infty \subset \mathbb{C}$ . So  $\text{Gal}(K_\infty/K) = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) = \Gamma$ .

Let  $L_\infty$  denote the maximal unramified pro- $p$  abelian extension of  $K_\infty$ . Let  $X := \text{Gal}(L_\infty/K_\infty)$ , which, since it is a projective limit of  $p$ -abelian groups, we can view naturally as a  $\mathbb{Z}_p$ -module. Also,  $\Gamma = \text{Gal}(K_\infty/K)$  acts naturally (and  $\mathbb{Z}_p$ -linearly) on  $X$ . The action of an element in  $\Gamma$  on  $X$  is defined by lifting it to an element in  $\text{Gal}(L_\infty/K)$  and then noting that conjugation by that lifted element doesn't depend on the lifting and induces a well-defined automorphism of  $X$ .

One knows that  $V := X \otimes_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p$  is a finite-dimensional vector space. If  $\chi : \text{Gal}(K/k) \hookrightarrow \mathbb{C}^*$  is an odd (faithful) character cutting out the field extension  $K/k$ , we will be considering the action of  $\Gamma$  on the  $\chi$ -part of  $V$ , i.e., on  $V^\chi := \{v \in V \mid g(v) = \chi(g) \cdot v\}$ . We want to get as full an understanding of the  $V^\chi$  as possible and specifically the eigenvalues of  $1 - \gamma$  on  $V^\chi$  where  $\gamma \in \Gamma$  is a topological generator.

### $L$ -Functions

By interpolating special values of the classical complex  $L$ -functions, Kubota and Leopoldt defined the  $p$ -adic  $L$ -functions over the field  $k = \mathbb{Q}$ , and subsequently Deligne and Ribet defined them over totally real fields  $k$ . Briefly, in the context above, let  $\zeta_k(\sigma, s)$  denote the partial zeta-function of  $k$  associated to elements  $\sigma \in \text{Gal}(K/k)$ . The special values  $\zeta_k(\sigma, 1-n)$  are rational numbers, as proved by Klingen and Siegel. Let  $\psi$  be a one-dimensional character over  $k$  with values in  $\bar{\mathbb{Q}}_p^*$  and  $\chi_n := \psi^{-1}\omega^n$

<sup>1</sup>For example, from abelian extensions of  $p$ -power degree unramified over those alluded-to extensions of  $k$ .

where  $\omega$  is the Teichmüller character. For any integer  $n \geq 1$  one puts

$$L_p(1-n, \psi) = \sum_{\sigma \in \text{Gal}(K/k)} \chi_n(\sigma) \zeta_k(\sigma, 1-n) \cdot \prod_{P|p} (1 - \chi_n(P) N(P)^{1-n}),$$

where “ $\prod_{P|p}$ ” is taken over all primes  $P$  of  $k$  lying above  $p$ , and  $N(P)$  is the norm of the ideal  $P$ .

These special values,  $\{1-n \mapsto L_p(1-n, \psi) \in \bar{\mathbb{Q}}_p\}$ , interpolate to produce a  $p$ -adic analytic function  $L_p(s, \psi)$  on  $\mathbb{Z}_p$  if  $\psi$  is not trivial and on  $\mathbb{Z}_p - \{1\}$  with a simple pole at  $s = 1$  when  $\psi$  is trivial. Moreover, there is a unique power series  $G_\psi(T) \in \bar{\mathbb{Z}}_p[[T]]$  such that for the topological generator  $\gamma \in \Gamma \subset \mathbb{Z}_p^*$  (viewed as an element of  $\mathbb{Z}_p^*$ ) we have

$$L_p(s, \psi) = G_\psi(\gamma^s - 1).$$

The Main Conjecture, then, for  $\chi$  odd and  $p > 2$  identifies the characteristic polynomial of  $\gamma$  acting on  $V^\chi$  as described above, with the Weierstrass polynomial of the power series  $G_\psi(\gamma(1+T)^{-1} - 1)$  where  $\psi = \chi^{-1}\omega$ . That is, as Andrew proves [3], *the eigenvalues of  $\gamma$  acting on  $V^\chi$  are identified with the zeroes of  $L_p(s, \psi)$ .*

## Method

The essential issue in proving the Main Conjecture is to construct (by means of Galois representations attached to modular forms) as many abelian unramified extensions as would be predicted from the analytic side of the conjectured formula. A version of the classical analytic formula then allows one to conclude the conjecture. When the field  $k$  is  $\mathbb{Q}$ , we did this [1] by a two-step approach, starting from a marvelous idea of Ken Ribet relating divisibility of the  $p$ -adic  $L$ -function by  $p$  to a similar divisibility of the order of a specific ideal class group by  $p$ . (For a leisurely discussion of Ribet’s method and connection with the earlier work of Herbrand, see [2].) Briefly, the  $p$ -adic  $L$ -function  $L_p(s, \chi)$  (times an elementary nonzero factor) occurs as the constant term of a  $p$ -adic Eisenstein series (of  $p$ -adic weight determined by the value of  $s$ ). Whenever the  $p$ -adic  $L$ -function vanishes, this Eisenstein series has constant term zero and can be shown, therefore, to be congruent modulo arbitrarily high powers of  $p$  to cuspidal eigenforms. Moreover, since these cuspidal modular eigenforms are congruent modulo a high power of  $p$  to Eisenstein series, their associated  $p$ -adic Galois representations are extremely well behaved modulo those powers of  $p$  and can be seen to cut out larger and larger unramified  $p$ -power abelian extensions of  $K_\infty$ . The procedure employed by Wiles [3] for the totally real case is a good deal more delicate than in the case of  $k = \mathbb{Q}$ , in that one now uses the Galois representations of eigenforms on Hilbert–Blumenthal moduli spaces to construct the desired unramified abelian extensions. Andrew works systematically over the relevant weight space constructing the appropriate cuspidal  $\Lambda$ -adic Hilbert modular forms, using Hida’s theory, and [3] proves much more.

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## Mirela Çiperiani

### Solvable Points on Genus One Curves

In the spring of 2002, as a graduate student at Princeton, I approached Andrew Wiles to ask about the possibility of working under his supervision. He suggested that I think about a problem that was thrillingly natural and beautiful: *Does every genus one curve, defined over the rational numbers  $\mathbb{Q}$ , have a point over some solvable extension of  $\mathbb{Q}$ ?*

Solvable extensions of  $\mathbb{Q}$ —Galois extensions with solvable Galois group—are, concretely, fields contained in root extensions of  $\mathbb{Q}$ , i.e., in fields obtained by starting with  $\mathbb{Q}$  and considering successive extensions of the form  $F(\sqrt[n]{\alpha})/F$  for some  $\alpha \in F$  and  $n \in \mathbb{N}$ . This relationship between solvability of the Galois group and iterated adjunction of roots is the Galois-theoretic criterion for solvability by radicals of a polynomial equation in one variable. Thus, as shown by Abel and Galois, equations in one variable with coefficients in  $\mathbb{Q}$  and degree at least 5 need not be solvable by radicals. This connection to classical Galois theory is one of the appeals of the problem.

An obvious stumbling block is that, while this is evidently a diophantine equations problem, it is much less tangible than that of solving a quintic: we don’t have a generic way of writing the equations that describe genus one curves defined over  $\mathbb{Q}$ . A priori, a genus one curve defined over  $\mathbb{Q}$  is cut out by several homogeneous polynomial equations with rational coefficients. The famous Weierstrass cubic equations  $y^2z = x^3 + axz^2 + bz^3$  (here  $a, b \in \mathbb{Q}$  such that  $x^3 + ax + b$  has no repeated roots) describe *elliptic* curves, i.e., genus one curves which *do* have a point over their field of definition, namely,  $(0 : 1 : 0)$ . However, each genus one curve is a torsor for an elliptic curve, its Jacobian; i.e., the Jacobian acts on the genus one curve, and over  $\bar{\mathbb{Q}}$  that action becomes freely transitive. Conversely, every torsor for an elliptic curve  $E$  is a genus one curve with Jacobian  $E$ . In light of this correspondence, genus one curves over  $\mathbb{Q}$  with fixed Jacobian  $E$  are parametrized by a (Galois) cohomology group, the Weil–Châtelet group  $H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), E(\bar{\mathbb{Q}}))$ . So now we dispense with defining equations of genus one curves and work with elements of the Weil–Châtelet group.

In 2006, in joint work with Wiles, we showed that under certain restrictions, Wiles’s original question has a positive answer, as stated momentarily. The removal of these restrictions is the subject of ongoing joint work.

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**Theorem ([CW]).** *Let  $C$  be a genus one curve defined over  $\mathbb{Q}$ , with Jacobian  $E$ , such that*

- (1)  *$C$  has a point defined over the  $\ell$ -adics  $\mathbb{Q}_\ell$  for all rational primes  $\ell$ ; and*
- (2) *one of the following conditions holds:*
  - (a) *the analytic rank of  $E/\mathbb{Q}$  is less than or equal to 1 or*
  - (b)  *$E$  is semistable over  $\mathbb{Q}$ .*

*Then  $C$  has a point over a solvable extension of  $\mathbb{Q}$ .*

The analytic rank of an elliptic curve  $E$  is the order of vanishing of its  $L$ -series  $L(E, s)$  at  $s = 1$ . As for semistability, after lifting an elliptic curve from  $\mathbb{Q}$  to  $\mathbb{Z}$  and then reducing modulo a prime, it may remain nonsingular or it may acquire a node or cusp. Semistability disallows cusps.

I will now describe two fundamental ideas of Wiles which provide the frame of the proof. The first is referred to as the “unramified under ramified principle.” Fix an elliptic curve  $E$  defined over  $\mathbb{Q}$ . Genus one curves  $C$  defined over  $\mathbb{Q}$  with  $E$  as their Jacobian, and with points over  $\mathbb{Q}_\ell$  for every rational prime  $\ell$ , form a subgroup of the Weil–Châtelet group. (This subgroup contains the Tate–Shafarevich group, which consists of genus one curves that have both  $\ell$ -adic and real points.) We in fact work with their preimages in the *Selmer group*. These Selmer classes become trivial after an extension of  $\mathbb{Q}$  unramified at all but a finite set of primes  $\ell$ . According to the unramified under ramified principle, if we find a finite set of primes  $\mathcal{Q}$  obeying certain conditions and if we have sufficiently many cohomology classes which are ramified at primes in  $\mathcal{Q}$ , then under the cohomology group structure they will generate all the unramified classes.

We will apply this principle in the case where the genus one curve  $C$  satisfies conditions (1) and (2a) of our theorem by constructing ramified classes which correspond to genus one curves with points over some solvable extension of  $\mathbb{Q}$ . These ramified classes will then generate a group containing all genus one curves that have the same Jacobian as  $C$  and satisfy conditions (1) and (2a). Hence  $C$  has a solvable point. Thus the task is to choose the set  $\mathcal{Q}$  and to construct sufficiently many ramified classes which have solvable points. This is achieved by using the cohomology classes constructed by Kolyvagin. The construction of these classes makes use of *Heegner points*. These points are defined on modular curves and pushed forward to the elliptic curve via its modular parametrization—whose existence is known by the work of Wiles extended by Breuil, Conrad, Diamond, and Taylor.

The unramified under ramified principle is sufficient to prove the theorem in the case when the analytic rank of  $E/\mathbb{Q}$  is less than or equal to 1. It is not sufficient to prove it in general, for the following two reasons:

- (i) If the analytic rank of  $E/\mathbb{Q}$  is greater than 1, then the relevant Heegner points are trivial. Hence, in this case we cannot construct enough nontrivial Kolyvagin classes.
- (ii) The set of primes  $\mathcal{Q}$  depends on the order of the curve  $C$  viewed as an element of the Weil–Châtelet group of  $E/\mathbb{Q}$ , and the existence of the

sufficiently many cohomology classes ramified at primes in  $\mathcal{Q}$  depends on the finiteness of all the  $p$ -primary components of the Tate–Shafarevich group of  $E/\mathbb{Q}$ . This is only known for elliptic curves  $E/\mathbb{Q}$  of analytic rank less than or equal to 1 (by the modularity theorem, and the combined and celebrated work of Kolyvagin, Gross–Zagier, Bump–Friedberg–Hoffstein, and Murty–Murty).

When conditions (1) and (2b) hold, we can still find nontrivial Heegner points, using work of Cornut–Vatsal, by viewing  $C$  as a genus one curve over a nontrivial extension of  $\mathbb{Q}$ . We now attempt to apply the unramified under ramified principle. However, while this field extension enables us to construct ramified Kolyvagin classes, because of issue (ii) we are not able to see that we have sufficiently many of them.

It is in circumventing the potential existence of an infinite  $p$ -primary component of the Tate–Shafarevich group that we need the second idea. It is referred to as the “patching method.” A similar method was used in the proof of the modularity theorem. Wiles’s idea is the following. We construct as many ramified classes as we can for each in an infinite sequence of field extensions  $\{F_n\}$ . Observe that in order to do this we must choose unrelated sets of primes  $\mathcal{Q}_n$  for the fields  $F_n$ , all with the same cardinality. We consider groups  $M_n$  of cohomology classes over  $F_n$  ramified at primes in  $\mathcal{Q}_n$  (actually,  $M_n$  is viewed as a module over a ring related to  $F_n$ ). There is no natural containment between these modules, but each of them contains all the classes over  $\mathbb{Q}$  that we want to capture. However, our Kolyvagin classes generate a submodule  $M'_n \subseteq M_n$ , and for no  $n$  can we see that  $M'_n$  contains the desired classes. By considering their module structure (and ignoring their content), we construct injective maps  $M_n \rightarrow M_{n+1}$ . This gives rise to a module  $\varinjlim M_n$  over an Iwasawa algebra. Miraculously, a structure

theorem from Iwasawa theory shows us that each of our genus one curves  $C$  satisfying conditions (1) and (2b) lies in some  $M'_n$  for some  $n$ . Thus  $C$  has a solvable point.

Working with Wiles has been a wonderful experience. At the beginning, when I barely knew what an elliptic curve was, I felt lucky but humbled and daunted to be entrusted with such a fantastic problem. I had to rapidly learn

*I developed enormous admiration of Wiles for his generosity and modesty, and awe for his ability to see to the heart of the matter.*

the background material needed to begin thinking about the problem and to understand Wiles’s proposal to use the unramified under ramified principle. Later, at times when the problem seemed impossible, I was buoyed by Wiles’s confidence that we could solve it. I developed

enormous admiration of Wiles for his generosity and modesty, and awe for his ability to see to the heart of the matter—feelings that stay with me today.

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# Chandrashekhara Khare

## Modularity of $\mathrm{GL}_2$ Galois Representations and the Work of Andrew Wiles

The work of Andrew Wiles on the modularity of elliptic curves provided some of the key ideas and techniques which led to the proof of Serre’s modularity conjecture. Moreover, and equally importantly, it psychologically made it possible to imagine that there could be a strategy to prove results that before Wiles’s work seemed completely inaccessible.

To retrace the path from Wiles’s work [5] on modularity of elliptic curves to the proof of Serre’s conjecture [2], we describe briefly Wiles’s modularity lifting theorem and his strategy for proving it.

Let  $G_{\overline{\mathbb{Q}}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  be the Galois group of an algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$ . Wiles’s modularity lifting theorem is about 2-dimensional  $p$ -adic Galois representations for a prime  $p > 2$ . These are certain homomorphisms

$$\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{Z}_p),$$

including those that arise from the action of  $G_{\mathbb{Q}}$  on the torsion points of an elliptic curve defined over  $\mathbb{Q}$ . Wiles proved [5], [4] that under certain hypotheses on  $\rho$  and on its residual representation

$$\bar{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{F}_p)$$

(the reduction of  $\rho$  modulo  $p$ ), the representation  $\rho$  is modular.

The hypotheses on the residual representation  $\bar{\rho}$  in Wiles’s theorem require that  $\bar{\rho}$  be odd (the image of complex conjugation has eigenvalues  $+1$  and  $-1$ ), irreducible, and modular. For  $\bar{\rho}$ , being modular means that there is a cuspidal modular eigenform (an analytic function on the complex upper-half-plane with many symmetries)

$$f(\tau) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi n \tau}$$

such that for all but finitely many primes  $\ell$ , the Fourier coefficient  $a_{\ell}(f)$  can be matched with the trace of  $\bar{\rho}$  evaluated on a Frobenius element in  $G_{\mathbb{Q}}$  for the prime  $\ell$ . More precisely, the Fourier coefficients  $a_n(f)$  are all algebraic integers (so belong to  $\overline{\mathbb{Q}}$ ) and for some embedding  $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$  of  $\overline{\mathbb{Q}}$  into an algebraic closure  $\overline{\mathbb{Q}}_p$  of  $\mathbb{Q}_p$ , for almost all primes  $\ell$  the image of  $\iota(a_{\ell}(f))$  in the residue field  $\mathbb{F}_p$  of  $\overline{\mathbb{Q}}_p$  equals the trace of  $\bar{\rho}$  on a Frobenius element for  $\ell$ . Similarly,  $\rho$  is modular if there is an eigenform  $f$  and an embedding  $\iota$  such that for almost all primes  $\ell$ ,  $\iota(a_{\ell}(f))$  equals the trace of  $\rho$  on a Frobenius element for

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$\ell$ . Wiles’s theorem “lifts modularity” in that it asserts that if  $\bar{\rho}$  is modular, then the lift  $\rho$  of  $\bar{\rho}$  is also modular.

Wiles’s theorem allows for representations  $\rho$  where  $\mathbb{Z}_p$  is replaced by more general  $p$ -adic rings. In particular, it allows for homomorphisms  $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ ; a continuity argument associates to such a  $\rho$  a residual representation  $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{F}_p)$ . Much earlier results of Shimura and Deligne attach to each eigenform  $f$  and embedding  $\iota : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$  a Galois representation

$$\rho_{f,\iota} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p).$$

The spirit of Wiles’s modularity lifting theorem is: if  $\bar{\rho} \cong \bar{\rho}_{f,\iota}$  for some eigenform  $f$ , then  $\rho \cong \rho_{g,\iota'}$  for some eigenform  $g$  (not necessarily the same as  $f$ ). More about this theorem and its context can be found in Darmon’s lecture in this issue and in the article by Thorne.

Wiles’s strategy for proving his marvelous modularity lifting theorem can very roughly be paraphrased as follows. The kernel of the mod  $p$  reduction map  $\mathrm{GL}_2(\mathbb{Z}_p) \rightarrow \mathrm{GL}_2(\mathbb{F}_p)$  is prosolvable (the inverse limit of solvable groups), which enables the use of duality theorems in Galois cohomology and congruences between modular forms to bootstrap the modularity property from  $\bar{\rho}$  to  $\rho$ .

One spectacular application of Wiles’s modularity lifting theorem was the proof of the modularity of semistable elliptic curves defined over  $\mathbb{Q}$  (and hence Fermat’s Last Theorem!). Elliptic curves are often encountered as the projective curves defined by an (affine) Weierstrass equation:  $y^2 = x^3 + ax + b$  for constants  $a$  and  $b$ . The curve is defined over  $\mathbb{Q}$  if  $a, b \in \mathbb{Q}$ . The points on an elliptic curve  $E$  form an abelian group, with the addition law defined by rational functions in  $x$  and  $y$ ; the identity element of the group is the unique point at infinity (the point  $(0 : 1 : 0)$  in projective coordinates). The torsion points  $E[N]$  on  $E$  of (positive integer) order  $N$  form a group isomorphic to  $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ . If the elliptic curve is defined over  $\mathbb{Q}$ , then the Galois group  $G_{\mathbb{Q}}$  acts on  $E[N]$  by its action on the coordinates of the points. The  $p$ -adic Tate module  $T_p E$  of  $E$  is then the inverse limit of the groups  $E[p^n]$  of  $p$ -power torsion points:  $T_p E = \varprojlim_n E[p^n] \cong \mathbb{Z}_p^2$ . The action of  $G_{\mathbb{Q}}$  on  $T_p E$  determines a  $p$ -adic Galois representation

$$\rho_{E,p} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{Z}_p).$$

The residual representation  $\bar{\rho}_{E,p}$  of  $\rho_{E,p}$  is just the representation of  $G_{\mathbb{Q}}$  on the group of  $p$ -torsion points  $E[p] \cong \mathbb{F}_p^2$ . The modularity of an elliptic curve can be interpreted as the  $p$ -adic Galois representation  $\rho_{E,p}$  being modular for some prime  $p$  (equivalently, all primes  $p$ ).

The application of the modularity lifting theorem to modularity of elliptic curves over  $\mathbb{Q}$  comes by taking  $\rho = \rho_{E,p}$ . For  $p = 2$  or  $3$  the image of the representation  $\rho_{E,p}$  is prosolvable (the inverse limit of finite solvable groups). This is one of the two places in his argument where Wiles uses lucky accidents which happen for small primes. He uses results of Langlands and Tunnell which imply that an odd irreducible representation  $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{F}_3)$  is modular. This uses the solvability of  $\mathrm{GL}_2(\mathbb{F}_p)$  for  $p = 3$  (indeed,  $\mathrm{PGL}_2(\mathbb{F}_3)$  is isomorphic to  $S_4$ ) and the fact that the map  $\mathrm{GL}_2(\mathbb{Z}_p) \rightarrow \mathrm{GL}_2(\mathbb{F}_p)$  splits for  $p = 3$ ; both these statements are false for  $p > 3$ !

At this point, the modularity lifting theorem allows Wiles to conclude modularity of semistable  $E$  unless  $\bar{\rho}_{E,3}$  is reducible. To deal with the case when  $\rho_{E,3}$  is reducible, Wiles plays a “3-5” trick. He constructs another semistable elliptic curve  $E'$  over  $\mathbb{Q}$ , whose mod 3 representation surjects onto  $\mathrm{GL}_2(\mathbb{F}_3)$  and such that the mod 5 representations arising from  $E$  and  $E'$  are isomorphic and irreducible. Here again the use of the small prime 5 is vital, as the moduli space he considers of elliptic curves with level 5 structure isomorphic to  $E[5]$  turns out to be of genus zero, and in fact the projective line over  $\mathbb{Q}$ , and thus has many rational points. Then applying the modularity lifting theorem he deduces that  $\rho_{E',3}$  is modular. This implies that  $\rho_{E',5}$  is also modular, hence  $\bar{\rho}_{E,5} = \bar{\rho}_{E',5}$  is modular, and then by another application of the theorem, that  $\rho_{E,5}$ , and therefore  $E$ , is modular.

Wiles’s work was generalized by Breuil, Conrad, Diamond, and Taylor (2001) to prove the modularity of all elliptic curves over  $\mathbb{Q}$ . Recently, Freitas, Hung, and Siksek (2015) proved the modularity of elliptic curves defined over all real quadratic fields, using a “3-5-7” trick.

Before the work of Wiles there was no path from a  $p$ -adic Galois representation to a modular form. His completely new method of modularity lifting showed that if only a small quotient of a  $p$ -adic representation arose from a modular form, then the entire Galois representation did. This has turned out to be a very powerful method to prove modularity of Galois representations!

### Conjectures of Serre and Fontaine–Mazur

Serre conjectured [3] that for any  $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$  that is continuous, odd, and irreducible, there is an eigenform  $f$  such that  $\bar{\rho} \simeq \bar{\rho}_{f,i}$ ; i.e.,  $\bar{\rho}$  is modular.

J.-M. Fontaine and B. Mazur conjectured [1] that if  $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{Q}_p)$  is continuous, odd, irreducible, unramified outside a finite set of primes, and potentially semistable with Hodge–Tate weights  $(a, b)$ , say  $a \leq b$ , then the cyclotomic twist  $\rho(-a)$  is modular.

Wiles’s modularity lifting results were in the direction of showing that the conjecture of Serre implies that of Fontaine–Mazur. These modularity lifting results were improved in various crucial ways by Diamond, Fujiwara, and Kisin, including generalizations with the base field  $\mathbb{Q}$  replaced by a totally real field  $F$ . In an important development, Skinner and Wiles lifted the condition that  $\bar{\rho}$  is irreducible in the case when the lift  $\rho$  is ordinary at  $p$ . All these developments were crucial in our later work on Serre’s conjecture.

### Potential Version of Serre’s Conjecture

The automorphic descent results of Saito and Shintani, and Langlands are an important ingredient in the applications of modularity lifting theorems. These show that given a cyclic extension of totally real number fields  $K/F$  of prime degree and a cuspidal automorphic representation of  $\pi$  of  $\mathrm{GL}_2(\mathbb{A}_K)$  which is a discrete series at the infinite places and invariant under  $\sigma \in \mathrm{Gal}(K/F)$ , then  $\pi$  is the base change of a cuspidal automorphic representation of  $\mathrm{GL}_2(\mathbb{A}_F)$ . When combined with modularity lifting results,

these descent results imply that modularity of  $\bar{\rho}$  follows by showing that  $\bar{\rho}|_{G_F}$  arises from a Hilbert modular form for a solvable, totally real extension  $F/\mathbb{Q}$ .

To study general representations  $\bar{\rho}$  as in Serre’s conjecture, Taylor considered moduli spaces over  $\mathbb{Q}$  whose points over a number field  $K$  correspond to abelian varieties over  $K$  (with real multiplication) which give rise to  $\bar{\rho}|_{G_K}$ , and at an auxiliary place  $\ell \neq p$  give rise to a mod  $\ell$  dihedral representation. The latter are known to be modular by an old result of Hecke. Taylor used a theorem of Moret–Bailly to produce points of the moduli spaces he considered over totally real fields  $F$  (but which could not be guaranteed to be solvable over  $\mathbb{Q}$ ). Then modularity lifting theorems for representations of  $G_F$  yield that  $\bar{\rho}|_{G_F}$  arises from a Hilbert modular form over  $F$ . This may be regarded as a potential version of Serre’s conjecture. Together with automorphic descent for cyclic prime degree extensions of totally real number fields, this led to the meromorphic continuation of the Hasse–Weil  $L$ -series attached to elliptic curves over all totally real fields.

To proceed along these lines to prove Serre’s conjecture in the general case, it would be necessary either to find a general procedure to show existence of totally real solvable points on geometrically irreducible smooth projective varieties of general type over  $\mathbb{Q}$  or to prove nonsolvable automorphic descent for Hilbert modular forms.

### Proof of Serre’s Conjecture

My proof with Wintenberger of Serre’s conjecture [2] took a different path and used as a starting point results of Tate and Serre that proved Serre’s conjecture for representations  $\bar{\rho}$  of residue characteristic  $p \leq 3$  with limited ramification, with no a priori assumptions on the image of  $\bar{\rho}$ . Tate and Serre proved that any representation  $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$  unramified outside  $p$  and with  $p \leq 3$  is reducible. This is another instance of the magic of small primes!

Our proof measured the complexity of the representation  $\bar{\rho}$  in terms of its ramification at  $p$  (Serre’s weight) and the ramification away from  $p$  (its Artin conductor  $N$ ). A double induction on  $(p, N)$  was used to reduce Serre’s conjecture to the results of Tate and Serre. A principle of the proof is to use potential modularity to produce compatible systems of representations that lift  $\bar{\rho}$  such as would exist if it were known that  $\bar{\rho}$  were modular.

Our proof of Serre’s conjecture owes its existence to the modularity lifting theorems initiated by Wiles, a tool to attack modularity which was as powerful as it was unexpected when it was introduced, and also to Wiles’s prime switching trick in his proof of the modularity of elliptic curves.

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## Jack Thorne

### Modularity of $n$ -Dimensional Galois Representations

More than twenty years have passed since Andrew Wiles proved the first modularity lifting theorems and deduced Fermat's Last Theorem as a consequence. His theorems focussed on proving the modularity of certain 2-dimensional Galois representations. These were the homomorphisms

$$(1) \quad \varrho : G_{\mathbb{Q},S} \rightarrow \text{GL}_2(\mathbb{Z}_\ell)$$

arising from elliptic curves  $E$ ,  $G_{\mathbb{Q},S}$  being the Galois group of the maximal extension of  $\mathbb{Q}$  unramified outside some finite set of primes  $S$ . Despite this focus, the ideas of the proof have proved powerful and flexible enough that they are still a driving force of our understanding of much more general Galois representations today.

The first key hypothesis imposed on  $\varrho$  is the modularity of the residual representation

$$(2) \quad \bar{\varrho} : G_{\mathbb{Q},S} \rightarrow \text{GL}_2(\mathbb{F}_\ell)$$

that is the reduction of  $\varrho$  modulo  $\ell$ . In other words, one assumes at the outset the existence of a cuspidal modular eigenform

$$(3) \quad f = \sum_{n \geq 1} a_n(f) q^n$$

which is matched with  $\bar{\varrho}$ , in the sense that for almost all primes  $p$ , the Fourier coefficient  $a_p(f)$  is congruent modulo  $\ell$  to the integer  $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$ . This allows the introduction of a first key player, the Hecke algebra  $\mathbb{T}_{\bar{\varrho},S}$ , which acts faithfully on a space of cuspidal modular forms, all of whose Fourier coefficients agree modulo  $\ell$  with those of  $f$  and which have level supported at the primes of  $S$ .

The second key player is the universal deformation ring of  $\bar{\varrho}$ , which we call  $R_{\bar{\varrho},S}$ . It is a complete local ring which is characterized by a universal property. For example, the set of homomorphisms  $R_{\bar{\varrho},S} \rightarrow \mathbb{Z}_\ell$  is in bijection with the set of equivalence classes of lifts  $\varrho' : G_{\mathbb{Q},S} \rightarrow \text{GL}_2(\mathbb{Z}_\ell)$  of  $\bar{\varrho}$  that one hopes to prove are modular.

These key players are related by a surjective ring homomorphism

$$(4) \quad R_{\bar{\varrho},S} \rightarrow \mathbb{T}_{\bar{\varrho},S}.$$

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Passing to spectra, one can think of  $\text{Spec } \mathbb{T}_{\bar{\varrho},S}$  as a closed subspace of  $\text{Spec } R_{\bar{\varrho},S}$ : it is the locus of the modular Galois representations inside the space of all Galois representations. In particular, the existence of this map expresses the *existence* of Galois representations attached to modular forms, itself a highly nontrivial fact. Diagrams such as (4) have achieved an iconic status in algebraic number theory.

The truth of the modularity lifting theorem is implied by the much more refined statement that the map (4) is an isomorphism. This goes some way towards explaining the importance of the universal deformation ring  $R_{\bar{\varrho},S}$ , which was first introduced by Mazur. A large part of Wiles's fundamental 1995 work is taken up with introducing the tools necessary to effectively study the map (4), putting him in a position to prove ' $R = \mathbb{T}$ ' in many cases.

One tool that has turned out to be surprisingly versatile is the Taylor–Wiles method, introduced in the companion paper [3]. Roughly speaking, this is an effective machine to study (4) in the so-called minimal case where  $S$  is as small as possible. To pass from this case to the general case, Wiles introduced a numerical isomorphism criterion to compare the situation for varying sets  $S$ . The verification of this criterion then involves delicate calculations in Galois cohomology and with modular forms.

The first modularity lifting theorems for Galois representations of dimension  $n > 2$  were proved by Clozel, Harris, and Taylor in a paper published in 2008, which was heavily influenced by an earlier unpublished manuscript of Harris and Taylor. This was made possible thanks to the construction of  $n$ -dimensional Galois representations attached to modular forms on unitary groups, itself initiated by Clozel and Kottwitz and then studied in great detail by Harris and Taylor in their proof of the local Langlands conjectures for  $\text{GL}_n$ , allowing one to write the  $n$ -dimensional analogue of the map  $R_{\bar{\varrho},S} \rightarrow \mathbb{T}_{\bar{\varrho},S}$ .

The Taylor–Wiles method was generalized by Clozel, Harris, and Taylor in order to prove a modularity lifting theorem in the minimal case. However, such a restriction on ramification makes these theorems very difficult to apply in interesting situations. The general case was treated using a generalization of the numerical criterion of Wiles, but only conditional on a conjecture (referred to colloquially as Ihara's lemma) that remains unproven today.

Kisin had earlier developed a generalization of the Taylor–Wiles method in his study of modularity for  $\text{GL}_2$ , in a work published in 2009. He enlarged the diagram (4) to a diagram

$$(5) \quad \widehat{\bigotimes}_{p \in S} R_{\bar{\varrho},p} \rightarrow R_{\bar{\varrho},S} \rightarrow \mathbb{T}_{\bar{\varrho},S},$$

where each ring  $R_{\bar{\varrho},p}$  is an object parameterizing deformations of the restriction of  $\bar{\varrho}$  to a decomposition group at  $p$  (in other words, a local Galois group  $D_p = \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$ ). In this point of view, the geometry of the rings  $R_{\bar{\varrho},p}$  begins to play a key role. The Taylor–Wiles–Kisin method finally allows one to link the modularity of Galois representations  $\varrho_1, \varrho_2 : G_{\mathbb{Q},S} \rightarrow \text{GL}_n(\mathbb{Z}_\ell)$  which have the property that the representations  $\varrho_1|_{D_p}, \varrho_2|_{D_p}$  determine points on

the *same* irreducible component of  $\mathrm{Spec} R_{\bar{\varrho},p}$  for each prime  $p \in S$ .

Building on this, Taylor [2] made a detailed study of the irreducible components of certain ‘local’ deformation rings and introduced a very surprising trick that allowed him to circumvent completely the conjectural Ihara’s lemma and prove the first modularity lifting theorems for  $\mathrm{GL}_n$  without restriction on the permitted ramification of  $\varrho$  relative to  $\bar{\varrho}$ .

These theorems have had spectacular applications. Combined with the earlier work of Harris, Shepherd-Barron, and Taylor, they implied that the even-dimensional symmetric power Galois representations

$$(6) \quad \mathrm{Sym}^{n-1} \varrho : G_{\mathbb{Q},S} \rightarrow \mathrm{GL}_n(\mathbb{Z}_\ell)$$

are potentially modular: there exists a number field  $F/\mathbb{Q}$  such that the restriction  $\mathrm{Sym}^{n-1} \varrho|_{G_F}$  to the absolute Galois group of  $F$  is modular, in the sense of being associated to modular (or automorphic) forms on  $\mathrm{GL}_{n,F}$ . These ideas in turn led to the proof of the Sato-Tate conjecture for elliptic curves over  $\mathbb{Q}$ :

**Theorem 1.** *Let  $E$  be an elliptic curve over  $\mathbb{Q}$  without complex multiplication. Then the quantities  $a_p(E)/2\sqrt{p} \in [-1, 1]$  are equidistributed as  $p \rightarrow \infty$  with respect to the Sato-Tate measure*

$$(7) \quad \frac{2}{\pi} \sqrt{1-t^2} dt.$$

There are many fruitful directions that remain to be explored. For example, can one show that the symmetric powers (6) are modular and not just potentially modular? In joint work with Clozel [1], I showed that the answer to this question is affirmative for  $n \leq 9$ . A major part of our proof is a generalization of an important modularity lifting theorem of Skinner and Wiles which applies to Galois representations  $\varrho$  for which the residual representation  $\bar{\varrho}$  is *reducible*. This modularity lifting theorem also played a major part in the proof of Serre’s conjecture.

*The work of Wiles  
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effect.*

The proof of the Skinner-Wiles theorem employs many ideas which appear in Andrew Wiles’s most famous works, such as  $p$ -adic families of modular forms and congruences between Eisenstein series and

cuspidal modular forms, as well as many other ideas whose importance would become apparent only later: we mention in particular an emphasis on the geometry of Galois deformation rings. The work of Wiles has had a transforming effect on this corner of number theory, and his influence continues to be felt throughout the subject.

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## ABOUT THE AUTHORS

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**Christopher Skinner** studied at Princeton with Andrew Wiles in the mid-1990s.



**Christopher Skinner**



**Karl Rubin**

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**Karl Rubin** was Andrew Wiles's first PhD student.

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**Barry Mazur** delights in the memories of the times Andrew and he had at Harvard, as colleagues.



**Barry Mazur**



**Mirela Çiperiani**

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**Mirela Çiperiani** completed her PhD in 2006 under the supervision of Andrew Wiles.

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Using techniques and ideas that grew out of Wiles's proof of Fermat's Last Theorem, **Chandrashekhhar Khare** (with Jean-Pierre Wintenberger) proved Serre's Conjecture.



**Chandrashekhhar Khare**



**Jack Thorne**

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**Jack Thorne** studied at Harvard with Richard Taylor, himself a student of Andrew Wiles.

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# Interview with New AMS President Kenneth A. Ribet

Kenneth A. Ribet received his PhD from Harvard University in 1973. He was a lecturer and assistant professor at Princeton University for five years before joining the faculty at the University of California, Berkeley, where he is professor of mathematics. His honors include election to the US National Academy of Sciences (1990) and the Prix Fermat of the Institut de Mathématiques de Toulouse (1989). Ribet took office as President of the AMS on February 1, 2017. This interview was conducted in fall 2016.

## Education and the AMS

**Notices:** First I'd like to ask you about education. You have been devoted to teaching all your career. You won two teaching awards at UC Berkeley, one early on in your time as a faculty member there, and another one just recently, in 2013. How do you see the role of the AMS in education?

**Ribet:** The AMS has a big voice and can look at trends in education and make policy statements about practices we should be adopting in our classrooms. For example, many people are interested in strategies for active learning.<sup>1</sup> Students today are uneasy going into a classroom and having a passive experience. It is a challenge to engage students and get them involved in the course and the material. If the AMS has a role in shaping how people see such challenges, that would be a very positive development.

One of the committee meetings I have enjoyed the most is that of the AMS Committee on Education, which meets in Washington DC, typically every October. The committee brings in speakers who talk about educational practices.

*There is a real hunger among the general public for fairly sophisticated mathematics.*

Allyn Jackson is senior writer and deputy editor of the Notices. Her e-mail address is [axj@ams.org](mailto:axj@ams.org).

<sup>1</sup> See "What Does Active Learning Mean For Mathematicians?" by Benjamin Braun, Priscilla Bremser, Art M. Duval, Elise Lockwood, and Diana White, Notices, February 2017.

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Ken Ribet (back row, third from left) often meets for breakfast with groups of students, such as this one that met with him in November of last year (from left to right): Xuefei Lei, Rahul Malayappan, Aileen Ho, Karam Samplay, Gauri Powale, Emily Hsiao, and Genaro Hernandez Salgado.

It's thoroughly illuminating. I am by training a completely traditional teacher. Nonetheless, the exposure that I've had to these different points of view has influenced the way that I interact with my students.

Currently I am teaching a class where the enrollment is about 375. It's my job to engage with those students as well as I can. I try to make sure the students come to the lectures—I call them class meetings instead of lectures—and to make them as interactive as possible. I run around the room and solicit questions, I make an effort to call students by name when they raise their hands. This seems to limit absences and eliminate texting and web surfing in class.

**Notices:** *You said students don't want a passive experience in the classroom. Is that very different from when you first started at Berkeley?*

**Ribet:** When I first started at Berkeley, students would go into this old room with lots of wood and chalkboards and sit there. The professor would lecture, and the students would take notes. The experience was completely scripted. No one questioned the procedure. Now, students get input through a constant barrage of media—through their tablets and phones and laptops. It's perceived as cool to ditch class and try to make up for it at the very last minute by binge-watching recorded lectures.

In my courses, I look for ways to interact with the students informally, to get a sense of what would appeal to them, what reaches them. I try to make mathematics something that they can relate to.

### Public Awareness of Mathematics

**Notices:** *Andrew Wiles's proof of Fermat's Last Theorem [featured in this issue of the Notices], to which your work contributed crucially, attracted huge attention from the general public. Since that time, the public's interest in mathematics has grown quite a bit. Why? And what role can the AMS play here?*

**Ribet:** I was in Cambridge, England, in June 1993, when Wiles announced a proof of Fermat's Last Theorem. When I returned to Berkeley, people at MSRI [Mathematical Sciences Research Institute]—Bill Thurston, Lenore Blum, and others—suggested doing a public event around FLT. There was a huge gathering in San Francisco where a group of mathematicians—I was one of them, as were Karl Rubin, Bob Osserman, Lenore Blum, and others—got up in front of 2,000 people and talked about Wiles's proof.<sup>2</sup> We had a post-event discussion at MSRI the next week, and Thurston said: "This is great. We have to do more things like this. What do we do now?" I remember being skeptical. I thought the proof of FLT was such a singular development that there would be no way to maintain the public's interest in research mathematics. Now, over twenty years later, it's been incredibly gratifying to see the growth in interest in FLT and many other wonderful mathematical results.

Recently I watched a Numberphile<sup>3</sup> video on a combinatorial problem called the Josephus problem. The video featured a young mathematician named Daniel Erman. He explained something that is fairly sophisticated, using induction and binary expansion. At the time I watched it, the number of views of that video was up to about a quarter of a million, and it had been out only a few days. That's one of many indications of a real hunger among some segments of the general public for fairly sophisticated mathematics.

<sup>2</sup>See "Fermat Fest Draws a Crowd," by Allyn Jackson, *Notices*, October 1993.

<sup>3</sup>Sponsored by MSRI, Numberphile ([www.numberphile.com](http://www.numberphile.com)) is a website featuring videos made by Brady Haran, in which mathematicians talk about mathematical ideas and concepts. The videos are personal, humorous, quirky, insightful, and a lot of fun.

I think we mathematicians have gotten much better at presenting ourselves to the outside world. Mathematicians, who fifty years ago were solitary figures pacing the floor, are interacting with people outside their immediate circle. It's perhaps part of the general trend where mathematics has become more collaborative. People are more used to explaining themselves to others who are not necessarily "living inside their brains," as my wife says. More and more mathematicians have taken up positions in business and industry and government, where they have to explain their work to colleagues who have no mathematical training. And they have to listen to descriptions of problems that are fundamentally mathematical, from people who don't view those problems in mathematical terms.

*The world of math  
is much less  
orderly than it  
used to be.*

There is a related trend. More and more mathematics is applied. Thirty years ago, if you talked about applied mathematics, it meant differential equations and fluid dynamics. Nowadays, applied math can be

anything—number theory, or combinatorics, or algebraic geometry applied to robotics. There is a wonderful report, *Mathematical Sciences in 2025* [National Academy Press, 2013], that talks about all the ways in which math has influenced daily life. Quaternions in electric toothbrushes and video games—this is just great! Obviously we have something to offer to the outside world. The AMS is right at the center of the picture.

**Notices:** *What do you think the AMS can do to capitalize on this upswell of public interest in math?*

**Ribet:** The AMS is part of the reason there is this public interest. We have the wonderful AMS Public Awareness Office, which is tweeting all the time and has a big presence on Facebook. We also have the AMS office in Washington DC, which helps communicate about mathematics with people on Capitol Hill.<sup>4</sup> It's one of the fundamental activities of the AMS now, to look beyond the immediate community of academic mathematicians.

### New Generations, New Technologies

**Notices:** *You mentioned in your candidacy statement for AMS president that technology is bringing rapid changes and unprecedented opportunities to mathematics. Can you say more about what you mean by that, what kind of technology you are referring to?*

**Ribet:** It's clear that one of the challenges for the AMS now is to continue to maintain its relevance in a context where young mathematicians are using technology to share and gather information with less of the formal structure that older mathematicians like myself are used

<sup>4</sup>Karen Saxe, who started as director of the AMS Washington Office in January of this year, was interviewed by *Notices* associate editor Harriet Pollatsek in the December 2016 issue.

to. People ask questions and get answers on MathOverflow. Many people feel that the definitive version of a manuscript is the one deposited on arXiv. Traditional journals are sometimes seen as unimportant. People often bypass MathSciNet® when trying to get information. They collaborate through blogs. The world of math is much less orderly than it used to be. When I was a young mathematician, I would run to the library every month to look at the latest printed volume of *Math Reviews*. I would wait for the *Notices* and the *Bulletin* and the *Abstracts* to arrive in my mailbox. I would go to the library to look at the latest issues of *Inventiones*, the *Duke Mathematical Journal*, and *Annals of Math*. Now, many results are disseminated without that structure.

**Notices:** *What can the AMS do to make itself relevant to young people who are developing as mathematicians in this very different environment?*

**Ribet:** An initiative ongoing for the past three or four years, through strategic planning, aims to reinvent the AMS in some sense—to make sure it remains relevant and to meet changing needs of younger mathematicians. The AMS has carried out interviews and surveys to find out what mathematicians think of and need from the AMS. It's hard for me to summarize the findings in a few short sentences, but it is clear that in the next few years, the AMS will evolve in lots of ways in response to this strategic planning. The books will look different. There will be new journals. The AMS will respond to what we have heard from our membership.

### Preparing for Careers in Business, Industry, Government

**Notices:** *What other challenges face the AMS?*

**Ribet:** One challenge is that the majority of the young people who are in graduate programs or are new PhDs will pursue careers outside of academia. It's important for the AMS to make people aware of this reality and to prepare them for transitions they'll need to make, to ensure that graduate programs are providing good preparation.<sup>5</sup>

My wife, Lisa Goldberg, is a mathematician who works in industry as a financial economist and is on the faculty of UC Berkeley in economics and statistics. Graduate students come into those departments, and from Day One, the majority of them are thinking about jobs in industry, biotech, consulting, government. Very few of them consider that their goal is to be like their professors. In mathematics, which is a neighboring discipline, it's completely different. If you talk to a first- or second-year graduate student in mathematics, you find that person's goal is to, say, transform the study of Banach algebras through the use of derived algebraic geometry. With few exceptions, they think in terms of academic careers. What happens is that a majority of those people, soon after their PhDs, realize that they would be a lot happier in a business or industry or government setting. They



**Ken Ribet.**

often scramble to prepare themselves for such careers by increasing their programming skills or learning basic statistics.

People who make this transition report back and say, "It's fantastic, I should have done this from the beginning of my graduate career, this is a much more rewarding experience than I imagined when I started on that road." At every JMM [Joint Mathematics Meetings] there is a panel about industry jobs, and the panelists are just glowing, saying how wonderful their work is. But somehow, the word is not percolating to the level of first-year graduate students, who are still very much of the mindset that they want to be exactly like their professors. I think the AMS can have an important role in changing the mindset.

### Memories of Serge Lang

**Notices:** *You were a great friend of Serge Lang.<sup>6</sup> He resigned from the AMS about ten years before his death. What do you think he would have said about your becoming AMS president?*

**Ribet:** Serge had an expression he would use to describe, say, a speech by a prominent university administrator or some other august figure. He would refer to these speeches as "big time bullshit." This was one of his favorite descriptors, and he used it a lot. Maybe twenty-five years ago, I was asked to run for the AMS Council. Serge was in Berkeley every summer, and he was here when I wrote my candidate's statement. I showed it to Serge and asked him what he thought. He shook his head and looked very displeased. Using Serge's expression, I said, "Big time bullshit?" And he said, "Not big time!"

I lost that election. More recently, I was asked again to run for the Council. I had just been to an informal gathering of the Council after one of its long meetings, and I had an incredibly positive impression of the whole enterprise. I loved the people and what they were doing. I agreed enthusiastically to make a second run for the

<sup>5</sup> See "Math PhD Careers: New Opportunities Emerging Amidst Crisis," by Yuliy Baryshnikov, Lee DeVille, and Richard Laugesen in this issue of the *Notices*.

<sup>6</sup> Serge Lang (1927–2005) was a French-born mathematician who spent most of his career at Yale University. Read more about him in the obituary that appeared in the May 2006 issue of the *Notices*.

Council. I was elected to the Council and was later chosen for the Executive Committee. Then I was asked to run for president—and I was elected.

I think that Serge, despite the fact that he had negative views of various things, had a great deal of respect for people's passion. He liked me very much, and he respected me. If he were around today and saw that I was passionate about working for the AMS, he would respect that.

Just to summarize what happened with Serge's resignation from the AMS: In the 1970s, Serge started his "file-making," in which he developed the theme that there were systematic abuses of power by people at the top of the scientific establishment who would quash opposing views if those views conflicted with their own ambitions. He thought that HIV/AIDS was one of those subjects where the people in power were quashing opposing viewpoints. What prompted Serge's resignation from the AMS was one particular article in the *Notices* about the mathematics of HIV.<sup>7</sup> This was most unfortunate, because it was an abrupt break with an organization that he had supported through his whole career, over one particular incident. So it was certainly not the case that Serge was opposed to the AMS in general.

**Notices:** *He was an interesting character.*

**Ribet:** He really was. I still have stacks of his files in my office. When my office was renovated, I had a great deal of fun filling up barrels with paper to be recycled or shredded. But I kept all of Lang's files.

### Budding Sommelier?

**Notices:** *In 1988 you were named Vigneron d'Honneur de St. Emilion. What does this mean?*

**Ribet:** I have been interested in wine ever since I was a graduate student and then a Sloan Fellow in Paris. From the beginning, I had very strong opinions about the quality of wines. I could discern all kinds of tastes in wines and was completely impassioned. In Paris, through a chance encounter, I met someone who was a sommelier, and I ended up becoming an affiliate member of the Association des Sommeliers de Paris. Once I was on their mailing list, I got invited to a ton of industry tastings in Paris. There would be five or six per week. Of course I didn't go to every tasting, but I went to quite a few of them. They would typically be held at 4 pm or 5 pm in the afternoon, between the two meal services at restaurants. It was a fantastic education. I became acquainted with every single alcoholic product on the French market at the time—this was the late 1980s.

During this period, some of the mathematicians in Bordeaux felt that it was incredibly unusual to have a professional mathematician who also had a professional education in wine. So they brought my name to the attention of the people in St. Emilion who have an annual induction ceremony for honorary winemakers. Most of these winemakers are wine critics from the newspapers, or industry figures—people who have served in the wine

profession for many years. Somehow, they snuck my name in. I got to wear a strange cloak and participate in all sorts of amazing rituals. I really felt like an impostor! After the induction ceremony, we were invited to a seven-course banquet, and for every course they would bring out about a dozen St. Emilion wines.

**Notices:** *A dozen for each course?*

**Ribet:** A dozen for each course. There were many glasses on the table, and I came close to sampling all of the wines. A few years later I went back to one of their

*When I was a graduate student, I had a moment of doubt and thought about becoming a disc jockey.*

banquets with my wife, who was also completely amazed. I've never had anything comparable in my experience.

I no longer have the palate of the professional, and I no longer follow everything that's available, but I still have pretty strong preferences when looking at wine lists in restaurants. I typically go for wines that are exceptional values in a given price range.

**Notices:** *Did you think about becoming a sommelier?*

**Ribet:** I never thought about becoming a sommelier. But when I was a graduate student, I had a moment of doubt: "Why am I doing this? I'll never do any research. Even if I write a thesis, it won't be very good." I thought about becoming a disc jockey, because when I was an undergraduate I spent all my time on the campus radio. So I turned to my friends who had gone into the radio industry and asked if they had a job for me. They said, "You're really good in math, why don't you stick with that? The radio industry is pretty hard." So I decided to try again.

At the end of my graduate career, I was very surprised that people were actually offering me jobs. I thought, "Who cares? I'm doing this thing with abelian varieties, it's not likely to attract any attention."

**Notices:** *So everybody goes through those doubts.*

**Ribet:** Absolutely. Doing mathematics is just a permanent situation of being wracked with self-doubt. Where is my next theorem? Am I still any good? Is this proof really right? Will anyone care about it?

**Notices:** *Those are comforting words for the young people entering the profession. They know they are not alone.*

**Ribet:** You bet!

**Notices:** *And will you be in charge of the wine at the AMS banquet?*

**Ribet:** I'll be glad to help if they need me.

### Photo Credits

Photo of Ken Ribet with students is courtesy of Jerry Fowler.

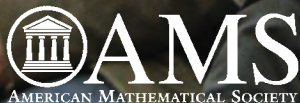
Photo of Ken Ribet is courtesy of Gauri Powale.

<sup>7</sup>"Using Mathematics to Understand HIV Immune Dynamics," *Notices*, February 1996; see also Lang's response on receiving the 1999 Steele Prize for Exposition, *Notices*, April 1999.

# *100 years from now you can still be advancing mathematics.*

When it's time to think about your estate plans, consider making a provision for the American Mathematical Society to extend your dedication to mathematics well into the future.

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# Math in Brazil: Sowing the Seeds

*Marcelo Viana*

Brazil is soon to host the first International Congress of Mathematicians ever to be held in the Southern Hemisphere, ICM 2018. About a year before that, Rio de Janeiro will also be the stage for the first International Mathematical Olympiad to happen in the country, IMO 2017. Taking advantage of this circumstance, the mathematical community is promoting the Biennium of Mathematics 2017–2018, a broad and ambitious initiative sponsored by the National Congress to disseminate and popularize mathematics in the whole society—children and their families, students and their teachers.

All this pays tribute to the remarkable development experienced by Brazilian mathematics over the last decades. And none of it could have been foreseen only a generation ago, back in 1986 when I joined IMPA—Instituto de Matemática Pura e Aplicada for grad school. Back then,



**Figure 1.** Rio de Janeiro will host the International Mathematical Olympiad in 2017 and the International Congress of Mathematicians in 2018.

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**Figure 2.** The first Brazilian Mathematical Colloquium was held in 1957.

should anyone have told us that in less than three decades a young Brazilian raised and educated in the country would win the Fields Medal, we would all have smiled in utter disbelief. Except, perhaps, for then seven-year-old Artur Avila himself.

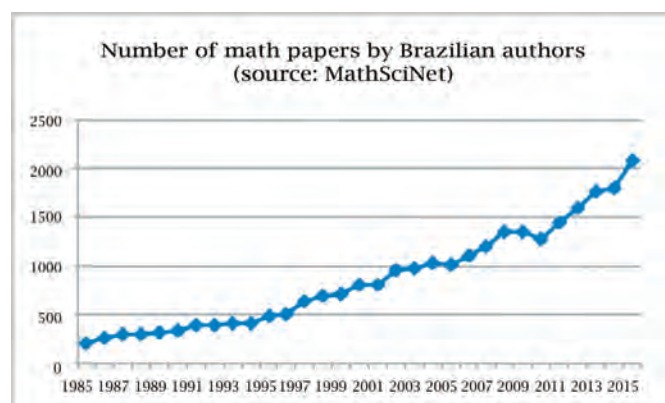
Because the fact is that mathematics, and science in general, in Brazil is a very young idea. Brazilians love to blame it on the kings of Portugal. The conventional story goes that a jealous colonial power, eager to prevent the development and spread of “subversive ideas,” forbade the printing and circulation of journals and newspapers, thus hampering the development of a knowledgeable society. A more nuanced truth is that many such policies remained in place after independence under the two emperors and, to a lesser extent, even after the republic had replaced the monarchy. As a result, most scientific institutions were created rather late. Besides, not surprisingly, the earlier ones were concerned with such issues as tropical diseases, sanitation, and public health. Mathematics was not a priority at that stage.

The first person to go on record as a mathematical researcher was Joaquim Gomes de Souza, born in 1829 in the northeastern state of Maranhão. After getting a degree from the Military School in Rio de Janeiro, the only institution in the country offering higher education in mathematics, Souza traveled to Paris, where he presented his

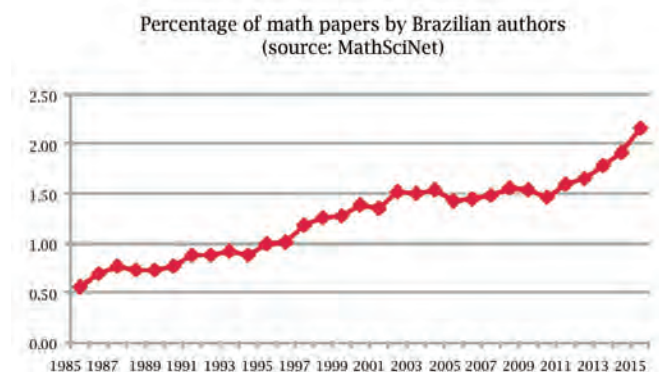
mathematical papers to the Académie des Sciences. Deeply disappointed with the lack of response (the committee, chaired by Cauchy, never even met), Souza returned to his country, quit mathematics for politics and poetry, and died at the age of thirty-five, leaving no academic heirs.

Another century would pass before regular activities in mathematics could take off in the 1950s. That is when Brazil joined the IMU (International Mathematical Union), the Brazilian Mathematical Colloquium was first held (see Figure 2), and a number of important institutions were founded, including IMPA. By then a network of public universities was in place around the country, most notably the University of São Paulo and the University of Brazil in Rio de Janeiro. Leopoldo Nachbin and Mauricio Peixoto graduated in engineering from the latter and went on to co-found IMPA and to be the first Brazilian mathematicians giving invited addresses at the ICM in 1962 and 1974, respectively.

The contribution of Brazilian mathematicians to the ICMs has become more and more regular, with a current total of three plenary lectures and fourteen invited addresses. A record was broken in Seoul 2014 with one plenary (Fernando Codá Marques) and three invited speakers (in dynamics, geometry, and probability).



**Table 1. The number of mathematics papers by Brazilian authors has risen from 255 in 1985 to 2,076 in 2015.**



**Table 2. The percentage of mathematics papers by Brazilian authors has risen from a half percent in 1985 to over 2 percent today.**



**Figure 3. Students of the indigenous school Tuparã, at Nova Ubiratã, state of Mato Grosso, participating in the 2015 Mathematical Olympiad.**



**Figure 4. Students participating in the 2014 Mathematical Olympiad: in the front row, triplets and gold medalists Fabia, Fabiele, and Fabiola Monteiro.**

All this reflects the rapid growth of mathematical research that is so clearly displayed in Tables 1 and 2: Brazil now accounts for about 2.1 percent of the world's total production. As a comparison, both the GDP and the population of Brazil are around 2.9 percent of the world's total.

There are now nearly sixty graduate programs in mathematics and statistics. They train an increasing number of Brazilian students and a substantial number of foreigners, especially from Latin America and, increasingly, Asia, Europe, and North America. A major concern is that, despite steady growth, the total number of doctoral degrees granted in mathematics and statistics every year (about 180 currently) still falls short of the needs presented by Brazil's expanding university system, not to mention industry. The difficulty in attracting good graduate students is also a reflection of deeper problems in basic education.

Indeed, official data about the performance of students in elementary, middle, and high schools just released by the Ministry of Education show that, while there has been



**Figure 5.** At the 2007 Mathematical Olympiad award ceremony, Brazil president Lula pays tribute to Ricardo Oliveira, a survivor of spinal amyotrophy, who was carried to school in a wheelbarrow and went on to win a gold medal. Ricardo will soon graduate in industrial mechatronics.

some improvement in early years, progress at the high school level has stalled. Results in the PISA (Programme for International Student Assessment), an international performance test held every third year for fifteen-year-old students, are equally unflattering. Despite some modest progress, Brazil still performs well below the OECD (Organization for Economic Cooperation and Development) average. In fact, in 2012 Brazil ranked 58th among the 65 countries taking the test. These facts prompted the federal government to propose a reform of high school education, which is currently being discussed in Congress.

The SBM (Brazilian Mathematical Society) was founded in 1969. Other mathematical societies have been established since: the Brazilian Society for Applied and Computational Mathematics (SBMAC), the Brazilian Statistics Association, the Brazilian Society for Mathematical Education, and the Brazilian Society for the History of Mathematics.

The SBM has about two thousand associates and is a nonprofit publishing house for mathematical books and journals. It also runs several initiatives of broad interest,

*Brazilian  
mathematics  
was born  
open to  
international  
cooperation.*

such as PROFMAT, the nationwide master's program for school-teachers. PROFMAT operates through a nationwide network of over seventy universities and institutes in all twenty-seven Brazilian states and has already granted Master's degrees to more than 2,600 high school teachers. It is also helping bring universities and schools together

for a dialogue that has been missing for decades and is crucial for dealing with the problems of education, especially at the high school level.

Brazil has two major mathematical Olympiads. The OBM (Brazilian Mathematical Olympiad) was created in 1979 at the initiative of the SBM, promoting regional and national mathematical competitions involving about 500,000 students from middle school to college level as well as Brazil's very successful participation in International Mathematical Olympiads.

The OBMEP (Brazilian Mathematical Olympiad for Public Schools) was started by IMPA and the federal government in 2005 and now reaches about eighteen million secondary school students every year. Over little more than a decade, it has become a major event in our academic calendar and one with a particularly high profile. As a sign of its prestige, the award ceremony for the gold medalists is usually chaired by the president of Brazil. Efforts are currently under way to integrate the two competitions more effectively, starting in 2017.

Brazilian mathematics was born open to international cooperation by necessity as much as by design. IMPA keeps a vigorous visitor program totaling over eight hundred visitor-months per year. A highlight is the cycle of meetings run by the SBM and the SBMAC jointly with their counterparts in some of our major partner countries. The first Brazil-Spain meeting was held in Fortaleza in December 2015, and the Brazil-Italy meeting hosted by IMPA in August 2016 was also remarkably successful. Preparations for the Brazil-France meeting in 2019 are well advanced. The idea of a joint meeting with the United States in 2020 has been floated, and conversations were initiated with our Portuguese colleagues, having in mind that 2022 will mark the 200th anniversary of Brazil's independence from Portugal.

Preparations for the ICM 2018 and the IMO 2017, as well as for the IMU General Assembly 2018, in São Paulo, are well under way.

Rio de Janeiro has gotten used to hosting major international events: the 2013 Catholic Journey of Youth,



**Figure 6.** Around 300 mathematicians attended the Brazil-Italy joint meeting at IMPA in August 2016.

the 2014 FIFA World Cup, and the 2016 Olympic and Paralympic Games, just to mention the largest ones. Major infrastructure was overhauled in several parts of town, most especially in the western neighborhood of Barra da Tijuca, pictured in Figure 1, where ICM 2018 will take place. Barra da Tijuca was a major site for the Olympic Games and now boasts over 12,000 hotel rooms and a brand-new transportation system.

Following a rough year in 2015 and despite an economic situation that remains adverse, the general mood in Brazil has been improving substantially as the country's political and economic troubles seem to be subsiding. Both the achievements of Brazilian mathematicians and the shortcomings of mathematical education in the country regularly put mathematics in the headlines, which makes the timing for the upcoming IMO and ICM especially fortunate. All this makes me confident that we will have a great Biennium of Mathematics 2017–2018 and that it will sow the seeds of a new era for mathematics in Brazil. I'm really looking forward to the official launching of the Biennium at the first Brazilian Math Festival, April 27–30, 2017. Check out the website [festivaldamatematica.org.br](http://festivaldamatematica.org.br) and come participate!

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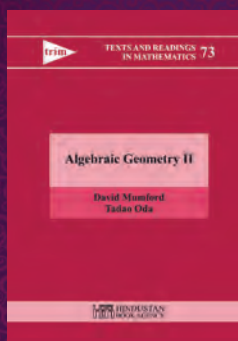


Marcelo Viana

### ABOUT THE AUTHOR

**Marcelo Viana**, director of IMPA, received the inaugural ICTP Ramanujan Prize in 2005 and served as president of the Brazilian Mathematical Society from 2013 to 2015. Being a father of two (six and nine) is reintroducing him to mathematics from a whole new angle.

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**David Mumford**, *Brown University, Providence, RI*, and  
**Tadao Oda**, *Tohoku University, Japan*

Several generations of students of algebraic geometry have learned the subject from David Mumford's fabled "Red Book," which contains notes of his lectures at Harvard University. Their genesis and evolution are described by Mumford in the preface.

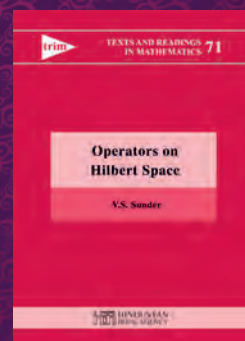
2015; 516 pages; Hardcover; ISBN: 978-93-80250-80-9; List US\$76; AMS members US\$60.80; Order code HIN/70

### OPERATORS ON HILBERT SPACE

**V. S. Sunder**, *Institute of Mathematical Sciences, Chennai, India*

This book's principal goals are: (i) to present the spectral theorem as a statement on the existence of a unique continuous and measurable functional calculus, (ii) to present a proof without digressing into a course on the Gelfand theory of commutative Banach algebras, (iii) to introduce the reader to the basic facts concerning the various von Neumann-Schatten ideals, the compact operators, the trace-class operators and all bounded operators, and finally, (iv) to serve as a primer on the theory of bounded linear operators on separable Hilbert space.

2015; 110 pages; Softcover; ISBN: 978-93-80250-74-8; List US\$40; AMS members US\$32; Order code HIN/69



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## Ryan Haskett Interview



**Ryan Haskett is a mathematician and finance professional and currently about halfway through a year of extreme vagrancy. He and his partner have just finished an RV tour of the Rocky Mountains and are off with backpacks to Europe and North Africa.**

**Diaz-Lopez:** When/how did you know you wanted to be a mathematician?

**Haskett:** That is a hard question for me. I'm not sure I remember a time when I wasn't interested in math or computer science. Games were the early draw for understanding computers, as in those days often a good amount of expertise was required just to get games working on a PC.

In math, maybe the draw was orderliness. I like how systems work together. Also, at my college (at the time) the professors of mathematics were much more engaging than the professors of computer science.

**Diaz-Lopez:** Who encouraged or inspired you (mathematically or otherwise)?

**Haskett:** My parents for sure. I had good high school teachers who were very supportive and many college professors who took the teaching aspect of their jobs very seriously. My PhD advisor was instrumental in keeping me in grad school.

**Diaz-Lopez:** How would you describe your work to a graduate student?

**Haskett:** I'm currently taking a career break and traveling, but let me talk about my last job. My main goal was to build a system that would minimize the effects of random currency price spikes on our global portfolio of investments, at a reasonable cost. Building and improving this semi-automated system kept me fairly busy but also left time to explore some research questions for the firm. For instance, what are metrics to judge whether a hedge fund was good, or just lucky? Or, how do you understand the interactions between really long-term investments, like private equity, and more-common financial investments, like stocks and bonds?

I mainly use linear algebra and scientific computing and little else to directly justify all the differential equations and asymptotics I studied in graduate school. Still, my time as a grad student taught me how to work with messy data sets and also apply results from academic papers, even from other fields. These skills have been extremely useful in finance.

**Diaz-Lopez:** You finished your PhD, and shortly afterward you started working in the financial sector. What message would you give to doctoral students and professional mathematicians thinking about having a career in this sector?

**Haskett:** The first thing to note is just how easy the transition was. I had taken only one economics class during college and grad school combined, but companies were more than willing to help to get more employees with strong mathematical backgrounds. The Chartered Financial Analyst (CFA) program was useful as well, and most companies will cover the fees. The CFA designation is billed as an entry to a community of financial professionals. However, the financial community seems to think of it as a series of three hard tests that are meant to create a class of people who passed, versus a class who did not. That sounds fairly awful, but what the CFA does exceptionally well is assemble the whole background one needs to

DOI: <http://dx.doi.org/10.1090/noti1492>

# THE GRADUATE STUDENT SECTION

really get into finance and a series of tests to make sure you learn it thoroughly.

To answer your question: obviously, the salaries are fairly exceptional, but the big draw for me was the time-frame of projects. In grad school, projects took years, and I would often get bored working on the same thing for that long. In finance, projects usually take weeks to months, which is enough time to really dive into an area, but not too long to make it tedious. Also, the feedback on your work tends to be fast and numerically obvious.

**Diaz-Lopez:** All mathematicians feel discouraged occasionally. How do you deal with discouragement?

**Haskett:** This seems unrelated, but regular exercise and sufficient sleep seem to make the real difference between those who succeed and those who don't, both as grad students and professionals. It works wonders for people's mental states. A lot of research has come out recently showing the variety of benefits from both.

**Diaz-Lopez:** *If you were not a mathematician, what would you be?*

**Haskett:** Computer programmer for sure, probably in the Valley.

**Diaz-Lopez:** *If you could recommend one lecture (book, paper, article, etc.) to graduate students, what would it be?*

**Haskett:** For fun, I love the "Crash Course World History" lecture series on YouTube.

## Photo Credit

Photo of Ryan Haskett is courtesy of Ryan Haskett.

## Looking Back on a Year of Interviews

The Graduate Student Section was launched with the January 2017 issue of the *Notices*. Since then, each installment of the section has included an interview with a mathematician. Below is a list of the Graduate Student Section interviews that have appeared to date.

Comments and suggestions for future interviews are welcome. Please post comments on the *Notices* website at [www.ams.org/notices](http://www.ams.org/notices). Please send suggestions for interviews, or for other topics you would like to see covered in the Graduate Student Section, to [noti-gradsec@ams.org](mailto:noti-gradsec@ams.org).

January 2016: Ian Agol

February 2016: Fernando Codá Marques

March 2016: Elisenda Grigsby

April 2016: Arlie Petters

May 2016: Melanie Wood

June/July 2016: Jordan Ellenberg

August 2016: Helen Moore

September 2016: Po-Shen Loh

October 2016: Timothy Gowers

November 2016: Colin Adams

December 2016: Gigliola Staffilani

January 2017: Arthur Benjamin

February 2017: Tom Grandine



**Alexander Diaz-Lopez**, having earned his PhD at the University of Notre Dame, is now visiting assistant professor at Swarthmore College. Diaz-Lopez was the first graduate student member of the *Notices* Editorial Board.

# ? WHAT IS...

## an Elliptic Curve?

Harris B. Daniels and Álvaro Lozano-Robledo

Communicated by Steven J. Miller and Cesar E. Silva

Elliptic curves are ubiquitous in number theory, algebraic geometry, complex analysis, cryptography, physics, and beyond. They lie at the forefront of arithmetic geometry, as shown in the feature on Andrew Wiles and his proof of Fermat's Last Theorem that appears in this issue of the *Notices*. The goal of arithmetic geometry, in general, is to determine the set of  $K$ -rational points on an algebraic variety  $C$  (e.g., a curve given by polynomial equations) defined over  $K$ , where  $K$  is a field, and the  $K$ -rational points, denoted by  $C(K)$ , are those points on  $C$  with coordinates in  $K$ . For instance, Fermat's Last Theorem states that the algebraic variety

$$X^n + Y^n = Z^n$$

has only trivial solutions (one with  $X$ ,  $Y$ , or  $Z = 0$ ) over  $\mathbb{Q}$  when  $n \geq 3$ . Here we will concentrate on the case of a 1-dimensional algebraic variety, that is, a curve  $C$ , and a number field  $K$  (such as the rationals  $\mathbb{Q}$  or the Gaussian rationals  $\mathbb{Q}(i)$ ). Curves are classified by their geometric genus as complex Riemann surfaces. When the genus of  $C$  is 0, as for lines and conics, the classical methods of Euclid, Diophantus, Brahmagupta, Legendre, Gauss, Hasse, and Minkowski, among others, completely determine the  $K$ -rational points on  $C$ . For example,

$$C_1 : 37X + 39Y = 1 \text{ and } C_2 : X^2 - 13Y^2 = 1$$

have infinitely many rational points that can be completely determined via elementary methods. However, when the genus of  $C$  is 1, we are in general not even able to decide whether  $C$  has  $K$ -rational points, much less determine all the points that belong to  $C(K)$ .

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Álvaro Lozano-Robledo is associate professor at the University of Connecticut. His e-mail address is [alvaro.lozano-roblado@uconn.edu](mailto:alvaro.lozano-roblado@uconn.edu).

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Figure 1. A curve of genus 1 over the complex numbers is a Riemann surface with one hole.

For example, the curve

$$C : 3X^3 + 4Y^3 = 5$$

has no  $\mathbb{Q}$ -rational points, but the local methods we use in the genus 0 case to rule out global points fail here.<sup>1</sup> A goal of the theory of elliptic curves is to find all the  $K$ -rational points on curves of genus one.

*The study of elliptic curves grew in the 1980s.*

An **elliptic curve**  $E$  is a smooth projective<sup>2</sup> curve of genus 1 defined over a field  $K$ , with at least one  $K$ -rational point (i.e., there is at least one point  $P$  on  $E$  with coordinates in  $K$ ). If the field  $K$  is of characteristic 0 (e.g.,

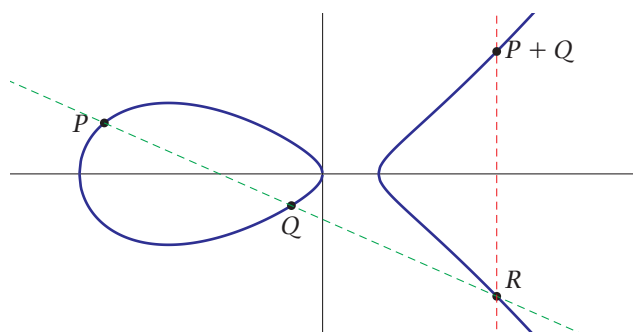
<sup>1</sup> $C : 3X^3 + 4Y^3 = 5$  is an example of Selmer where the local-to-global principle fails. This means that there are points on  $C$  over every completion of  $\mathbb{Q}$ —i.e., over  $\mathbb{R}$  and the  $p$ -adics  $\mathbb{Q}_p$  for every prime  $p$ —but not over  $\mathbb{Q}$  itself.

<sup>2</sup>Curves are considered in projective space  $\mathbb{P}^2(K)$ , where, in addition to the affine points, there may be some points of the curve at infinity.

number fields) or characteristic  $p > 3$ , then every elliptic curve can be given by a nice choice of coordinates, called a *short Weierstrass model*, of the form

$$E: y^2 = x^3 + Ax + B,$$

with  $A$  and  $B$  in  $K$  (and  $4A^3 + 27B^2 \neq 0$  for smoothness). In this model there is only one  $K$ -rational point *at infinity*, denoted by  $\mathcal{O}$ . One aspect that makes the theory of elliptic curves so rich is that the set  $E(K)$  can be equipped with an Abelian group structure, geometric in nature (see Figure 2), where  $\mathcal{O}$  is the zero element (in other words, elliptic curves are 1-dimensional Abelian varieties).



**Figure 2.** The addition law on an elliptic curve.

The Abelian group  $E(K)$  was conjectured to be finitely generated by Poincaré in the early 1900s and proved to be so by Mordell for  $K = \mathbb{Q}$  in 1922. The result was generalized to Abelian varieties over number fields by Weil in 1928 (a result widely known as the *Mordell-Weil Theorem*). The classification of finitely generated Abelian groups tells us that  $E(K)$  is the direct sum of two groups: its torsion subgroup and a free Abelian group of rank  $R \geq 0$ , i.e.,

$$E(K) \cong E(K)_{\text{tors}} \oplus \mathbb{Z}^R.$$

We then call  $R = R_{E/K}$  the *rank* of the elliptic curve  $E/K$ . For instance, for

$$E: y^2 + y = x^3 + x^2 - 10x + 10,$$

the group  $E(\mathbb{Q})$  is generated by  $P = (2, -2)$  and  $Q = (-4, 1)$ . Here  $P$  is a point of order 5 and  $Q$  is of infinite order, and so  $E(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}$ .

Which finitely generated Abelian groups can arise as the group structure of an elliptic curve over a fixed field  $K$ ? The possible torsion subgroups  $E(K)_{\text{tors}}$  that can occur have been determined only when  $K = \mathbb{Q}$ , or when  $K$  is a quadratic or cubic number field (e.g.,  $K = \mathbb{Q}(i)$ , or  $K = \mathbb{Q}(\sqrt[3]{2})$ ). For  $K = \mathbb{Q}$ , the list of torsion subgroups was conjectured by Levi in 1908, later reconjectured by Ogg in 1970, and finally proved in 1976 by Mazur:

$$E(\mathbb{Q})_{\text{tors}} \cong \begin{cases} \mathbb{Z}/N\mathbb{Z} & \text{for } 1 \leq N \leq 10, \text{ or } N = 12, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2M\mathbb{Z} & \text{for } 1 \leq 1M \leq 4. \end{cases}$$

In contrast, the list of possible ranks  $R_{E/K}$  is completely unknown, even over  $\mathbb{Q}$ . We do not even know if this list is finite or infinite for any fixed number field. The largest rank known over  $\mathbb{Q}$  is 28, for a curve found by Elkies.

The open questions about the rank of an elliptic curve are central to what makes the  $K$ -rational points on elliptic curves so hard to determine. The difficulty arises from the failure of the local-to-global principle (or Hasse principle) on curves of genus greater than or equal to 1 (see footnote 1). For an elliptic curve  $E/K$ , one defines the Tate-Shafarevich group  $\text{III} = \text{III}(E/K)$  to measure the failure of the Hasse principle on  $E$ . In a sense,  $\text{III}$  plays the role of the ideal class group of a number field. However, we do not know that  $\text{III}(E/K)$  is always a finite group.<sup>3</sup> If we knew that  $\text{III}$  is always finite, then a method Fermat inaugurated, called *descent*, would presumably yield an algorithm to determine all the  $K$ -rational points on  $E$ .

In the 1960s, Birch and Swinnerton-Dyer conjectured an analytic approach to computing the rank of an elliptic curve. Later, their conjecture was refined in terms of the Hasse-Weil  $L$ -function of an elliptic curve  $E$  (over  $\mathbb{Q}$  for simplicity), which is defined by an Euler product:

$$L(E, s) = \prod_{p \text{ prime}} L_p(E, p^{-s})^{-1},$$

where  $L_p(E, T) = 1 - a_p T + pT^2$  for all but finitely many primes,  $a_p = p + 1 - \#E(\mathbb{F}_p)$ , and  $\#E(\mathbb{F}_p)$  is the number of points on  $E$  considered as a curve over  $\mathbb{F}_p$ . Thus defined,  $L(E, s)$  converges as long as  $\text{Re}(s) > 3/2$ . In fact, Hasse conjectured more: any  $L$ -function of an elliptic curve over  $\mathbb{Q}$  has an analytic continuation to the whole complex plane. This has now been proved as a consequence of the modularity theorem that we discuss below. The Birch and Swinnerton-Dyer conjecture (BSD) claims that the order

*An elliptic curve can be equipped with an Abelian group structure.*

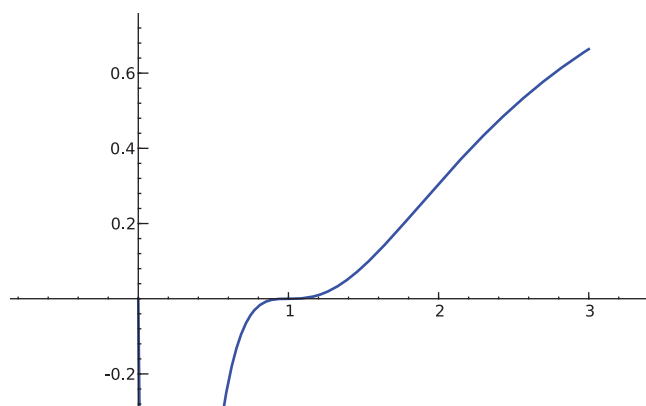
of vanishing of  $L(E, s)$  at  $s = 1$  is equal to  $R_{E/\mathbb{Q}}$ , the rank of  $E(\mathbb{Q})$ . In fact, the conjecture also predicts the residue at  $s = 1$  in terms of invariants of  $E/\mathbb{Q}$ .

For instance, the curve  $E: y^2 + y = x^3 - 7x + 6$  is of rank 3, with  $E(\mathbb{Q}) \cong \mathbb{Z}^3$ , and the graph of  $L(E, x)$  for  $0 \leq x \leq 3$  is displayed in Figure 3. The BSD conjecture is known to hold only in certain cases of elliptic curves of rank 0 and 1, by work of Coates and Wiles, Gross and Zagier, Kolyagin, Rubin, Skinner and Urban, among others. However, Bhargava, Skinner, and Zhang have shown that BSD is true for at least 66 percent of all elliptic curves over the rationals.

The study of elliptic curves grew in popularity in the 1980s when Hellegouarch, Frey, and Serre outlined a road map to prove Fermat's Last Theorem by proposing that a certain elliptic curve cannot exist. Roughly speaking, if  $p \geq 11$  and  $a^p + b^p = c^p$  is a nontrivial solution of Fermat's equation  $X^p + Y^p = Z^p$ , then the so-called Frey-Hellegouarch curve  $y^2 = x(x - a^p)(x + b^p)$  would have two properties thought to be contradictory. First, the curve would be *semistable*, which is a mild technical

<sup>3</sup>The finiteness of  $\text{III}$  is known only in certain cases with rank  $\leq 1$ , by work of Kolyagin and Rubin.

condition about the type of curves  $E/\mathbb{F}_p$  that we get by reducing the coefficients of  $E$  modulo  $p$ . Second, the curve would be *modular*, a property we explain in the next paragraph. The statement that the Frey–Hellegouarch curve is semistable but not modular was first formalized by Serre and then proved by Ribet. With Ribet’s result, to prove Fermat’s Last Theorem, one needed to prove “only” that all semistable elliptic curves over  $\mathbb{Q}$  are modular. This statement emerged in the 1950s and is sometimes called the modularity conjecture, or Taniyama–Shimura–Weil conjecture.<sup>4</sup> The modularity conjecture relates two seemingly very distinct objects: elliptic curves and modular forms.



**Figure 3.** An  $L$ -function  $L(E, x)$  with a zero of order 3 at  $x = 1$ .

A modular form is a complex-analytic function  $f$  on the upper-half complex plane that satisfies certain symmetries. In particular,  $f(s)$  admits a Fourier series expansion  $f(s) = \sum_{n \geq 0} a_n q^n$ , where  $q = e^{2\pi i s}$ , and we can attach to the modular form  $f$  an  $L$ -function  $L(f, s) = \sum_{n \geq 1} a_n / n^s$ . The modularity conjecture says that every elliptic curve  $E$  is associated to a modular form  $f$  such that  $L(E, s) = L(f, s)$ ; i.e., their  $L$ -functions coincide. In particular, this implies that  $L(E, s)$  has an analytic continuation to  $\mathbb{C}$ , because  $L(f, s)$  is known to have one. In 1993 Wiles [2] announced a proof of the modularity conjecture in the semistable case, but a flaw was found in the proof, which was fixed in 1995 by Taylor and Wiles. In 2001 the full conjecture was proved for all elliptic curves over  $\mathbb{Q}$  by Bruel, Conrad, Diamond, and Taylor. In 2015

*Modularity fits into a much larger context, together with the Langlands program and the Fontaine–Mazur conjecture.*

Freitas, Le Hung, and Siksek extended the modularity theorem to real quadratic fields. Modularity fits into a much larger context, together with the Langlands program and the Fontaine–Mazur conjecture, which was described in Mark Kisin’s “What Is a Galois Representation?” (*Notices*, June/July 2007).

The canonical starting point for a graduate student interested in learning more about elliptic curves is Silverman’s *The Arithmetic of Elliptic Curves* [1]. A more elementary approach is Silverman and Tate’s *Rational Points on Elliptic Curves*.

## References

- [1] J. H. SILVERMAN, *The Arithmetic of Elliptic Curves*, 2nd Edition, Springer-Verlag, New York, 2009. MR2514094
- [2] ANDREW WILES, Modular elliptic curves and Fermat’s last theorem, *Ann. of Math.* **141** (1995), no. 3, 443–551. MR1333035

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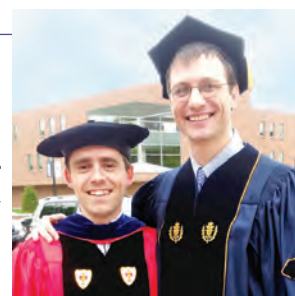
Figure 1/public domain, [https://commons.wikimedia.org/wiki/File:Torus\\_illustration.png](https://commons.wikimedia.org/wiki/File:Torus_illustration.png)

Figure 3 was created by Álvaro Lozano-Robledo with SageMath.

Photo of Harris B. Daniels and Álvaro Lozano-Robledo is by Keith Conrad, courtesy of Álvaro Lozano-Robledo.

## ABOUT THE AUTHORS

**Harris B. Daniels** (right) received his PhD in mathematics in 2013 from the University of Connecticut and is currently assistant professor at Amherst College in Massachusetts.



**Álvaro Lozano-Robledo** (left) and Harris B. Daniels is a father of two real children, one academic child, and is expecting a second (academic) descendent in spring 2017. He is the author of *Elliptic Curves, Modular Forms, and Their L-functions* (AMS, 2011) and has published over twenty-five research articles related to the theory of elliptic curves.

<sup>4</sup>See Lang’s article in the *Notices*, November 1995, for a detailed historical account of the modularity conjecture.

# Math in Moscow Scholarship Program

## Study mathematics the Russian way in English

The American Mathematical Society invites undergraduate mathematics and computer science majors in the U.S. to apply for a special scholarship to attend a semester in the Math in Moscow program, run by the Independent University of Moscow.

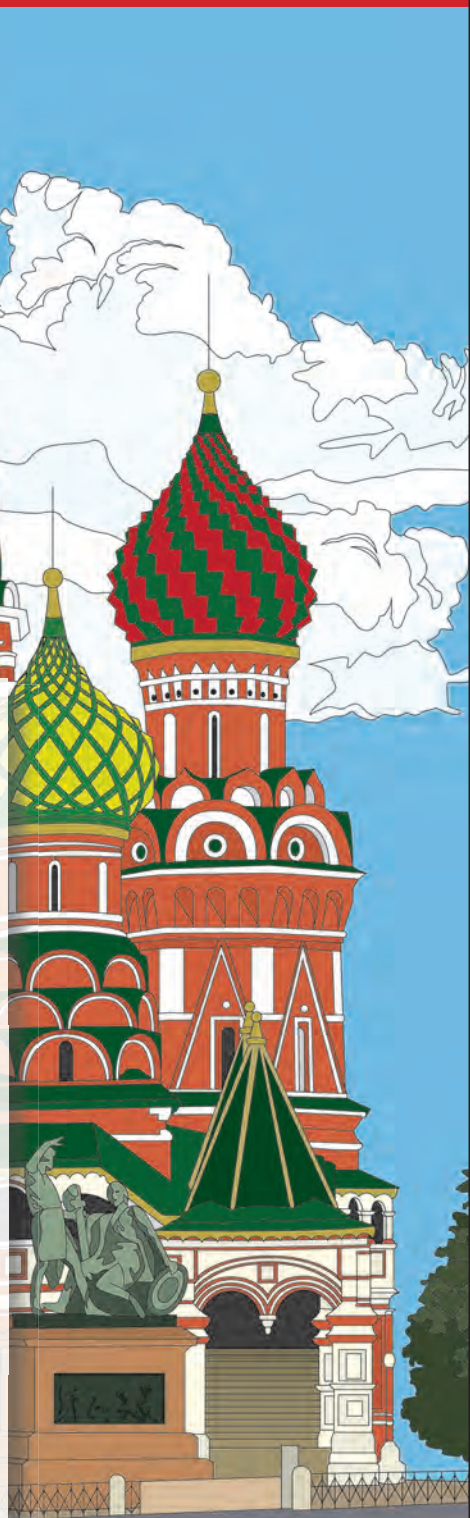
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# The *Common Vision* Project: Four Reactions

*Communicated by Harriet Pollatsek*

*Note: The opinions expressed here are not necessarily those of Notices.  
Responses on the Notices webpage are invited.*

The *Common Vision* project seeks to modernize undergraduate programs in the mathematical sciences, especially courses taken in the first two years. It is a joint effort of the American Mathematical Association of Two-Year Colleges, the American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics. Their report<sup>1</sup> reflects a synthesis of common themes in seven curricular guides published by these five associations along with more recent research and input.

The report calls on the community to:

- (1) update curricula,
- (2) articulate clear pathways between curricula driven by changes at the K-12 level and the first courses students take in college,
- (3) scale up the use of evidence-based pedagogical methods,
- (4) find ways to remove barriers facing students at critical transition points (e.g., placement, transfer), and
- (5) establish stronger connections with other disciplines.

It urges institutions to provide faculty with training, resources, and rewards for their efforts to meet these goals.

The *Notices* asked four mathematicians to write short pieces summarizing their reactions to the *Common Vision* report. These pieces appear below.



**A group of students from the University of Arizona with William Yslas Vélez at Montezuma Castle National Monument. They were en route to the 2011 SUNMARC conference at Northern Arizona University.**

## *William Yslas Vélez*

A document that brings together the thoughts of different mathematical societies into one united vision is certainly useful. This is especially important when these societies express a common vision. The data presented in the

<sup>1</sup>[www.maa.org/common-vision](http://www.maa.org/common-vision)

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DOI: <http://dx.doi.org/10.1090/noti1482>

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introduction are compelling and serve notice that much needs to be done to be more inclusive.

*There appears to be a disconnect between what the societies think is important and how rank-and-file faculty operate.*

As someone who has devoted considerable energy to increasing diversity in mathematics and increasing the number of mathematics majors, I am well aware of the content described in the first thirty pages of this report. There was only one piece of information that I found surprising: “A primary point emphasized by all the guides is that *the status quo is unacceptable.*”

My opinion is that faculty do not seem to have bought into such a strong statement. There appears to be a disconnect between what the societies think is important and how rank-and-file faculty operate. I give talks to many mathematics departments and regularly am part of academic program reviews of departments, yet I don’t encounter such strong feelings expressed during those interactions.

This difference is reflected in the section on “Scaling.” The authors pose three questions concerning the issue of sustainability/transportability of successful programs. I think that to these three questions must be added the *first* question: “How do we get departments interested in these questions?” How do we convince departments that they should reallocate their limited resources to address these questions when they have so many other pressing matters to address?

The last section on “Moving Forward” is of central importance. Indicating what the problems are is the first step; the second step should be how to mobilize departments to be concerned about finding solutions to these problems. Then and only then will we be able to begin to develop solutions.

In looking toward the future, we recognize that departments have different missions and that the issues they choose to address will be different, depending upon their goals. Bachelor’s-granting institutions are not the same as doctoral-granting institutions. However, they have one thing in common. Doctoral mathematical science departments train all these faculty, and it is this training that should be central to moving forward.

In an opinion piece in the 2015 June/July *Notices*, I spoke of gathering data on our graduate programs, and I want to promote that idea again here because of its impact on creating the next generations of faculty. Departments should record where their graduates are going. If these graduates are going into the teaching profession, then their training should reflect this path. On-the-job training in the many facets indicated in this *Common Vision* is simply not practical. If all a student sees in their graduate training is the standard lecture format and lack of

technology in the teaching of mathematics, how do we expect these same new faculty to adopt new teaching techniques? Those departments that have a higher percentage of graduates pursuing teaching should be taking the lead in providing the training that is necessary to move forward.

*Amy Cohen*

Although “workforce preparation and faculty development” is one of the four areas identified as requiring “significant further action,” it is not until page 26 that the *Common Vision* report mentions changes in instructional staffing over the last two decades that challenge the achievement of the project’s goals and more generally challenge our profession.

Tenured research-active faculty members now teach a smaller proportion of students in the calculus sequence, linear algebra, and differential equations.<sup>2</sup> Their place is increasingly taken by full-time non-tenure-track (FT NTT) doctoral faculty or by graduate students. Hiring in the category of FT NTT doctoral faculty by PhD-granting departments is growing. Hiring of “tenure-eligible” faculty has stagnated since 2008. Even upper-level math major courses in universities are frequently taught by FT NTT mathematicians.

Departmental culture and policy about course offerings, hiring, and promotion are still set by tenured faculty. FT NTT faculty with teaching loads of four courses per term cannot reasonably be expected to design or implement new courses, coordinate courses, do advising, and/or mentor new colleagues without support from senior faculty.

The growing enrollment in mathematics from sophomore year onward is largely driven by the lure of a growing STEM workforce. Students whose primary majors are outside math realize that a second major or a strong minor in math will enhance their preparation for graduate education and/or employment after college. Mathematicians whose undergraduate studies were in elite schools (inside or outside the United States) are often surprised that future high school teachers of mathematics enroll in upper-level math courses (e.g., analysis, algebra, geometry, and/or probability) alongside future engineers, microbiologists, and economists.

Mathematicians recognize the power of generalization and abstraction for a broad variety of applications. Undergraduates need to learn how to generalize from the concrete to the abstract and then how to specialize back to concrete problems. Pure mathematicians may have difficulty in teaching this process successfully without

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<sup>2</sup>R. Blair, E. Kirkman, J. Maxwell, Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the US, *Conference Board of the Mathematical Sciences, AMS, 2013*. See also Cohen’s article on “Disruptions of the academic math employment market” in the October 2016 *Notices*.

a culture of professional development including collaboration across disciplines. Without concurrent “faculty development” I fear that the new courses the *Common Vision* report advocates, if they find instructors at all, may fail for lack of effective instruction.

What forms might effective faculty development take? Professional societies in the mathematical sciences sponsor relevant conference sessions, short courses, and related projects. A very limited fraction of faculty (whether on tenure track or not) find institutional funding to participate. Professional societies should offer such programs also at their local and regional meetings. Individual institutions and their statewide systems could do the same. Online professional learning communities potentially offer another mode of outreach. Collaboration among university administration, departmental leadership and faculty, and funding sources is essential so that faculty development occurs and improves teaching and learning in undergraduate mathematics.

*Priscilla Bremser*

The *Common Vision* report puts forth an ambitious agenda for change in mathematics instruction. It sets expectations for administrators, departments,

*How might  
we design  
a workshop  
grounded  
in modern  
understanding of  
how people learn?*

and instructors, some of whom will need to be convinced that their participation is vital. Fortunately, the report is a collaborative effort, endorsed by all of the relevant major professional organizations. That in itself is no guarantee of success, however. The community, meaning those of us who are

eager to move into the implementation phase, will have to be intentional and creative in approaching our work. In order to modernize undergraduate programs in the mathematical sciences, we must bring modern approaches to curricular design, professional development, and outreach activities.

The verdict that “the status quo is unacceptable” won’t get much argument from the various constituencies. The data presented in the report confirm what we all know: too many college students fail mathematics courses, and too many others lose interest in mathematics. To its credit, the report does not place blame for the situation on any group. Chapter 3, “Moving Forward,” presents concrete “ways in which faculty members, deans, and other administrators can create an environment that supports improvements in undergraduate mathematical sciences education” (page 31).

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In the “Course Structure” section of the report the authors identify “a general call to move away from the use of a traditional lecture as the sole instructional delivery method in undergraduate mathematics courses.” The evidence for the efficacy of active-learning approaches is strong and growing; it should inform our professional development efforts as well. I’m encouraged that the term “workshop” appears often in Chapter 3, but I’ve participated in “workshops” that were essentially lectures and minimally effective as a result. How might we design a workshop grounded in modern understanding of how people learn? If we insist that the goal of a short course be not mere transmission of information but rather genuine cognitive engagement with the problem at hand, what changes in format and execution can we expect to see?

For that matter, how might we invite administrators to be active participants in the improvements we seek? At the end of the first year in the Vermont Mathematics Initiative, a professional development program for teachers, each teacher is required to bring a supervising administrator to a leadership training retreat. It’s intriguing to imagine a Modeling Across the Curriculum workshop on economic mobility with deans working alongside faculty and students. Instead of poster sessions for policymakers, how about bring-your-own-data service-learning projects, where statistics students and faculty work with legislators to interpret information relevant to current issues?

A corollary then to “the status quo is unacceptable” is this: traditional ways of promoting improvements in undergraduate mathematics education are not enough. Our potential collaborators in and out of the academy will not hop on board if all we offer is a passive experience, whether it’s a printed report, a PowerPoint presentation, or a MOOC. Instead, let’s invite them into carefully structured and intellectually challenging activities.

*Douglas Mupasiri*

The Department of Mathematics at the University of Northern Iowa, of which I am head, has just undergone its required septennial academic program review, and that process has brought the issues discussed in the *Common Vision* project into sharp focus. Here I discuss four of the biggest challenges I see, organized by the themes mentioned in all seven guides written by the five professional associations that participated in the *Common Vision* project. I also consider student diversity, mentioned in some but not all the guides.

*Curricula:* It is hard to see how departments will be able to modernize their curricula in the ways contemplated in the report without an infusion of new resources. Many departments will have to make some hard choices about which changes to make and which not, based on their local contexts. An essential part of any effort with a chance of succeeding will be developing appropriate curriculum models for different types of institutions, publicizing and disseminating them, and then scaling them up.

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## The onus is on the entire mathematical sciences community to heed the call.

**Course structure:** Most mathematics faculty still use the traditional lecture, but there is increasing recognition of the need for change to more active forms of instruction (Freeman et al., 2014). The recently published CBMS statement on active learning is but the latest example. The projects Progress through Calculus, Transforming-Post Secondary Education in Mathematics, and others are promising

efforts to address this challenge.

**Workforce preparation:** Should mathematical sciences departments train their students for specific jobs or concentrate their attention on providing them with the competencies and critical thinking skills that will give them the flexibility to successfully move from one job to another? The fact is: It is possible to do both. Departments can complement their existing course offerings with devices such as guided research and strategically arranged internships to design programs that aim to “narrow the gap between today’s mathematics as it is practiced in the academy, industry and government and how it is experienced in higher education’s instructional programs.”<sup>3</sup>

**Student diversity:** The authors of the *Common Vision* report write, “The fact that our community has been unable to attract and retain a diverse student population in the mathematical sciences is a dreadful shortcoming that must be remedied.” Beyond the moral case for recruiting students from groups that have been traditionally underrepresented in the mathematical sciences, including women, there is growing consensus that there is also an economic and national security imperative. But addressing this issue won’t be easy, as anyone who has tried it will attest. There are, however, some potentially hopeful signs on the horizon. Increasing press coverage of the growing income gap between the rich and the poor the world over is bringing more attention to this issue. Professional organizations are also giving greater focus to the matter. The AMS recently named Helen G. Grundman director of its newly created Department of Education and Diversity. SIAM has a Diversity Advisory Committee.

The *Common Vision* project has brilliantly laid a predicate for a compelling call to action. The onus is on the entire mathematical sciences community to heed the call. “Are we equal to the challenge?” I think yes.

Related *Notices* articles:

*A Common Vision for Undergraduate Mathematics*, by Tara Holm and Karen Saxe, June/July, 2016.

*Are Mathematics Faculty Ready for Common Vision?* by Marcus Jorgensen, November 2016.

*What Does Active Learning Mean for Mathematicians?* by Benjamin Braun, Priscilla Bremser, Art M. Duval, Elise Lockwood, and Diana White, February 2017.

Also:

AMS Blog *On Teaching and Learning Mathematics*, September 10, 20; October 1, 10, 2015.

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Photo of Priscilla Bremser is courtesy of Priscilla Bremser.

### ABOUT THE AUTHORS

**William Yslas Vélez** has a simple goal in life: to convince every student at his university to be a math major or math minor.



**William Yslas Vélez**

**Amy Cohen’s** major mathematical interests are PDEs with soliton solutions and making education in mathematics more satisfying for faculty as well as for students.



**Amy Cohen**

**Priscilla Bremser** enjoys bicycling, cross-country skiing, and hiking in Vermont.



**Priscilla Bremser**

**Douglas Mupasiri** has been on the faculty of the University of Northern Iowa for twenty-three years and head of the department of mathematics for the past six years. He was born in Zimbabwe and came to the United States to go to college and eventually became a naturalized citizen.



**Douglas Mupasiri**

<sup>3</sup> From the Executive Summary of the Common Vision Report.

# MATHEMATICAL MOMENTS

**MATHEMATICAL MOMENTS**  
**Making Art Work**

Great art looks good no matter where you're standing, but for art with perspective there's a special point where your view of the scene will match that of the artist. He or she uses geometry to maintain proportions and geometry to find exactly where a dimensional surface expands into a printout, unfortunately, your eye is not from the red dot. This helps explain why large-screen movie theaters

**MATHEMATICAL MOMENTS**  
**Trimming Taxiing Time**

It's hard to choose which part of air travel is the most fun—body scans, removing your shoes, fighting for the armrest, middle seats—but waiting on the runway is a worthy candidate. Controllers often let jets leave the gate when ready, regardless of existing runway lines, which can lead to long waits. Mathematical models that rely on probability and dynamic programming estimate travel time to the runway and options open to them have on flights' departure times. In tests at different airports the models have demonstrated the ability to shorten runway wait times, which reduces congestion and saves tons of fuel.

The models are very accurate: always predicting the number of aircraft in runway queues to within two. And despite their complexity (they involve many variables, quick controllers can get real-time updates on anticipated queues every 15 minutes. The models aren't yet in use everywhere, but they may be soon because with an air traffic managing departures is a good way to improve airport and airline efficiency.

**For More Information:** "A Queuing Model of the Airport Departure Process," Transportation Science, Ioannis Simalakis and Harsha Balakrishnan, Vol. 50, No. 1 (2015).

**Explaining Rainbows**

**Maintaining a Balance**

**Farming Better**

**7 ways math is making a difference**

[www.ams.org/mathmoments](http://www.ams.org/mathmoments)

See over 100 Mathematical Moments, hear people talk about how they use math on the job in the modern world, and read translations in 13 languages at

[www.ams.org/mathmoments](http://www.ams.org/mathmoments)



MM2016



# Women Doing Mathematics

Some work independently, others work in collaboration as members of interdisciplinary teams. Many of them also teach at the college or university level, while others are employed in industrial or government laboratories.

Whether they do mathematics for the sheer intellectual challenge, or for the critical insights it brings to solving important theoretical and real-world problems, women mathematicians love what they do.

Highlighted here are just a few of the women doing mathematics today. Please visit [www.ams.org/samplings/posters](http://www.ams.org/samplings/posters) to order this in its original poster format, the upcoming 2017 Women Doing Mathematics, or any other AMS poster that promotes awareness of mathematics, its beauty, and its many applications.

\*edited for Notices 64:03 spread

## Rebecca Goldin, PhD



Her research is in symplectic geometry, group actions and related combinatorics. She is Director of Research for the Statistical Assessment Service, a nonprofit media education and watchdog group affiliated with George Mason University concerned with the media's use of statistics and mathematics.

*"Mathematics for me touches on all the core joys of the human mind. It has rules, patterns, and structure, yet leaves so much room for creativity and invention. It impacts society deeply—from the roots of our education to the leading edge of science and technology—yet distinguishes itself by its sheer purity and abstraction. And there is almost nothing like the 'aha' moments that come with learning, teaching, understanding, or discovering something new in mathematics."*

## Trachette Jackson, PhD

She uses mathematical models, computer simulations and model-driven experiments to advance the current understanding of tumor growth and angiogenesis and to quantify the relative impact of new, cell-specific treatment strategies on the pathobiology of cancer. She received the Blackwell-Tapia Prize in 2010.

*"Many of the challenges of contemporary biology and medicine lie at the intersection of the mathematical and biological sciences. Working at this interface and continually striving to further integrate the fields of mathematics and biology is both exciting and rewarding. There's nothing I'd rather be doing with my career!"*



## Sommer Gentry, PhD



Her research in optimization in kidney transplantation has been profiled in *Science* and *TIME* magazine and on television. She serves as an advisor to both the US and Canada in their efforts to create national paired donation registries, and her research group helped lobby Congress to clarify the legal status of kidney paired donation.

*"I chose operations research because I wanted to make a difference to people's lives. Operations research is like a toolkit that's used everywhere: transportation, network security, credit cards, international relations, robotics, transplantation—I've worked in all these areas! If you master mathematical modeling you can be a contributor to almost any area of human endeavor."*

## Abigail Thompson, PhD

Her current research is in low-dimensional topology and knot theory. As a consequence of her work, the concept of 'thin position' has emerged as a major tool for attacking some of the fundamental problems in the study of 3-manifolds.

*"Some human constructions are unreasonably appealing, resonating with the part of our brain that recognizes beauty: suspension bridges, kites, sailboats, cellos, and elegant mathematical arguments."*



## Ivelisse Rubio, PhD

Her research interests are applications of computational algebra, finite fields, latin squares, and coding theory, which has applications in the internet, deep-space telecommunications, satellite broadcasting, and data storage. She has also organized and directed many undergraduate research programs and projects.

*"Mathematics is a world where there is no certainty: Nothing is true until you prove it. I love the challenge of facing a problem that has not been solved, that it is not easy to solve and maybe no one will solve! Working in mathematics is also my way of being different."*



photo by Angel Garcia

## Andrea L. Bertozzi, PhD

She develops mathematical methods and frameworks necessary to solve a diverse host of modern problems such as analyzing crime patterns, control of robotic vehicles, and fundamental physics of complex fluids. Her research brings together ideas from differential equations, inverse problems, and statistical physics.

*"I really enjoy working with students on applied mathematics research. It's very rewarding to train students to make an impact in diverse areas of science and engineering using the mathematics that they develop."*



photo by Mary Watkins/UCLA

## Bryna Kra, PhD

Her research lies in dynamical systems and ergodic theory, with a focus on problems related to combinatorics and number theory. She was awarded a Centennial Fellowship of the American Mathematical Society in 2006 and the Conant Prize in 2010. She is a member of the AMS Board of Trustees.

*"Every addition to our collective mathematical knowledge is a small triumph, from a child discovering a pattern, to a student solving an exercise, to a researcher taking a step in the proof of a new theorem. But nothing compares to the pure exhilaration that comes with proving an old conjecture or drawing a connection between seemingly unrelated concepts. Mathematics is the language for communicating such insights, connecting centuries of past research with future advances."*



## Melanie Wood, PhD

While a high school student she became the first female American to make the US International Math Olympiad Team. While at Duke University she won a Gates Cambridge Scholarship, Fulbright Fellowship, and a National Science Foundation Graduate Fellowship, became the first American woman to be named a Putnam Fellow, and also pursued her interest in theater. Wood is assistant professor at the University of Wisconsin–Madison and an American Institute of Mathematics Five-Year Fellow.

*"Insight. Originality. Inspiration. New perspectives. Opening your mind. Finding a different way. Playing around. That is mathematics. There is a myth that mathematics is about memorization, technicalities, formulas, and equations—there is only one correct answer. This picture utterly fails to describe the creative process that is professional mathematics."*



## Maria Chudnovsky, PhD

She studies the properties of graphs and was part of a team of researchers that proved a hypothesis in graph theory that had stumped mathematicians for 40 years. In addition to its mathematical beauty, graph theory can be a useful tool in operations research, computer science, and engineering. Chudnovsky is a MacArthur Fellow.

*"One of the best things about mathematics is that it teaches you to think clearly, no matter what you are thinking about."*



For more information on women mathematicians and their work see [www.ams.org/women-mathematicians](http://www.ams.org/women-mathematicians)

# 100 Years Ago: Joan Clarke

One hundred years ago, in June 1917, Joan Clarke was born in London. Her outstanding mathematical talent led her to become one of the very few women cryptanalysts working at Bletchley Park during World War II. Bletchley was the nerve center of a mammoth operation, carried out in utmost secrecy, to decode German and other enemy military communications. Decades later, as the secrecy surrounding Bletchley Park has lifted, the impact of the work of codebreakers like Clarke has become clear.

Clarke's story echoes that of the women in *Hidden Figures: The American Dream and the Untold Story of the Black Women Mathematicians Who Helped Win the Space Race*, a book by Margot Lee Shetterly that appeared last fall and was also made into a movie. The book recounts the stories of several mathematically talented women who worked as data analysts for NASA from the 1940s to the 1960s. Like the "hidden figures" in Shetterly's book, Clarke has also become better known through a movie, when her character was portrayed by Keira Knightley in the 2014 film *The Imitation Game*.

*Her equality with  
the men was never  
in question, even  
in those  
unenlightened  
days.*

Clarke entered Cambridge University in 1936 and the following year obtained a first in Part I of the university's legendary Mathematics Tripos. It had not been very long since women were even permitted to take the Mathematical Tripos; the first was Charlotte Angas Scott in 1881. In 1939, Clarke received a first in Part II, under the supervision of W. Gordon Welchman, and that same year she earned her BA degree. It was actually called a "title of degree"; Cambridge began awarding full-fledged degrees to women only after the end of World War II.

Clarke started at Bletchley in 1940. By 1942, the number of people working there reached over 7,000, and two-thirds of them were women. They worked in a multitude of roles, as intercept operators, transcribers, typists, encoders, linguists, punched card machine operators, administrators, secretaries—and as codebreakers. A week after joining the staff of Hut 8 at Bletchley, Clarke was moved to a room where she worked directly with Alan Turing and two other cryptanalysts, Tony Kendrick and Peter Twinn. In an essay about her work at Bletchley,

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**Women working on Enigma ciphers in Hut 6 at Bletchley Park. An Enigma machine is on the table in the foreground. By 1942, 7,000 people worked at Bletchley, two-thirds of them women.**

which appeared in the 1993 book *Codebreakers*, Clarke modestly noted that her quick rise from the ranks of "the girls," as the female staff was universally called, "was obviously because of my degree, and before I had had any chance of proving myself."

Clarke earned less than her male counterparts but worked with them as an equal. Her talent, ingenuity, and perseverance earned her great respect among her coworkers. After the war she went to work at Bletchley's successor organization, the Government Communications Headquarters (GCHQ). In his essay "Hut 8 from the Inside," which appeared in *The Bletchley Park Codebreakers* (2011), Rolf Noskwith writes of Clarke: "It was a tribute to her ability that her equality with the men was never in question, even in those unenlightened days."

What follows is a condensed and slightly edited version of an obituary for Clarke that appeared in *IEEE Annals of the History of Computing*, January-March 2001<sup>1</sup>. The obituary was written by Ralph Erskine, I. J. (Jack) Good, and Eric A. Weiss. Erskine is a leading historian of cryptography specializing in the history of World War II cryptography and was co-editor (with Michael Smith) of the *The Bletchley Park Codebreakers*. In 2002, Erskine gave the Gauss Lecture of the Deutsche Mathematiker Vereinigung. Good, who died in 2009, worked as a cryptanalyst at

<sup>1</sup>The original obituary is copyrighted by the IEEE Computer Society, which has kindly given permission for the Notices to publish the version appearing here. The Notices also thanks Ralph Erskine for his permission.

Bletchley Park in Hut 8, together with Clarke, and later became a professor of statistics at Virginia Tech. Weiss, who died last year at the age of 99, was a writer, editor, and electrical engineer who served as the biographies editor for *IEEE Annals of the History of Computing*.

— Allyn Jackson

### Obituary: Joan Elisabeth Lowther Clarke Murray

Joan Elisabeth Lowther Clarke Murray was born June 24, 1917 in London, England, and died September 4, 1996 in Headington, Oxfordshire, England. She was a cryptanalyst while at the Government Code and Cypher School (GCCS) at Bletchley Park, Buckinghamshire, England, from 1940 to 1945 and worked at the Government Communications Headquarters, England (1945 to 1952 and 1962 to 1977).

#### Before Bletchley Park

Clarke was the youngest daughter of William Kemp Lowther Clarke (a London clergyman) and Dorothy Elisabeth Clarke. She matriculated to Cambridge University in 1936. She chose Newnham College, half a mile from the university's heart, and not Girton, then Cambridge's only other women's college.

Early in 1940, Gordon Welchman, who joined the GCCS, recruited Clarke to join the codebreakers. Without telling her what the job was, he said that it did not really need mathematics but that mathematicians tended to be good at the job. Cryptanalysis requires imagination and accurate thinking. Some mathematics is often used, especially probability and statistics, but at that time the mathematics needed was usually not very advanced.

She was to start work in June 1940 after she had completed Part III of her studies and was to be paid 2 pounds (then about US\$8) a week. Men with her qualifications were getting paid much more.

#### Summer 1940

On June 17, 1940, the day before France capitulated, and after receiving a "pass" in Part III at Cambridge, Clarke arrived at Bletchley Park. Alan Turing, six years her senior, whom she had previously met as a friend of her older brother Martin, recruited her to work in his Hut 8 instead of with Welchman in Hut

*Clarke fully held  
her own in the  
male-dominated  
world of  
codebreaking.*

6. The grounds of the big estate of Bletchley Park were gradually being covered with hastily constructed, barn-like huts—one-story, non-uniform, and wooden—usually about 18 by 48 feet in size, internally partitioned into rooms of various shapes. A hut number designated a building and also indicated the work being done in it. Hut 8 was attacking the Enigma signals of the German Navy,

while Hut 6 was working on those of the German Air Force and Army.

#### Decoding Enigma-coded Messages

The German military believed that the signal contents were secret since they had been encrypted on the Enigma machines before being transmitted by radio. The electro-mechanical Enigma machine—which had been available in commercial form since 1919—changed each letter in a complex manner depending on a daily changing key. The German military knew that the commercial Enigma could be broken, but believed that the military machine, which had a plugboard (*Stecker* board), was unbreakable in practice. The plugboard interchanged letters in pairs, which added 150 trillion possible settings to a key (for the ten letter pairs that were generally used).

Even Alastair Deniston (head of GCCS at Bletchley at the time) shared the German belief that the military Enigma was invincible, telling his codebreakers that "all German codes were unbreakable." But the new codebreakers at Bletchley Park, Clarke among them, were determined to prove him wrong by reading German Enigma signals. However, German confidence in the machine was not wholly misplaced. Some Enigma cipher versions, such as Aegir (code-named Pike by Bletchley Park



Joan Clarke in 1936.

and used by merchant raiders) and TGD (used by the Gestapo), were never broken during the war.

Matched plaintext and Enigma ciphertext for April 23–26, 1940, captured from the German patrol boat *Schiff 26* off the Norwegian coast, had become available in early May and were being used to test a new key-finding aid called the Bombe. Clarke started checking the output of the first Bombe—optimistically named Victory—on the data from *Schiff 26*. The Bombe had entered service on March 18, 1940. The plugboard connections for one day were then apparently found to be already available on a piece of paper from *Schiff 26*, enabling the solution of that day to be completed manually.

#### More Decoding and Banburismus

Early in July 1940, Turing figured out how to make the Bombes test 26 hypotheses at once, using a process known as "simultaneous scanning," rather than one at a time. He explained his brainstorm to Clarke and discovered that he first had to tell her how electrical relays worked. Later

that summer, when Turing wrote his account of Enigma theory for the use of new recruits to Huts 6 and 8, locally called the “Prof’s book,” Clarke was the guinea pig who had to read it to test whether it was understandable to lesser mortals.

Hut 8 made little progress against the Naval Enigma until the capture, in February and May 1941, of the keys and special bigram (digraph) substitution tables used to encipher Naval Enigma message keys, which were the starting positions for Enigma’s rotors in specific signals. Hut 8 cryptanalysts were therefore sometimes deployed against other versions of Enigma and other cryptographic machines. Following Lieutenant-Colonel John Tiltman’s recovery of the plaintext of signals enciphered on the Railway Enigma in July 1940, Clarke helped to find the wiring of its specially wired rotors and worked on breaking traffic using it. Based on the commercial Enigma, the Railway Enigma-cipher—named Rocket by Bletchley Park—lacked a plugboard, making the traffic relatively easy to solve.

In August 1941, following another series of codebook captures, Hut 8 started to use the codebreaking technique called Banburismus that Turing had developed in mid-1940. It had been impossible to employ it earlier, since it depended on Bletchley Park having the bigram tables. Banburismus was a Bayesian sequential procedure for producing and handling a probability network consisting of thousands of weights of evidence. It used paper sheets that were punched with representations of the messages under attack and then moved across each other to find both single-letter and multi-letter repeats of coincident holes to obtain a probabilistic piece of information about the two messages.

Banburismus was invaluable in breaking the main Naval Enigma cipher, *Heimisch*—later called Hydra and code-named Dolphin by Bletchley Park. Without Banburismus, breaking the Naval Enigma would have been greatly slowed down, since Bombes were in short supply until the US Navy’s four-rotor Bombes came into service in August 1943.

Clarke was good at Banburismus and was so enthusiastic and enthralled that she would sometimes not hand over her work to the next shift but would stay to see if a few more tests would give a result. One of Clarke’s important contributions to the work of Hut 8 accelerated the solution of Naval Enigma *Offizier* signals, which were often extremely difficult to break. Some were never solved, since they were reenciphered with a second set of plugboard settings. Leslie Yoxall devised the first stage of an ingenious statistical method for recovering *Stecker* settings from messages of about two hundred letters or more. It involved a graphical procedure and was called a *dottery*, because the cryptanalyst penciled dots into the cells of a square matrix to represent letter correspondences.

A day or two after Yoxall’s invention, Clarke invented a method for greatly speeding up the “routine method,” the first stage of the dottery procedure. Clarke’s contribution was the second stage of the procedure. It took the

*Stecker* of the letter E, as determined in the first stage, assumed that all the plaintext consisted of the letter E,

*Her work on the Naval Enigma helped to shorten the war and saved many lives on both sides of the conflict.*

and proceeded with this through a triangular dottery. Her name was not attached to the invention, so the entire *Stecker*-recovery procedure was named Yoxallismus, although Yoxall now disclaims all memory of inventing the second stage.

At the time, Clarke was told that she had merely rediscovered “pure Dillyismus,” a reference to the highly talented Dillwyn (Dilly) Knox, a founding member of GCCS and a former member of Room 40, the Royal Navy’s highly successful codebreaking unit during World War I.

### Engagement on and off

In the spring of 1941, Alan Turing and Joan Clarke got engaged. He gave her a ring, and they arranged for formal introductions to both families. However, they kept the engagement secret, and Clarke did not wear her ring in the Hut. After taking their holiday together at the end of August, they broke off the engagement by mutual agreement. Turing believed the marriage would be a failure owing to his homosexuality. Clarke and Turing continued to be friends and corresponded during Turing’s trip to the United States in the winter of 1942-1943. Clarke said that they had a special friendship, and her warm feelings for Turing lasted throughout her life.

### End of the War

The labor-intensive Banburismus attack was discontinued in mid-September 1943 when three-rotor Bombes became more freely available following the entry into service of the US Navy’s four-rotor Bombes. By the end of 1943, the US Navy codebreaking unit (Op-20-G) had assumed responsibility for breaking the four-rotor cipher, Triton (Shark to Bletchley Park), that the Atlantic U-boats used. This led to many of the Hut 8 staff being transferred to other work, but Clarke remained in Hut 8 as a highly capable member of a small team that broke Naval Enigma ciphers until the end of the war.

### Postwar Period

After the war, Clarke transferred to the successor of both Bletchley Park and GCCS, the Government Communications Headquarters (GCHQ) at Eastcote, where she met a colleague named Lieutenant-Colonel J. (Jock) K.R. Murray, whom she married in 1952. She was appointed a Member of the British Empire in 1947 for her codebreaking work during the war. Because of Murray’s poor health, the Murays moved to Scotland, where they developed a shared interest in Scottish history.

After her retirement, Clarke helped Sir Harry Hinsley on what became Appendix 30 to volume 3, part 2 of the 1988 *British Intelligence in the Second World War*—a substantially revised assessment of the Polish, French, and British contributions to breaking Enigma.

### Summary

Clarke was a lady in the tradition of her day. She was congenial but shy, kind, gentle, truthful, nonaggressive, agreeable to all, and always subordinate to the men in her life, except in Hut 8, where she was treated as an equal. Although she recognized and later mentioned some of the impediments and unfairness she met as a woman at Cambridge and Bletchley Park, she never directly confronted them.

She was a highly intelligent person of exceptional gifts. Clarke was fully able to hold her own in the male-dominated world of outstanding British wartime codebreaking—she was one of only three female cryptanalysts who worked on the different Enigma machines. She was an enthusiastic and encouraging colleague who was much admired by all who worked with her at Bletchley Park and GCHQ.

Although the remaining secrecy associated with cryptanalysis still makes it impossible to be more specific about her accomplishments, it is clear that her work on the Naval Enigma helped to shorten the war and saved many lives on both sides of the conflict.

—Ralph Erskine, I.J. (Jack) Good, Eric A. Weiss

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Photo of women working on codebreaking in Bletchley Park are ©Crown. Reproduced by kind permission, Director, GCHQ.

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## Celebrating

# Women's History Month

## Looking Back a Quarter-Century

Twenty-six years ago, the September 1991 issue of the *Notices* carried a special issue devoted to the theme “Women in Mathematics.” Below are some of the pieces from that issue:

### “In Her Own Words: Six Mathematicians Reflect on Their Lives and Careers”

(with contributions by Joan S. Birman, Deborah Haimo, Susan Landau, Bhama Srinivasan, Vera Pless, and Jean E. Taylor)

### “The Past, Present, & Future of Academic Women in Math Sciences” by L. Billard

### “Top Producers of Women Mathematics Doctorates” by Allyn Jackson

### “Mathematics and Women: The Undergraduate School and Pipeline” by D. J. Lewis

### “Merging and Emerging Lives: Women in Mathematics” by Claudia Henrion

### “The Escher Staircase” by Jenny Harrison

### “Mathematics and Women: Perspectives and Progress” by Alice T. Schafer

### “A Brief History of the Association for Women in Mathematics: The Presidents’ Perspectives” by Lenore Blum [Available on the AWM web site at [www.awm-math.org/articles/notices/199107/blum/](http://www.awm-math.org/articles/notices/199107/blum/)]

# Commutative Algebra Provides a Big Surprise for Craig Huneke's Birthday

*Irena Swanson*

*Communicated by Tom Garrity*

A commutative algebra conference in July 2016 on the occasion of Craig Huneke's sixty-fifth birthday, included a major mathematical surprise.

Craig Huneke has been at the forefront of research in commutative algebra, introducing and advancing several influential notions, such as  $d$ -sequences, licci ideals,

*During the banquet, Hochster read his poem honoring Huneke.*

symbolic powers, homological methods, computational methods, tight closure, uniform bounds, and prime characteristic methods. He has mentored twenty-four PhD students and many postdocs; he has

coorganized conferences, such as the Kansas-Missouri-Nebraska KUMUNU commutative algebra conference; he has served on the Executive Committee of the AMS and on the Board of Trustees of the MSRI.

Almost all the talks were directly related to Huneke's work, and most speakers started their talks describing how Huneke affected their work as a mathematician and as a person: they talked about his mathematical productivity, extensive collaborations, productive and precious lunches with napkin notes, excellent and influential talks, his advising, mentoring, his friendly competitiveness, and so on. Claudia Polini addressed how commutative algebra in general is friendly to women (possibly due to Emmy Noether being one of us) and in particular how Huneke has been a tremendous role model as a teacher, collaborator, mentor, and organizer. His own family life, with wife Edith

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**Group photo of the conference participants, Craig Huneke and Melvin Hochster in the center. Claudia Polini said that commutative algebra in general is friendly to women (possibly due to Emmy Noether being one of us) and in particular how Huneke has been a tremendous role model as a teacher, collaborator, mentor, and organizer.**

Clowes, professor of Slavic languages and literatures, and their children Sam and Ned, has been a shining example for possibilities in home and professional life for women in academia. During the banquet, Hochster read his poem honoring Huneke. The most colorful roast was given by Hailong Dao, who mused how he had turned down a postdoc offer from the University of Kansas, where Huneke was based at the time, and when Hailong later joined the University of Kansas in a tenure-track position, Huneke "won" the postdoc standoff by leaving after Hailong's three years there. Neal Epstein, Karl Schwede, and I wrote a song that the whole room sang for Huneke to the tune of Woodie Guthrie's "This Land Is Your Land," one of Huneke's favorite songs.



Melvin Hochster read his touching poem and presented his plaque to Craig Huneke.



Huneke's postdocs, from left to right: Karen Smith, Alberto Corso, Graham Leuschke, (Craig Huneke), Luis Núñez-Betancourt, Ian Aberbach, Adela Vraciu, Jeff Mermin. Standing in the background: PhD students Janet Striuli and Branden Stone.

There were twenty-nine talks over six days, but the really big deal was two consecutive talks on day five by Jason McCullough and Irena Peeva on Rees-like algebras. As an organizer I was aware that the two wanted to give consecutive talks, but I was not aware of their research collaboration, nor did their abstracts hint at anything field-changing. Whereas their talks were spectacular in content and in the changes they are bringing to the field, their abstracts were clever decoys, Peeva's just mentioning "some open questions."

It turns out that Rees-like algebras are very interesting and powerful and that the discussion of the open questions was the big deal!

McCullough started with an announcement that Peeva and he were coordinating their talks and tried hard to get them ready for the conference as a birthday surprise of a sort, which meant that the audience would have to come back for Peeva's talk to get the rest of the story. Early in his talk McCullough brought up the Castelnuovo-Mumford regularity (the smallest integer such that related cohomology modules vanish in shifted higher degrees) and some open questions in this area, such as the Eisenbud-Goto conjecture and the Bayer-Stillman conjecture. The audience was now abuzz: is the surprise about resolving one of these two? Which one? Or is it something else? Is the answer positive or negative?

The Eisenbud-Goto conjecture ([EG], 1984) estimates the Castelnuovo-Mumford regularity  $\text{reg}(P)$  of a homogeneous prime ideal  $P$  in a polynomial ring  $S$ , such as  $\mathbb{C}[x_1, \dots, x_n]$ , in terms of the degree (multiplicity) of  $S/P$  and the codimension of  $P$ :

$$\text{reg}(P) \leq \deg(S/P) - \text{codim}(P) + 1.$$

Bayer and Mumford (1993) had discussed that the missing link between the sharp doubly exponential bound on regularity of arbitrary homogeneous ideals and the nice bounds on regularity in the smooth case is that

we do not yet have a decent bound on the regularity of all reduced equidimensional ideals. The Eisenbud-Goto Conjecture was aiming to fill in this gap. Special cases have been proved.

In their talks at the conference, McCullough and Peeva reported on their new techniques for handling projective dimension and Betti numbers: Rees-like algebras and step-by-step homogenization.

The classical **Rees algebra** of an ideal is a workhorse for analyzing powers of that ideal in one fell swoop, and the projective scheme of a Rees algebra is the blowing up of the spectrum of the ring along the subscheme defined by the ideal. Explicitly, the **Rees algebra** of an ideal  $I$  in a commutative ring  $S$  is a subring of the polynomial ring  $S[t]$  in one variable  $t$ , and it is generated over  $S$  by elements  $at$  as  $a$  varies over  $I$ . The classical **extended Rees algebra** of  $I$  is  $S[It, t^{-1}]$ , which is generated over  $S[It]$  by  $t^{-1}$ . Huneke has done great work with Rees algebras, including analyzing them for ideals generated by  $d$ -sequences, determining their Cohen-Macaulay and integral closure properties, and exploring their analogues for modules with Eisenbud and Ulrich (2002).

McCullough and Peeva defined and developed a new **Rees-like algebra**  $S[It, t^2]$  of  $I$ , generated over  $S[It]$  by  $t^2$ . This notion is a generalization of a local example due to Hochster where he produced a family of prime ideals with fixed embedding dimension and fixed Hilbert-Samuel multiplicity but arbitrarily many minimal generators.

For the context of the conjectures, let  $S$  be a polynomial ring over a field; let  $I$  be an ideal in  $S$  generated by homogeneous elements  $f_1, \dots, f_m$ ; let  $T$  be the polynomial ring generated over  $S$  by  $m+1$  variables  $y_1, \dots, y_m, z$ ; let  $\varphi: T \rightarrow S[It, t^2]$  be the  $S$ -algebra map taking  $y_i \mapsto f_i t$  and  $z \mapsto t^2$ ; and finally, let  $Q$  be the kernel of  $\varphi$ . McCullough and Peeva proved the following:

1.  $Q$  is a prime ideal (minimally) generated by the elements  $\{y_i y_j - z f_i f_j : 1 \leq i, j \leq m\}$  and



Huneke's PhD students at the conference, from left to right: Yongwei Yao, 2002; Manoj Kummini, 2008; Neil Epstein, 2005; Giulio Caviglia, 2004; Branden Stone, 2012; Craig Huneke; Irena Swanson, 1992; Janet Striuli, 2005; Ilya Smirnov, 2015; Elóisa Grifo, current student; Alessandro De Stefani, 2016.

$\{\sum_{i=1}^m c_{ij}y_i : \sum_{i=1}^m c_{ij}f_i = 0\}$ . Thus we have an explicit set of generators of  $Q$ , the latter kind arising from the first syzygies of  $I$ . (In contrast, the generators of the presenting ideal of the classical Rees algebras are not well understood at all.)

2. If we assign  $\deg(y_i) = \deg(f_i)$  and  $\deg(z) = 2$ , then  $Q$  is a homogeneous ideal in  $T$ . Furthermore, if  $\deg(f_i) \geq 2$  for all  $i$ , the graded Betti numbers of  $T/Q$  over  $T$  equal the graded Betti numbers of  $T/(Q + (z))$  over  $T/(z)$  since  $z$  is a non-zerodivisor. These Betti numbers are well understood because the minimal free resolution of  $T/(Q + (z))$  over  $T/(z)$  is a mapping cone with ingredients being the minimal  $S$ -resolution of  $I$  and the well-known Koszul resolution and the Eagon-Northcott resolution.

The only “problem” with Rees-like algebras is that the constructed polynomial ring  $T$  is not standard graded and so not immediately relevant for resolving the Eisenbud-Goto and the Bayer-Stillman conjectures. The classical approach is to change the degrees of the variables to 1 and then homogenize the ideal. However, Betti numbers change under homogenization, and so all of the information about the Rees-like algebra would be lost in this way. McCullough and Peeva invented a way to remove the nonstandard obstacle, and they called their process **step-by-step homogenization**. They homogenize the ideal repeatedly, one variable at a time. The striking discovery is that this process preserves the Betti numbers if the ideal is prime, as it is in their case. In contrast to classical homogenization, where one needs to homogenize a Gröbner basis in order to obtain generators of the homogenized ideal, for the step-by-step homogenization it suffices to homogenize the generators of  $Q$ . Without going into too much detail, the homogenized  $Q$  is a prime ideal in a standard graded polynomial ring, it has the same

number of generators as  $Q$ , and its dimension, depth, projective dimension, Castelnuovo-Mumford regularity, and degree are all well understood and well behaved. For example, the homogenized prime's quotient has degree equal to  $2 \prod_{i=1}^m (\deg(f_i) + 1)$ , and its projective dimension equals  $m - 1$  plus the projective dimension of  $S/I$ .

This is a beautiful and powerful construction, all by itself!

We got this far in the second talk, delivered by Peeva. We were close to the inevitable big announcement, but Peeva paused right there, leaving us in suspense a little longer, to address the honoree of the conference with high praise for his research, service, and enormous impact on the field. The context made her words all the more powerful.

And then she went for what their birthday surprise was all about. Namely, starting with an arbitrary homogeneous ideal in a polynomial ring  $S$ , via step-by-step homogenization of the defining ideal of its Rees-like algebra, one gets a prime ideal  $P$  in a polynomial ring  $S$  with precise control on the dimension of  $S$ , on the number of generators of  $P$  and their degrees, on the dimension, depth, multiplicity, regularity, and projective dimension of  $S/P$ . In particular, if they start with the famous Mayr-Meyer ideals (1982) or variant ideals due to Koh (1998) and Bayer and Stillman (1988), the prime ideals constructed via this machinery *yield counterexamples to the Eisenbud-Goto conjecture and the Bayer-Stillman conjecture*. And worse, McCullough and Peeva showed that the regularity of reduced irreducible schemes is not bounded above by any polynomial in the degree. Peeva finished her talk with a discussion of compelling future redirections.

Peeva's and McCullough's lectures were masterfully delivered: they carried amazing content, and the suspense was just right. The two got huge applause.

Huneke, instead of the usual questions for the speakers, turned to the audience and said: “When I was seven years old, I wanted a bicycle. I really, really wanted a bicycle. And I got it for my birthday, and that was the best birthday present to me ever—until this present right now.”

These results of McCullough and Peeva are reshaping the field of commutative algebra: their counterexamples to the Eisenbud-Goto conjecture not only show that the conjectured bound does not hold but also that there is no bound on the regularity of irreducible varieties that is a polynomial function of the degree. Furthermore, McCullough and Peeva produced a beautiful and general machinery that is worthy of deeper study, and in fact they are working with collaborators on further papers.

The summer of 2016 was amazing for commutative algebra: McCullough and Peeva resolved a thirty-two-year-old problem, then Yves André proved the direct summand conjecture which was posed in its general form by Mel Hochster forty-seven years ago, Bhargav Bhatt significantly shortened the proof, and Tigran Ananyan and Mel Hochster gave a positive answer to Stillman's question.

ACKNOWLEDGMENTS. I thank Melvin Hochster for the lion's share of the organization of the conference, including convincing Huneke that a conference was in order. I also thank Craig Huneke, Jason McCullough, Irena Peeva, Marilina Rossi, and Bernd Ulrich for feedback on early versions of this article, and Bernd Ulrich for sharing the slides of his talk.

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### Photo Credits

Group photo of the conference participants is courtesy of Margie Morris.

Photo of Melvin Hochster reading his poem is courtesy of Margie Morris.

Photo of Huneke's postdocs is courtesy of Irena Swanson.

Photo of Huneke's PhDs students at the conference is courtesy of Margie Morris.

Photo of Irena Swanson is courtesy of Irena Swanson.



Irena Swanson

### ABOUT THE AUTHOR

Irena Swanson sports the Algebra\* T-shirt designed on the occasion of Craig Huneke's sixtieth birthday by Amelia Taylor, with help from Ananthnarayan Hariharan and Manoj Kummini.

## Celebrating

## Women's History Month

### Links for Women in Mathematics

#### African Women in Mathematics Association:

[africanwomeninmath.org/](http://africanwomeninmath.org/)

#### Association for Women in Mathematics:

<https://sites.google.com/site/awmmath/home>

#### Association for Women in Science:

[www.awis.org](http://www.awis.org)

#### European Women in Mathematics:

[www.europeanwomeninmaths.org/](http://www.europeanwomeninmaths.org/)

#### IMU Committee for Women in Mathematics:

[www.mathunion.org/cwm](http://www.mathunion.org/cwm)

#### Women in Math Project:

[darkwing.uoregon.edu/~wmnmath/](http://darkwing.uoregon.edu/~wmnmath/)

#### 2017 Women in Mathematics, Science, and Technology Conference:

[www.millersville.edu/wmsc/](http://www.millersville.edu/wmsc/)

#### 2017 Workshop “WIN4 Women in Numbers 4”:

[www.birs.ca/events/2017/5-day-workshops/17w5083](http://www.birs.ca/events/2017/5-day-workshops/17w5083)

#### 2017 Workshop “Women in Control: New Trends in Infinite Dimensions”:

[www.birs.ca/events/2017/5-day-workshops/17w5123](http://www.birs.ca/events/2017/5-day-workshops/17w5123)

# Math PhD Careers: New Opportunities Emerging Amidst Crisis

*Yuliy Baryshnikov, Lee DeVille, and Richard Laugesen*

**ABSTRACT.** In light of the shortage of regular academic positions for recent PhDs and the underappreciated opportunities for mathematicians in industry, we describe a model program preparing young mathematicians for a broader range of careers.

One of the most troubling trends in the mathematics profession is the shortage of regular positions in academia for recent PhDs, as described in Amy Cohen's op-ed in the October *Notices* [Coh16].

Our view is that the demand for PhD mathematicians is there, but we just fail to recognize it. True, many PhD graduates will leave mathematical academia, but that does not mean they must leave mathematics. That mathematics is critically relevant to the needs and challenges of twenty-first-century society is a cliché, but still true. Scientific and engineering fields have become more quantitative and computational, and the opportunities for mathematicians to contribute wherever sophisticated models are used—be it in big data, neuroscience, climate science, telecommunications—are expanding rapidly. These opportunities are often not explicitly acknowledged by us in our roles as mentors and hence can be hard for our mentees to identify and pursue.

Thus mathematics graduates often pursue such opportunities on their own through a slow and sometimes painful osmosis-like process. Most of us know personally or by sight the students from our departments who have “gone into industry,” in many cases after growing weary of the rat race in pursuit of an elusive tenure-track position. Many become quite successful in their new careers (being

smart and industrious people), but few take this route well prepared. Often these graduates fall into their jobs by chance, after some struggle, and frequently with a feeling that an indelible stigma is attached to their nonacademic career choice. This is a sign of our failure as mentors. Regardless of one's opinion on the proper goals of PhD study, the fact that a significant fraction of graduates from institutions of all types will spend the bulk of their professional lives outside mathematical academia forces

*Academia is not a closed ecosystem.*

upon us the moral responsibility of providing opportunities that will prepare graduate students for intellectually challenging and professionally rewarding careers that continue to involve mathematics.

The path from academia to a new career is rarely smooth. Hurdles and challenges—from relatively trivial ones such as learning to code to such existential ones as abandoning the quest for pure knowledge—are abundant along the way. We as advisors rarely have relevant experience ourselves. More often than not, we faculty members are unable to guide our students at this critical juncture in their careers.

We cannot expect to overcome this hurdle by having all mathematical faculty suddenly develop industry connections (although it doesn't hurt to try). But we can choose to speak out in our own departments, in the common rooms, and in faculty meetings, and advising meetings, to say that academia is not a closed ecosystem and that our graduates should explore the full range of options, including the intellectually rewarding, satisfying, and stable jobs outside universities that rely on mathematical expertise and have tangible impacts on our economy and society. Students can seek careers in national labs, in other academic disciplines, in industrial research, in NGOs and governmental research organizations. Mathematics is vital for society to move forward. We just need to start believing our own words and putting them into practice through our students.

Another hurdle faced by students willing to venture “outside” to industry is the reluctance of many potential employers to hire a mathematician. Department heads

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**Teaching assistant Stefan Klajbor Goderich (center) helps graduate students William Linz (left) and Colleen Robichaux (right) with a Python programming assignment during the beginning-of-summer Computational Mathematics Bootcamp.**

and project managers in industry know what they get by hiring a statistician or an electrical engineer. But a mathematician? What are they good for?!<sup>1</sup> Unless the enterprise has previously employed PhD mathematicians, the hiring managers have little experience to draw upon. And indeed, few of our graduates really are ready to walk right in and start work in a typical industrial R&D environment. The training they need might be minor compared to what they have already learned in graduate school, but still it creates an additional risk and burden that industrial research units might be unwilling to bear.

We do have allies in industry, consisting at a minimum of those of our graduates who went there. A back-of-envelope calculation<sup>2</sup> shows that at least six hundred PhD graduates in mathematics leave academia yearly. The majority of them do not get the word “mathematician” in their job title (indeed, the total number of “mathematicians” outside academia in the US is a paltry three thousand five hundred [USB]). Nonetheless, these individuals work in demanding and technical environments that use their mathematical talent and training at a maximal level. These “hidden mathematicians” enable scientific and technological progress at a level we are generally unaware of. One of the long-term goals for the mathematical community should be to generate broad acceptance

<sup>1</sup>V. I. Arnold quoted a famous physicist, Y. Zeldovich, as remarking that formulating a problem precisely is valuable because it allows one to involve mathematicians, who, “like flies, can walk on the ceiling” [Sun04, p. 198]. This attitude is what we would like to see in potential employers!

<sup>2</sup>The number of new PhDs in mathematical sciences is about 1900 per year, and one third of these graduates take their first job in a nonacademic institution. Meanwhile, the number of tenure-track positions under recruitment is fewer than 1000 annually, and not all these positions get filled [AMSA], [AMSb].

of the notion of an “in-house” mathematician in every company facing technical or theoretical challenges.

Turning now from diagnosing the problem to proposing solutions, we ask: how can we help our students to overcome career transition hurdles in the short term, and in the longer term create an explicit appreciation in industry and government of the need for mathematically trained personnel?

We believe that at the very least, we need to begin providing to all interested students the skill sets that will enable them to transition more readily to careers outside mathematics departments and that we should educate them, in specific, hands-on ways, about the options outside mathematical departments.

To be clear: we do not advocate turning mathematical graduate education into a training funnel for corporate R&D or pushing any of our students away from theoretical mathematics or away from academia. However, we should work to change the general level of ignorance among academic mathematicians of what mathematicians can do outside universities and to change the ignorance prevalent in industry of the benefits of hiring mathematicians. These changes can together unlock a huge untapped reservoir of career opportunities for our graduate students.

The good news is that to instigate these changes, we need not radically reshape what our students do during graduate school or how we operate professionally as academics. Small but sustained tweaks in graduate training and a robust effort to connect with potential employers will suffice.

With these principles in mind, at the University of Illinois we created the Program for Interdisciplinary and Industrial Internships at Illinois (PI4), with much-appreciated support from the National Science Foundation under grant DMS MCTP 1345032.

The main goal of the PI4 program is to expose mathematics PhD students to alternative career paths early in their graduate careers so that they understand what skill sets they should acquire by the time they graduate. Interested students can then prepare for a career path in industry or government during the middle of their PhD program (rather than in a rush at the end), for example, by picking up coding skills from online MOOCs over winter break or by taking a course in machine learning from the statistics department and also doing one or more internships to develop the soft skills of teamwork and project management. The secondary goal of PI4 is to develop a cadre of supportive local employers who

*These changes can together unlock a huge untapped reservoir of career opportunities for our graduate students.*

have experienced first-hand the value provided to their organization by hiring a mathematics student.

Program components include: in-semester topics courses such as Top Ten Algorithms for the Twenty-First Century, a beginning-of-summer Computational Bootcamp, summer working groups for junior students on exploratory topics such as “Topologically constrained problems of statistical physics, and internship placements for both junior and senior students.

The two-week intensive Computational Bootcamp kicks off the summer, teaching practical techniques with Python programming to students having little or no programming experience. Each day, students:

- (i) learn new concepts,
- (ii) test their understanding on short programming exercises,
- (iii) present and critique student project solutions, and
- (iv) code the next project.

The examples typically center around numerical linear algebra, optimization, graph theory algorithms, differential equations, and data fitting. It is an intensive program, with most of the students’ days spent programming.

Students then split into three tracks for the remainder of the summer: Prepare, Train, and Intern. The Prepare cohort is typically made up of incoming students and students at the end of the first year of graduate school. They work in a group under the supervision of a faculty member (sometimes local, sometimes external), and it is useful to think of this group as an enhanced Research Experience for Undergraduates, an “REU++.” The students spend most of their summer learning about a subject and then some time at the end working on questions related to it. This subject usually has a computational tilt, but it is still open-ended research in some mathematical topic, such as random matrix pencils, for example. Members of the Train cohort tend to be more senior students, with one or two years of graduate school experience, who work in small groups on a focused research problem in mathematics, where simulations and experimentation must be combined with theoretical analysis. These parts of the summer program are useful for all mathematics graduate students as they transition from course-takers to researchers.

The Intern stage is the apex of the program. We directly support about twelve students each summer in two tracks: industrial and interdisciplinary internships.

Industrial internships are hosted by enterprises having a local R&D outlet. The financial support from the NSF grant enables us to achieve two objectives. First, we can work with smaller firms, such as startups or small research groups at companies that are not considered traditional players in research, that would not normally be able to support interns on their own. Second, the financial support reduces the perceived risk to new industrial partners when they hire the unknown quantity of a mathematics intern. By getting a foot in the door, we lay the groundwork for them to hire interns with company funds in future years. Internship examples include image recognition research at the start-up Personify, weed resistance modeling at



**Graduate students Derrek Yager (left) and Vaibhav Karve (not pictured) uncovered low-dimensional structure in New York City taxi data during a scientific internship with Professor Richard Sowers (right). Their work reduced the dimension of the problem from hundreds of thousands down to about fifty dominant roadway combinations.**

Dow AgroSciences, and customer data analytics work at utility company Ameren.

Interdisciplinary internships are something completely new for us. We embed a student in a scientific lab on our campus, where they tackle a mathematical problem that supports the agenda of that lab. Almost always the student works in the lab space or is colocated with the group in some other manner. We request that the mentors think of this student as a member of the lab, so that the student learns the techniques required and soaks up the culture of the field. We really want the student to get a feel for what it is like to work in that discipline and how one can contribute with mathematics.

*What we are doing here addresses a critical need in the mathematical community.*

Internships attract students from all areas of mathematics, not just those in traditionally applied fields. Recently, for example, a number theorist modeled ant colony behavior in the entomology department, and a logic student analyzed fMRI data as part of a larger project trying to find physiological imprints of tinnitus.

Once the internship culture is established, students start finding their own positions and funding. Our department has 210 graduate students, including 160 in the PhD program. Internship numbers have grown from six in 2013 to a total of forty-two interns in 2016, with thirty-one internships taking place over the summer and eleven during the fall or spring semesters.

# Intern Success Story

Byron Heersink (PhD in number theory expected 2017) did a summer internship through the PI4 program on “A Response Threshold Model for Ant Colonies,” a project supervised by faculty members Samuel Beshers (Entomology) and Lee DeVille (Mathematics). Byron developed a computational model of the division of labor in social insect colonies. There is a large literature modeling, calibrating, and checking the individual-based rules insects might use for specialization, but it is difficult to determine how these rules turn into colony-level behaviors. Byron’s simulations revealed two surprising effects in the models considered: the degree of specialization is mostly independent of the size of the colony, and the overall colony task behavior depends most strongly on relatively few workers most sensitive to each task and not on the average task sensitivity of the colony.

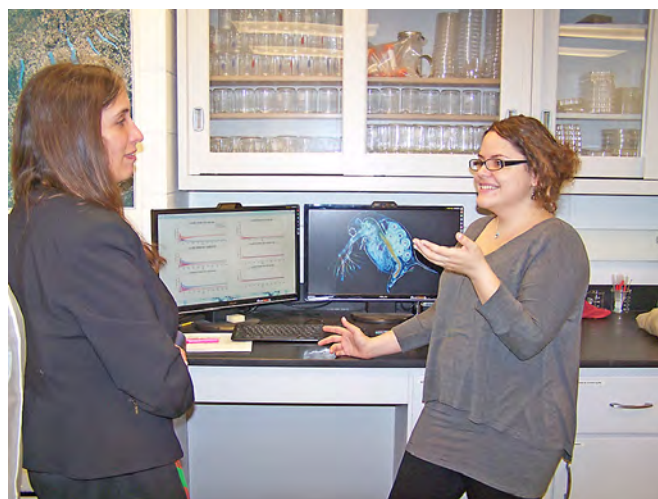
This project was Byron’s first experience of mathematics outside the mathematics department, and his first experience of computational modeling. He subsequently took these skills to an internship at Sandia National Laboratories and then to a second internship at HRL Laboratories, and he is now applying for postgraduation positions in industry as well as academia.

Internships are not slowing down the students’ time to degree here at Illinois. Students who aim at an industry or government career after doing an internship develop a sharpened focus that speeds up their academic progress.

The type of industry careers sought by PhD students at Illinois is shifting. Some still aim at the world of finance, but national labs and data science careers are increasingly valued. The PI4 student cohorts are moving through our PhD program, and we view the coming diffusion of mathematical talent into the broader world as a highly positive development.

We believe what we are doing here addresses a critical need in the mathematical community: the need to open to our graduates a variety of rewarding, creative, and intellectually challenging mathematical careers. Many graduate students will continue to follow a traditional academic career path, but having the option to choose careers in industry and governmental organizations will benefit all of them.

Mathematics departments must change their culture and (some of) their practices in order to remain relevant in the coming century. Certain departments around the country are already doing so. We hope this article about our experiences at Illinois will foster a conversation in many more mathematics departments about how the community can move forward from the current crisis toward new opportunities for our PhD graduates.



**Biology professor Carla Cáceres (left) mentored graduate student Vanessa Rivera Quiñones (right) in her lab for a summer scientific internship. This ongoing work aims to develop biologically realistic SIR population models, applied initially to evolutionary branching in organisms such as water fleas.**

# National Internship Efforts

The NSF-IPAM Mathematical Sciences Internship Workshop Report [IPA15] proposes a national effort at multiple scales to provide career and internship placement resources for mathematics graduate students:

- National: create a network to increase internship information exchange, data collection, access, and opportunities;
- Regional: establish internship centers to build internship contacts and organize training opportunities;
- Local: encourage and enable student participation in internships in mathematical sciences departments.

A new BIG Math Network is promoting this concept (BIG = Business + Industry + Government), which has generated enthusiasm at national societies including AMS, SIAM, MAA, ASA, AWM, NAM, AMATYC, INFORMS, and SACNAS. Supporters who want to volunteer time and effort or who simply want to stay in the loop can keep informed at the website [BIG] and sign up at [tinyurl.com/BIGmathnetwork](http://tinyurl.com/BIGmathnetwork).

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 Photo of Richard Laugesen is courtesy of Darrell Hoemann.



Yuliy Baryshnikov

### ABOUT THE AUTHORS

**Yuliy Baryshnikov** spent a couple of decades in R&D labs in the FSU and the US and now explores the boiling tedium of the academe.



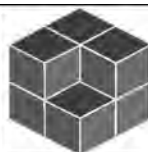
Lee DeVille

**Lee DeVille** is an applied(ish) mathematician with interests in dynamical systems, partial differential equations, and stochastic processes. He is greatly interested in finding connections—both in research and in employment—between academic mathematics and that practiced in industry and government.



Richard Laugesen

**Richard Laugesen** enjoys researching and teaching partial differential equations and spectral theory. In addition to promoting mathematical careers in industry and government, he works toward a future in which a lot more women and minority students will pursue graduate-level mathematics.



Institute for Computational and Experimental Research in Mathematics

SPRING SEMESTER 2018

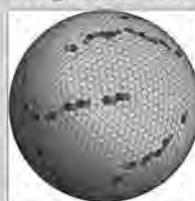
## Point Configurations in Geometry, Physics and Computer Science

February 1 – May 4, 2018

### Organizing Committee:

**Christine Bachoc**, University of Bordeaux  
**Henry Cohn**, Microsoft Research – New England  
**Peter Grabner**, Technische Universität Graz  
**Doug Hardin**, Vanderbilt University  
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**Achill Schürmann**, University of Rostock  
**Sylvia Serfaty**, Université Pierre et Marie Curie Paris  
**Salvatore Torquato**, Princeton University  
**Rob Womersley**, University of New South Wales

### Program Description:



The arrangement of point configurations in metric spaces, whether deterministic or random, is a truly interdisciplinary topic of great interest in mathematics, physics and computer science.

Mathematical aspects involve optimization, discretization of manifolds, best packing and cubature, among others. For physics, such configurations arise in the study of crystallization, point processes connected with random matrices, self-assembling materials, jammed states, hyperuniformity and phase transitions. For computer science, extremal point configurations play a fundamental role in coding and information theory, and lattice-based protocols in cryptography and related computational complexity issues are of growing importance. Furthermore, there has been recent and substantial progress on related age-old problems (such as the Kepler conjecture).

Topics for this program include random point configurations, computation and optimization of energy, packing and covering, multi-pole methods, sparsity, and frames, and the theory of lattices with applications to coding and cryptography.

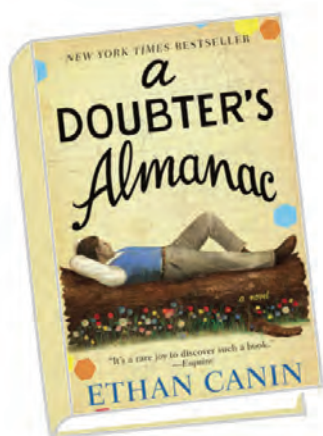
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  - summer undergrad program
- Apply for a:*
- semester program or workshop
  - postdoctoral fellowship
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# A Doubter's Almanac

*A Review by Sheldon Axler*

## *A Doubter's Almanac*

Ethan Canin

Random House, 2016

US\$28.00, 551 pages

ISBN: 978-1-4000-6826-5

Ethan Canin's novel *A Doubter's Almanac* follows fictional mathematician Milo Andret through several decades, including graduate school, winning a Fields Medal, and decline. The author is not a mathematician, although he was an engineering major at Stanford before switching his major to English. He then went on to obtain an MD degree before leaving medicine for a successful career in writing. Several of his novels, including the one under review, have made the *New York Times* bestseller list.

The first part of the book concentrates on Milo's life up to a few years after getting his PhD at UC Berkeley. A topologist, he receives his Fields Medal for proving the (fictional) Malosz conjecture. Now world-famous, he searches for another big open problem to tackle, finally deciding on the Abendroth conjecture (also fictional) after briefly considering the Goldbach conjecture (every even number greater than 2 is the sum of two primes). Milo's partial progress on the Abendroth conjecture leads to four excellent papers: one published in *Annals of Mathematics*, one in *Acta Mathematica*, and two in *Inventiones*. Milo's confidence that he will crack the Abendroth conjecture and reap even more fame is shattered when a proof of the Abendroth conjecture is discovered by a teenager.

Milo never recovers from losing the race to prove the Abendroth conjecture, and he is never again mathematically productive. The remaining 60 percent of the book deals with decades of decline and despair, focusing on

Milo's relationship with his family, especially his son, who is also a mathematician. At this point the book also has a dramatic shift of viewpoint, using a nice literary device that I will not disclose here because I enjoyed the surprise and do not want to spoil it for future readers.

Surely this novel's appearance on bestseller lists was not due to its mathematical context. I assume that non-mathematician readers were attracted by the beautifully written portrait of a seriously flawed genius and his family struggling with alcohol and drug problems in the midst of too many lies and deceptions. Although I appreciated the well-crafted prose and loved the mathematical setting, I did not like the nasty Milo, and I found the last 60 percent of the book to be depressing.

*The novel does a fine job of depicting the difficult creative process of mathematics.*

All three universities at which fictional mathematician Milo Andret was a student or faculty member are also places where I was a student or faculty member (Princeton, UC Berkeley, Michigan State University). Thus I paid special attention to the representations of the mathematical culture of those institutions. Here are the most significant departures from reality that I noticed in this book:

- The book gives the following description of mathematics graduate students at UC Berkeley at the time that Milo was a graduate student there:

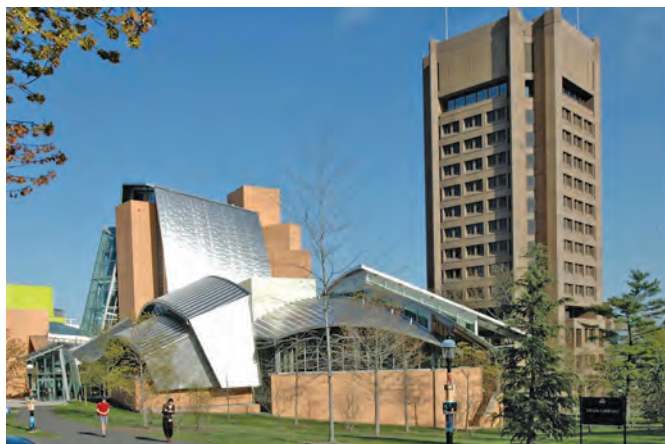
*At the time, most of his classmates hoped to find jobs at Xerox in Palo Alto or at IBM in White Plains or at one of the industry-funded think tanks that were popping up now along the coast.*

In fact, as I know because I was a graduate student at UC Berkeley at the same time as (the fictional) Milo, a large majority of our classmates wanted academic jobs.

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DOI: <http://dx.doi.org/10.1090/noti1491>



The main character in *A Doubter's Almanac* was a mathematics professor at Princeton University in the 1970s, with an office in Fine Hall, the tower on the right in this picture. Is the book accurate in suggesting that neckties were common attire among Princeton mathematicians in the 1970s? The reviewer, who earned his bachelor's degree at Princeton in 1971, remembers a largely necktie-free department.

- The book refers to the mathematics department at Princeton University as the

*Department of Broken Englishes—that's how they were known around campus.*

I was an undergraduate at Princeton a few years before the time of this remark, and I cannot recall ever hearing a complaint from a fellow student about the English language skills of any mathematics department faculty member.

- At one point when Milo is a tenure-track-but-still-untentured faculty member in the Princeton mathematics department, he is called into a committee meeting of the nine most senior faculty members of the department. Here is the book's description of Milo's perception of this committee:

*A startlingly uniform wall of bulbous Semitic features, threadbare sport coats, and colorless ties.*

I am at a loss as to how to deal with the author's use of the phrase "bulbous Semitic features." Thus I will just mention the inaccuracy at the end of the sentence above: my memory as a student at Princeton is that faculty rarely wore ties, and I never saw anything like nine of them wearing ties.

- The Fields Medals are awarded at the once-every-four-years meeting of the International Congress of Mathematicians (ICM). The Fields Medalists are notified in advance of their award so that they can be sure to attend the ICM, but otherwise the names of the winners are kept secret until the announcement at the ICM. That's how it goes in this novel. After Milo returns from the ICM, the chair of the Princeton mathematics department congratulates Milo on his award and on his discretion in not leaking it even to his colleagues, telling Milo:

*That's a pretty big secret to keep.*

Except it should not have gone that way because Milo was awarded the Fields Medal at the ICM held in Warsaw. That meeting of the ICM was originally scheduled for the summer of 1982 in Warsaw. However, in December 1981 martial law was declared in Poland and Solidarity was suppressed. The International Mathematical Union, which organizes the ICM, decided that the ICM could not be held in those conditions. Thus the ICM that was scheduled for 1982 was postponed until the summer of 1983, still in Warsaw. The International Mathematical Union announced the winners of the Fields Medal in 1982 rather than waiting until the postponed ICM in 1983. Thus the book is inaccurate in describing an element of secrecy concerning the announcement of Milo's Fields Medal at the Warsaw ICM.

- Late in the book, Milo's granddaughter is supposed to be reading *Swiss Family Robinson*, but her father discovers that she is actually reading *Zygmund and Fefferman's Trigonometric Series*.

This is a family of geniuses, so I think it's a reasonable stretch to have a child reading this classic and highly respected work. However, the authorship above is wrong. The first edition of this book appeared in 1935, with Antoni Zygmund as the sole author. In 1959 an expanded second edition was published in two volumes, again with Zygmund as the sole author. In 2003 a third (and so far final) edition was published, with a forward by Robert Fefferman but still with Zygmund as the sole author. Mathematicians working in Fourier series refer to these volumes as "Zygmund." I have never heard anyone call them "Zygmund and Fefferman."

I noticed four more tiny errors, all less consequential than those discussed above. Overall, I consider the number of errors in this 551-page novel written by a nonmathematician about a mathematician to be remarkably low.

The author gets many big issues right. In particular, the novel does a fine job of depicting the difficult creative process of mathematics. The reader will come away with the understanding that even a superstar mathematician's research breakthroughs come only after being stuck with little progress for long periods of time.

Mathematicians who read this book will appreciate numerous tidbits that other readers will miss. For example, all the chapter titles have a mathematical meaning, starting with "Induction" as the title of the first chapter and ending with "Proof" as the title of the last chapter. As another example, Milo's daughter, Paulette, is named after Paul Erdős (the book gets the Hungarian double acute accent correct, in contrast to the incorrect umlaut that one often sees decorating this mathematician's name). Although the novel explains that Paul Erdős is a famous mathematician, readers who are not mathematicians will probably be unaware of the startlingly unusual aspects of Erdős's career that Milo may have found attractive.

Normally, book reviewers should not read other reviews of the book in question until after finalizing their own review. In this case, I broke that rule because the public perception of mathematics deserves our attention. Too often we see negative comments about mathematics in the media. One of my least favorite examples comes from the *New York Times* editorial in 1984 endorsing Walter Mondale for US president over Ronald Reagan. Trying to balance its positive comments about Mondale with something negative, the *New York Times* wrote:

*Walter Mondale has all the dramatic flair of a trigonometry teacher.*

To check whether the reviews of this book would be filled with such sentiments, I chose eight large cities in widespread geographic areas of the United States and looked for reviews of this book in the main newspaper of each of those cities. Seven of those eight newspapers (all except the *Atlanta Journal-Constitution*) reviewed this book in 2016, showing that this book received good exposure in popular culture. The reviews contained less hostility to mathematics than I had expected. The worst offender was the *Washington Post*, whose review said that the book

*is a long, complex novel about math, which sounds like the square root of tedium, but suspend your flight instinct for a moment. Ethan*

---

*a very  
unpleasant  
and extremely  
unhappy  
mathematical  
genius...who  
is no role  
model*

---

*Canin writes with such luxuriant beauty and tender sympathy that even victims of Algebra II will follow his calculations of the heart with rapt comprehension.*

The *Boston Globe* review of this book assumed that some of its readers have forgotten arithmetic that they learned in elementary school:

*[O]ne need not have studied calculus (or even remember how to add and subtract fractions) to appreciate the book.*

Two of the seven reviews that I read justifiably raised issues about the role of women in this book. Milo horribly mistreats his lovers, his wife, and his daughter, and they keep coming back for more. Canin's blinkered conception of his female characters strains credulity. After summarizing one incident in the book when a departmental secretary finally admits a drunken Milo into her hotel room, the reviewer for the *New York Times* wrote:

*At this juncture of "no" meaning "yes" (not the only such in these pages), I took a moment to toss the book across the room.*

Milo has one great mathematical triumph when he proves the Malosz conjecture, but everything else about his life reeks of disaster. I cannot imagine that any reader would want to be Milo, not even in return for a Fields Medal. The author's creation of Milo as a very unpleasant and extremely unhappy mathematical genius may be a literary success, but Milo is no role model.

### Photo Credit

Photo of Princeton University is courtesy of Princeton University, Office of Communications; photo by Brian Wilson.



Sheldon Axler

### ABOUT THE REVIEWER

**Sheldon Axler** is the author of several books, including *Linear Algebra Done Right*, *Harmonic Function Theory* (with Paul Bourdon and Wade Ramey), and *Precalculus: A Prelude to Calculus*. He received the MAA's Lester R. Ford Award for expository writing in 1996. He was dean of the College of Science & Engineering at San Francisco State University for thirteen years before returning full time to mathematics in 2015. He has served as a member of the Council of the AMS and as a member of the Board of Trustees of MSRI.



# 2017 Class of Fellows of the AMS

Sixty-five mathematical scientists from around the world have been named Fellows of the American Mathematical Society (AMS) for 2017.

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics. Among the goals of the program are to create an enlarged class of mathematicians recognized by their peers as distinguished for their contributions to the profession and to honor excellence.

The 2017 class of Fellows was honored at a dessert reception held during the Joint Mathematics Meetings in Atlanta, Georgia. Names of the individuals who are in this year's class, their institutions, and citations appear below.

The nomination period for Fellows is open each year from February 1 to March 31. For additional information about the Fellows program, as well as instructions for making nominations, visit the web page [www.ams.org/profession/ams-fellows](http://www.ams.org/profession/ams-fellows).



**AMS Executive Director Catherine A. Roberts chats with 2017 Fellows, including Scott T. Chapman of Sam Houston State University.**

**Dan Abramovich**, Brown University  
*For contributions to algebraic geometry and service to the mathematical community.*

**Guillaume Bal**, Columbia University  
*For contributions to inverse problems and wave propagation in random media.*

**John T. Baldwin**, University of Illinois at Chicago  
*For contributions to model theory, exposition, and service to mathematics education.*

**Alexandra Bellow**, Northwestern University  
*For contributions to analysis, particularly ergodic theory and measure theory, and for exposition.*

**Aaron Bertram**, University of Utah  
*For contributions to algebraic geometry and for service to the mathematical community.*

**Jeffrey Brock**, Brown University  
*For contributions to Kleinian groups, low-dimensional topology and geometry, and Teichmüller theory.*

**Jim Bryan**, The University of British Columbia  
*For contributions to algebraic geometry and service to the mathematical community.*

**Gunnar Carlsson**, Stanford University  
*For contributions to algebraic topology, particularly equivariant stable homotopy theory, algebraic K-theory, and applied algebraic topology.*

**Mei-Chu Chang**, University of California, Riverside  
*For contributions to arithmetic combinatorics, analytic number theory, and algebraic geometry.*

**Sagun Chanillo**, Rutgers The State University of New Jersey, New Brunswick  
*For contributions to partial differential equations and geometric analysis.*

**Scott T. Chapman**, Sam Houston State University  
*For contributions to algebra and for service to the mathematical community.*

**Gui-Qiang G. Chen**, University of Oxford and Keble College  
*For contributions to partial differential equations, nonlinear analysis, fluid mechanics, hyperbolic conservation laws, and shock wave theory.*

**Jungkai Alfred Chen**, National Taiwan University  
*For contributions to algebraic geometry and for service to the mathematical community.*

**Mihai Ciucu**, Indiana University, Bloomington  
*For contributions to combinatorics, particularly relating gaps in lattice tilings to electrostatics.*

*(Continued on next page)*

**Miriam Cohen**, Ben Gurion University of the Negev  
*For contributions to Hopf algebras and their representations, and for service to the mathematical community.*

**Donatella Danielli**, Purdue University  
*For contributions to partial differential equations and geometric measure theory, and for service to the mathematical community.*

**Moon Duchin**, Tufts University  
*For contributions to geometric group theory and Teichmüller theory, and for service to the mathematical community.*

**Yalchin Efendiev**, Texas A&M University  
*For contributions to the field of multiscale finite-element methods with applications to porous-media fluid flow.*

**Kirsten Eisenträger**, Pennsylvania State University  
*For contributions to computational number theory and number-theoretic undecidability.*

**Mark Feighn**, Rutgers The State University of New Jersey, New Brunswick  
*For contributions to geometric group theory.*

**Rui Loja Fernandes**, University of Illinois, Urbana-Champaign  
*For contributions to the study of Poisson geometry and Lie algebroids, and for service to the mathematical community.*

**Yan Guo**, Brown University  
*For contributions to the mathematical theory of fluids and plasmas.*

**Piotr Hajlasz**, University of Pittsburgh  
*For contributions to analysis in metric spaces, in particular the notion of Sobolev spaces in metric-measure spaces also known as Hajlasz-Sobolev spaces.*

**Kathryn Hess**, École Polytechnique Fédérale de Lausanne (EPFL)  
*For contributions to homotopy theory, applications of topology to the analysis of biological data, and service to the mathematical community.*

**Nancy Hingston**, The College of New Jersey  
*For contributions to differential geometry and the study of closed geodesics.*

**Yulij Ilyashenko**, Cornell University and the National Research University Higher School of Economics  
*For contributions to dynamical systems and for service to the mathematical community.*

**Marius Junge**, University of Illinois, Urbana-Champaign  
*For contributions to the study of operator algebras, Banach spaces, harmonic analysis, and noncommutative probability, and for applications to quantum information theory.*



Attendants at the 2017 Fellows of the AMS Reception.

**Dmitry Kleinbock**, Brandeis University  
*For contributions to homogeneous dynamics and its applications to number theory, especially in metric Diophantine approximation.*

**Toshiyuki Kobayashi**, University of Tokyo  
*For contributions to the structure and representation theory of reductive Lie groups.*

**Alex Kontorovich**, Rutgers The State University of New Jersey, New Brunswick  
*For contributions to analytic number theory and for mathematical exposition.*

**Daniel Krashen**, University of Georgia  
*For contributions to the study of central simple algebras and local-global principles and for service to the mathematical community.*

**Henning Krause**, Universität Bielefeld  
*For contributions to representation theory and homological algebra, and for service to the mathematical community.*

**Michael Krivelevich**, Tel Aviv University  
*For contributions to extremal and probabilistic combinatorics.*

**Joseph M. Landsberg**, Texas A&M University  
*For contributions to differential geometry, geometry of projective varieties, representation theory, and complexity theory.*

**Congming Li**, University of Colorado, Boulder  
*For contributions to nonlinear partial differential equations and applications.*

**Jian-Guo Liu**, Duke University  
*For contributions to the analysis of numerical methods for fluid dynamics, kinetic theory, and nonlinear partial differential equations.*



**AMS Immediate Past President Robert L. Bryant with former AMS Executive Director and Fellow William "Bus" Jaco.**

**Ciprian Manolescu**, University of California, Los Angeles  
*For contributions to Floer homology and the topology of manifolds.*

**Kevin M. McCrimmon**, University of Virginia  
*For contributions to the theory of Jordan algebras, exposition, and service to the mathematical community.*

**Umberto Mosco**, Worcester Polytechnic Institute  
*For contributions to analysis and partial differential equations, in particular for introducing a theory of variational convergence.*

**Allen Moy**, Hong Kong University of Science and Technology  
*For contributions to representation theory of reductive groups over nonarchimedean local fields and for service to the mathematical community.*

**Isabella Novik**, University of Washington  
*For contributions to algebraic and geometric combinatorics.*

**Tony Pantev**, University of Pennsylvania  
*For contributions to algebraic geometry, mathematical physics, and string theory, and for service to the mathematical community.*

**Julia Pevtsova**, University of Washington  
*For contributions to modular representation theory.*

**Ami Radunskaya**, Pomona College  
*For contributions to mathematical oncology, immunodynamics, and applications of dynamical systems to medicine, and for service to the mathematical community.*

**Lasse Rempe-Gillen**, The University of Liverpool  
*For contributions to complex dynamics and function theory, and for communication of mathematical research to broader audiences.*

**Xiaochun Rong**, Rutgers The State University of New Jersey, New Brunswick  
*For contributions to Riemannian geometry.*

**Daniel Ruberman**, Brandeis University  
*For contributions to low-dimensional topology.*

**David Savitt**, Johns Hopkins University, Baltimore  
*For contributions to number theory and service to the mathematical community.*

**Richard Evan Schwartz**, Brown University  
*For contributions to dynamics, geometry, and experimental mathematics and for exposition.*

**Nimish A. Shah**, Ohio State University, Columbus  
*For contributions to ergodic theory and homogeneous dynamics and applications to number theory.*

**Peter B. Shalen**, University of Illinois at Chicago  
*For contributions to three-dimensional topology and for exposition.*

**Jie Shen**, Purdue University  
*For contributions to theoretical numerical analysis, scientific computing, computational fluid dynamics, and computational materials science.*

**Zuowei Shen**, National University of Singapore  
*For contributions to approximation theory, wavelet theory, and image processing.*

**Ivan Shestakov**, Universidade de São Paulo  
*For contributions to nonassociative algebra and affine algebraic geometry.*

**Sergei Starchenko**, University of Notre Dame  
*For contributions to model theory and its applications to geometry, analysis, number theory, and combinatorics.*

**Jason Starr**, Stony Brook University  
*For contributions to algebraic geometry.*

**Robert Strichartz**, Cornell University  
*For contributions to analysis and partial differential equations, for exposition, and for service to the mathematical community.*

**Daniel B. Szyld**, Temple University  
*For contributions to numerical and applied linear algebra.*

**Tao Tang**, Southern University of Science and Technology  
*For contributions to numerical methods for partial differential equations.*

FROM THE AMS SECRETARY

**Dinesh S. Thakur**

University of Rochester

*For contributions to the arithmetic of function fields, exposition, and service to the mathematical community.*

**Dylan Paul Thurston**, Indiana University, Bloomington

*For contributions to low-dimensional topology and geometry, and to cluster algebras.*

**Tatiana Toro**, University of Washington

*For contributions to geometric measure theory, potential theory, and free boundary theory.*

**Ben Weinkove**, Northwestern University

*For contributions to complex geometry and for service to the mathematical community.*

**Alexandru Zaharescu**, University of Illinois, Urbana-Champaign and Institute of Mathematics, Romanian Academy

*For contributions to analytic number theory.*


**Ofer Zeitouni**, Weizmann Institute of Science

*For contributions to probability theory.*

—See more at: [www.ams.org/profession/ams-fellows/new-fellows](http://www.ams.org/profession/ams-fellows/new-fellows)

—Photos courtesy of Kate Awetry/JMM 2017 photographer.

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

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# Mathematics People

## Viazovska Awarded Salem Prize



**Maryna Viazovska**

MARYNA VIAZOVSKA of the Berlin Mathematical School and Humboldt University of Berlin has been awarded the 2016 Salem Prize “for her breakthrough work on densest sphere packings in dimensions 8 and 24 using methods of modular forms.” She earned her PhD from the University of Bonn in 2013 with her dissertation “Modular functions and special cycles.” She works in the areas of number theory, discrete geometry, approximation theory, and physics. The prize, in memory of Raphael Salem, is awarded yearly to young researchers for outstanding contributions to the field of analysis.

*Editor’s note:* For more on the work of Maryna Viazovska, see the feature article “A Conceptual Breakthrough in Sphere Packing” by Henry Cohn in the February 2017 *Notices*.

—Elly Gustafsson,  
Institute for Advanced Study

## Presidential Mentoring Awards Given

The Presidential Awards for Excellence in Science, Mathematics, and Engineering Mentoring for fiscal year 2012 were awarded in 2015 by President Obama to fourteen individuals and one organization. Two scholars whose work involves the mathematical sciences were selected to receive awards: RAYMOND L. JOHNSON of the University of Maryland and J. TILAK RATNANATHER of Johns Hopkins University.



**Raymond L. Johnson**

Johnson was recognized “for his tireless and highly successful mentoring efforts with students from groups underrepresented in mathematics.” He was the first African American admitted to Rice University (1964), where he earned a doctorate in mathematics in 1969 for his dissertation, “A priori estimates and unique continuation theorems for second order parabolic equations.” He joined the faculty at Maryland in 1980, becoming the first African American to be pro-

moted to associate professor and the first to serve as Chair of the Department of Mathematics. His research began with work on non-well-posed problems, which led him to the study of Besov spaces and harmonic analysis. His interest in harmonic analysis continues today. After forty years at the University of Maryland, Johnson retired and returned to Rice as a visiting professor.

At the University of Maryland, Johnson personally mentored numerous graduate students, with the largest number of students advised in the period between 1990 and 2009. During that time, fifty-three underrepresented minority students pursued their MA and/or PhD degrees. All but one of these students was African American, and twenty-two of the students were African American women. Many of his graduate students graduated from smaller, historically black colleges and universities. Johnson’s mentoring plan involved regular group meetings to develop a sense of community, as well as course selection and counseling. Keenly aware of the need to familiarize minority students with interdisciplinary environments, Johnson encouraged his protégés to interact with minority graduate students across department lines.

Of his fifty-three graduate students, twenty-three completed their PhDs in mathematics. In 2000, the first African American woman earned a PhD in mathematics at the University of Maryland; in fact, three African American women graduated at the same time. Fourteen of his PhD recipients currently hold academic appointments at major US institutions of higher education, and three are tenured professors. Johnson was honored with the AAAS Lifetime Mentor Award in 2007.



**J. Tilak Ratnanather**

Ratnanather was recognized for his work “creating a system to support deaf and hard-of-hearing individuals in STEM.” He was the first congenitally deaf person to earn an undergraduate degree in mathematics from University College, London, in 1985, and in 1989 he became the first ever congenitally deaf individual in the world to be graduated with a doctorate in mathematics, which he earned from Oxford University. His interest in postdoctoral research in the auditory sciences brought him to Baltimore, Maryland, and the Johns Hopkins University School of Medicine. Ratnanather’s research interests include computational anatomy applications in neurodevelopmental and neurodegeneration disorders, as well as cochlear micromechanics and fluid mechanics.

There is a simple and powerful objective to Ratnanather’s mentoring programs: to provide opportunities for education and research in science, technology, engineering, and mathematics (STEM) for deaf and hard-of-hearing individuals who may not have otherwise been exposed to STEM and to achieve this objective through extensive and involved networking so that his protégés can later serve as mentors themselves.

The lack of accommodations for the deaf and hard of hearing at annual meetings of the Association for Research in Otolaryngology led him to begin his work to recruit deaf and hard-of-hearing students and pro-

vide the accommodation that allows them to participate in professional meetings. His success is clearly documented: In 1991, Ratnanather and one other graduate student were the only deaf and hard-of-hearing individuals pursuing studies in the auditory sciences. In 2011, there were seven deaf and hard-of-hearing faculty members in the auditory sciences, with more than fifteen currently pursuing graduate degrees in that field. He established the Hearing-Impaired Association for Research in Otolaryngology in 1992.

Ratnanather has personally mentored thirteen deaf or hard-of-hearing students in both STEM and medical school programs (five of whom are pursuing careers in medicine). His mentoring work includes twenty hearing female students, all of whom have pursued doctoral degrees in STEM. He continues to think about how new ideas in education and technology can broaden access by people with hearing loss.

—From a National Science Foundation announcement

## Radunskaya Receives AAAS Mentor Award



**Ami Radunskaya**

AMI RADUNSKAYA of Pomona College has been honored with the 2016 Mentor Award of the American Associa-

tion for the Advancement of Science for launching “dramatic education and research changes leading to an increase in the number of female doctorates in the field of mathematics.” She has mentored eighty-two students—eighty of whom are women—in earning their PhD degrees in mathematics. Twenty-three of her students are African American and five are Latino, and most of them are now affiliated with universities and colleges. Radunskaya is president-elect of the Association for Women in Mathematics and director of the Enhancing Diversity in Graduate Education (EDGE) program in mathematics. She earned her PhD in mathematics from Stanford University and has been honored with several awards for teaching and mentoring, as well as being named the AWM Falconer Lecturer in 2010.

—From an AAAS announcement

## Mnev Awarded Lichnerowicz Prize



**Pavel Mnev**

PAVEL MNEV of Notre Dame University has been awarded the André Lichnerowicz Prize in Poisson geometry for his work, which is “at the interface of Poisson geometry, topology, and mathematical physics.” Mnev received his PhD in 2008 from the Steklov Institute of Mathematics in St. Petersburg under the direction of Ludwig Fadeev and has held positions at the University of Zurich and the Max Planck

Institute. His research interests are in mathematical physics, in particular in the interactions of quantum field theory with topology, homological/homotopical algebra, and supergeometry. The award is given for outstanding work by a young mathematician in Poisson geometry.

—From a University of Notre Dame announcement

## Burkhardt and Swan Receive Castelnovo Award



**Hugh Burkhardt and Malcolm Swan**

HUGH BURKHARDT and MALCOLM SWAN of the Shell Centre for Mathematical Education at Nottingham University have been awarded the 2016 inaugural Emma Castelnovo Award for Excellence in the Practice of Mathematics Education “in recognition of their more than thirty-five years of development and implementation of innovative, influential work in the practice of mathematics education, including the development of curriculum and assessment materials, instructional design concepts, teacher preparation programs, and educational system changes.” The Castelnovo Award is given by the International Commission on Mathematical Instruction (ICMI) to recognize outstanding achievements in the practice of mathematics education.

—From an ICMI announcement

## Rizell Receives Wallenberg Fellowship



**Georgios Dimitroglou Rizell**

GEORGIOS DIMITROGLOU RIZELL of Uppsala University has been awarded a Wallenberg Academy Fellowship for his contributions to the development of symplectic geometry. According to the prize citation, “As a Wallenberg Academy Fellow he will continue to build upon the theory by investigating and classifying so-called Lagrangian submanifolds; these are subspaces whose properties are important for understanding the ambient symplectic space.” Wallenberg Fellows receive five-year grants of 5–9 million Swedish krona (approximately US\$542,000–976,000), depending on the field, and may apply for an additional five years. The program was established by the Knut and Alice Wallenberg Foundation in close cooperation with five learned academies and sixteen Swedish universities to give the most promising young researchers a work situation that enables them to focus on their projects and address difficult research questions over an extended period of time.

—From a Wallenberg Academy announcement

## Mathematical Society of Japan Prizes

The Mathematical Society of Japan (MSJ) awarded the following prizes at the MSJ Autumn Meeting in 2016.

The 2016 Autumn Prize was awarded to SHIGEYUKI MORITA, emeritus professor at the University of Tokyo and Tokyo Institute of Technol-



**Shigeyuki Morita**

ogy, for his outstanding contributions to work on cohomology theory of mapping class groups and outer automorphism groups of free groups. The MSJ Autumn Prize and the MSJ Spring Prize are the most prestigious prizes awarded by the MSJ to its members.

The 2016 Analysis Prizes were awarded to SOICHIRO KATAYAMA of Osaka University for studies on null structure in systems of nonlinear hyperbolic partial differential equations; to SHIGEKI KOIKE of Tohoku University for work on the theory of  $L^p$ -viscosity solutions for fully nonlinear elliptic and parabolic partial differential equations; and to TOMOHIRO SAKAMOTO of the Tokyo Institute of Technology for studies on nonequilibrium stochastic dynamical systems by exact solutions.

The 2016 Geometry Prizes were awarded to TERUHIKO SOMA of Tokyo Metropolitan University for a series of works on 3-manifold theory and to SHIGEHARU TAKAYAMA of the University of Tokyo for the algebro-geometric study of birationality of pluri-

canonical maps of algebraic varieties of general type.

The 2016 Takebe Katahiro Prizes were awarded to NORIHISA IKOMA of Kanazawa University for work on variational and nonvariational approaches for nonlinear elliptic problems; to TAKEFUMI NOSAKA of Kyushu University for work on algebraic topology of quandles and low-dimensional manifolds; and to MAKOTO YAMASHITA of Ochanomizu University for operator algebraic studies on quantum groups. The Takebe Katahiro Prize is awarded to young researchers who have obtained outstanding results.

The 2016 Takebe Katahiro Prizes for Encouragement of Young Researchers were awarded to KEN ABE of Kyoto University for work on the analysis of the Navier-Stokes equations by maximum norm; to YOSHIHIRO ABE of Kobe University for detailed estimates on cover times and local times of random walks on graphs; to TAKAHIRO OBA of Tokyo Institute of Technology for work on contact manifolds and their Stein fillings; to RYO KANDA of Osaka University for work on atom spectra of Grothendieck categories; to YU KITABEPPU of Kyoto University for work on geometry of spaces with Ricci curvature bounded below; and to YUTA WAKASUGI of Nagoya University for studies on the asymptotic behavior of solutions to damped wave equations. The Takebe Prize is intended for young mathematicians who are deemed to have begun promising careers in research by obtaining significant results.

—From an MSJ announcement

## Prizes of the New Zealand Mathematical Society

The New Zealand Mathematical Society (NZMS) has announced several awards for 2016.

DAVID BRYANT of the University of Otago and BERND KRAUSKOPF of the University of Auckland have been

named recipients of the NZMS Research Award. Bryant was honored for “work developing mathematical, statistical and computational tools for evolutionary biology, and work drawing on evolutionary biology to develop new theories in mathematics.” Krauskopf was honored for “outstanding contributions to dynamical systems, especially bifurcation theory and its application to diverse physical phenomena.”

GAVEN MARTIN of Massey University received the Kalman Prize for Best Paper for his article with T. H. Marshall “Minimal co-volume hyperbolic lattices, II: Simple torsion in a Kleinian group,” *Annals of Mathematics* 176(2012). [See Martin's cover story in the December 2016 *Notices*.]

ALEXANDER MELNIKOV of Massey University received the Early Career Award for “highly original contributions to the theory of computability in algebra and topology.”

NAOMI GENDLER of the University of Auckland was awarded the Aitken Prize for best contributed talk by a student at the annual NZMS Colloquium for her talk “Pulse dynamics of fibre lasers with saturable absorbers.”

—From an NZMS announcement

## Jackson Awarded 2016 Rosenthal Prize

TRACI JACKSON of Oak Valley Middle School, San Diego, California, has been awarded the 2016 Rosenthal Prize for Innovation in Math Teaching for her lesson “Creating color combos: Visual modeling of proportional relationships!” In the lesson, students explore proportional reasoning by mixing colored solutions, creating different color combinations to visualize ratios. Jackson received a cash award of US\$25,000. The second place award went to DENA LORDI of Diamond Bar High School in Diamond Bar, California, for her lesson “Where can I find a weightless stick?” In this lesson, students trace the changing balance point on a scale as weights are added

in order to identify the mean value of a set of numbers. Runners-up were CRYSTAL FROMMERT, JEMAL GRAHAM, and MARIA HERNANDEZ.

—From a National Museum of Mathematics announcement

## W. Wistar Comfort (1933–2016)



William Wistar Comfort

WILLIAM WISTAR “WIS” COMFORT, long-time associate secretary of the AMS, was known to many for his gallantry, dry wit, and humility. He was a widely published mathematician and scholar whose teaching career spanned five decades. A formidable runner and racquet sports athlete, Wis was beloved by his peers for his sense of fairness. As a lifelong Quaker, he could be counted on to speak for the marginalized or overlooked.

Wis held academic positions at Harvard University, the University of Rochester, and the University of Massachusetts at Amherst. He came to Wesleyan in July 1967 and from 1982 onwards was the Edward Burr Van Vleck Professor of Mathematics. He served as department chair three times before retiring in December 2007.

He continued to publish mathematical works in the weeks leading up to his death; more will be published posthumously. Wis mentored more than twenty-five graduate students and has about 150 publications with

fifty-seven coauthors from at least seventeen countries.

Wis worked principally in general topology, with a specialty in topological groups and cardinal invariants. He enjoyed infinitary combinatorics and their applications to topological structures. He held various editorial and advocacy positions for mathematical journals.

He had an extensive record of service to the AMS, including ten years, from 1973 to 1983, as associate secretary for the Eastern Section. He also served on the AMS Council, as well as several committees. He was the topology editor for the *Proceedings* from 1971 to 1974, and in the last year of that period served as managing editor. He was named a Fellow of the AMS in the 2013 inaugural class.

Born on April 19, 1933, in Bryn Mawr, Pennsylvania, Wis Comfort graduated from Haverford College (Pennsylvania) in 1954. His roots at Haverford run deep. His father, Howard Comfort II, was head of the Classics Department, and his grandfather, William Wistar Comfort, for whom he was named, was a noted Quaker scholar and president of Haverford from 1917 to 1940. Wis received the PhD degree from the University of Washington (Seattle) in 1958, where his thesis director was Edwin Hewitt. He married Mary Constance Lyon in March 1957, and the couple produced two children, Martha Wistar Comfort and Howard Comfort III, and enjoyed fifty-nine years of marriage before Mary Connie passed in May 2016.

Wis Comfort's principal postretirement avocation was Dixieland trombone, and he was affiliated with many groups in Connecticut and Maine. His rich, deep singing voice delighted many, and he worked assiduously at his music, living and breathing the old-time tunes.

—Martha Wistar Comfort

## Robert Seeley (1932–2016)



**Robert Seeley**

ROBERT SEELEY, professor emeritus of mathematics at the University of Massachusetts at Boston, was a pioneer in the theory of pseudo-differential operators. He earned his PhD at the Massachusetts Institute of Technology with Alberto Calderón and taught at Harvey Mudd College and Brandeis University before settling at the University of Massachusetts. He was as interested in the beginners as in the math majors, so he gladly taught the full range of courses. When students were ready for advanced work beyond what was available in the curriculum, he cheerfully and regularly supervised independent study. In retirement he taught mathematics in prisons.

Seeley was one of the first Fellows of the AMS and a member-at-large of the AMS Council from 1972 to 1974.

One of the great mathematical inventions of the 1960s was the theory of pseudo-differential operators. This theory made possible both the Atiyah-Singer Index Theorem and “microlocal” techniques with which one could apply the tools of symplectic geometry and dynamical systems to problems in partial differential equation theory. The origins of microlocal analysis were the papers on “singular integral operators” of Calderón and Zygmund in the late 1950s, but it was Seeley’s work in the next decade that turned these ideas into the modern theory of pseudo-differential operators.

Bob Seeley was a deep, thoughtful, kind family man, a mainstay of the Quaker community in Cambridge, and a world traveler who spent sabbaticals

in Peru, Mexico, Italy, and the Netherlands. He loved to sing, bike, run, and ski cross-country. Visitors to his home in Newton enjoyed both his hospitality and the marvels of carpentry he installed there to complement the furniture he built.

—Ethan Bolker,  
University of Massachusetts Boston

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# Mathematics Opportunities

## Project NExT 2017–2018

MAA Project NExT (New Experiences in Teaching) is a year-long professional development program of the Mathematical Association of America (MAA) for new or recent PhDs in the mathematical sciences. The program is designed to connect new faculty with master teachers and leaders in the mathematics community and address the three main aspects of an academic career: teaching, research, and service. Recent program sessions have included:

- Getting your research and grant writing off to a good start
- Innovative teaching and assessment methods and why they work
- Finding your niche in the profession
- Attracting and retaining underrepresented students
- Balancing teaching, research, and service demands
- Starting an undergraduate research program
- Preparing for tenure

MAA Project NExT Fellows join an active community of faculty who have gone on to become award-winning teachers, innovators on their campuses, active members of the MAA, and leaders in the profession. MAA Project NExT welcomes and encourages applications from new and recent PhDs in postdoctoral, tenure-track, and visiting positions. We particularly encourage applicants from underrepresented groups (including women and minorities). The deadline for applications for the 2017 cohort of MAA Project NExT Fellows is **April 15, 2017**. For applications and further information, see [projectnext.maa.org](http://projectnext.maa.org).

—From an MAA announcement

## Call for Nominations for Graham Wright Award

The Graham Wright Award for Distinguished Service of the Canadian Mathematical Society (CMS) recognizes individuals who have made sustained and significant contributions to the Canadian mathematical community and, in particular, to the CMS. The deadline for nominations is **March 31, 2017**. See [cms.math.ca/Prizes/dis-nom](http://cms.math.ca/Prizes/dis-nom).

—From a CMS announcement

## Intensive Research Programs at CRM

The Centre de Recerca Matemàtica (CRM) invites proposals for research programs consisting of one to five months of intensive research in any branch of mathematics or the mathematical sciences. Proposals are sought for programs to be organized preferably after August 2018. Preliminary proposals are due **February 28, 2017**; final proposals should be submitted before **April 28, 2017**. For guidelines and instructions, see [www.crm.cat/en/Host/SciEvents/IRP/Pages/CallApplication.aspx](http://www.crm.cat/en/Host/SciEvents/IRP/Pages/CallApplication.aspx).

—From a CRM announcement



**Renaissance Technologies,**  
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— Bloomberg Markets article, November 21, 2016

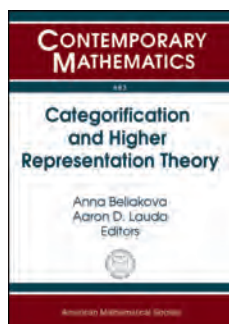
To apply, send your resume to [careers@rentec.com](mailto:careers@rentec.com).  
For more information, visit [www.rentec.com/careers](http://www.rentec.com/careers).

**Renaissance**

# New Publications Offered by the AMS

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## Algebra and Algebraic Geometry



### Categorification and Higher Representation Theory

**Anna Beliakova**, *Universität  
Zürich, Switzerland*, and  
**Aaron D. Lauda**, *University of  
Southern California, Los Angeles,  
CA*, Editors

The emergent mathematical philosophy of categorification is reshaping our view of modern mathematics by uncovering a hidden layer of structure in mathematics, revealing richer and more robust structures capable of describing more complex phenomena. Categorified representation theory, or higher representation theory, aims to understand a new level of structure present in representation theory. Rather than studying actions of algebras on vector spaces where algebra elements act by linear endomorphisms of the vector space, higher representation theory describes the structure present when algebras act on categories, with algebra elements acting by functors. The new level of structure in higher representation theory arises by studying the natural transformations between functors. This enhanced perspective brings into play a powerful new set of tools that deepens our understanding of traditional representation theory.

This volume exhibits some of the current trends in higher representation theory and the diverse techniques that are being employed in this field with the aim of showcasing the many applications of higher representation theory.

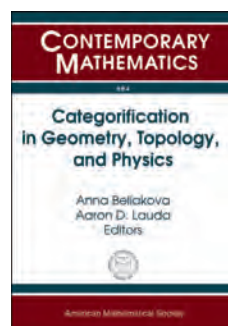
The companion volume (Contemporary Mathematics, Volume 684) is devoted to categorification in geometry, topology, and physics.

**Contents:** **I. Losev**, Rational Cherednik algebras and categorification; **O. Dudas**, **M. Varagnolo**, and **E. Vasserot**, Categorical actions on unipotent representations of finite classical groups; **J. Brundan** and **N. Davidson**, Categorical actions and crystals; **A. M. Licata**, On the 2-linearity of the free group; **M. Ehrig**, **C. Stroppel**, and **D. Tubbenhauer**, The Blanchet-Khovanov algebras; **G. Lusztig**, Generic character sheaves on groups over  $k[\epsilon]/(\epsilon^r)$ ; **D. Berdeja Suárez**, Integral presentations of quantum

lattice Heisenberg algebras; **Y. Qi** and **J. Sussan**, Categorification at prime roots of unity and hopfological finiteness; **B. Elias**, Folding with Soergel bimodules; **L. T. Jensen** and **G. Williamson**, The p-canonical basis for Hecke algebras.

**Contemporary Mathematics**, Volume 683

March 2017, approximately 363 pages, Softcover, ISBN: 978-1-4704-2460-2, 2010 *Mathematics Subject Classification*: 81R50, 17B10, 20C08, 14F05, 18D10, 17B50, 17B55, 17B67, **AMS members US\$88.80**, List US\$111, Order code CONM/683



### Categorification in Geometry, Topology, and Physics

**Anna Beliakova**, *Universität  
Zürich, Switzerland*, and  
**Aaron D. Lauda**, *University of  
Southern California, Los Angeles,  
CA*, Editors

The emergent mathematical philosophy of categorification is reshaping our view of modern mathematics by uncovering a hidden layer of structure in mathematics, revealing richer and more robust structures capable of describing more complex phenomena. Categorification is a powerful tool for relating various branches of mathematics and exploiting the commonalities between fields. It provides a language emphasizing essential features and allowing precise relationships between vastly different fields.

This volume focuses on the role categorification plays in geometry, topology, and physics. These articles illustrate many important trends for the field including geometric representation theory, homotopical methods in link homology, interactions between higher representation theory and gauge theory, and double affine Hecke algebra approaches to link homology.

The companion volume (Contemporary Mathematics, Volume 683) is devoted to categorification and higher representation theory.

*This item will also be of interest to those working in geometry and topology.*

**Contents:** **B. Webster**, Geometry and categorification; **Y. Li**, A geometric realization of modified quantum algebras; **T. Lawson**, **R. Lipshitz**, and **S. Sarkar**, The cube and the Burnside category; **S. Chun**, **S. Gukov**, and **D. Roggenkamp**, Junctions of surface

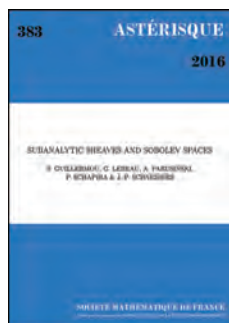
operators and categorification of quantum groups; **R. Rouquier**, Khovanov-Rozansky homology and 2-braid groups; **I. Cherednik** and **I. Danilenko**, DAHA approach to iterated torus links.

**Contemporary Mathematics**, Volume 684

March 2017, approximately 268 pages, Softcover, ISBN: 978-1-4704-2821-1, 2010 *Mathematics Subject Classification*: 81R50, 57M25, 14F05, 18D10, 58J28, 17B81, 20C08, 17B55, 17B67, **AMS members US\$88.80**, List US\$111, Order code CONM/684

## New AMS-Distributed Publications

### Algebra and Algebraic Geometry



### Subanalytic Sheaves and Sobolev Spaces

**Stéphane Guillermou**, *Université de Grenoble I, Saint-Martin d'Hères*, **Gilles Lebeau**, *Université Nice Sophia Antipolis, France*, **Adam Parusiński**, *Université Nice Sophia Antipolis, France*, **Pierre Schapira**, *Université Paris 6, Jussieu, France*, and **Jean-Pierre Schneiders**, *Université de Liège, Belgique*

Sheaves on manifolds are perfectly suited to treat local problems, but many spaces that one naturally encounter, especially in analysis, are not of a local nature. The subanalytic topology (in the sense of Grothendieck) on real analytic manifolds allows the authors to partially overcome this difficulty and to define, for example, sheaves of functions or distributions with temperate growth but not to make the growth precise.

In this volume, the authors introduce the linear subanalytic topology, a refinement of the preceding one, and construct various objects of the derived category of sheaves on the subanalytic site with the help of the Brown representability theorem. In particular, they construct the Sobolev sheaves. These objects have the nice property that the complexes of their sections on open subsets with Lipschitz boundaries are concentrated in degree zero and coincide with the classical Sobolev spaces.

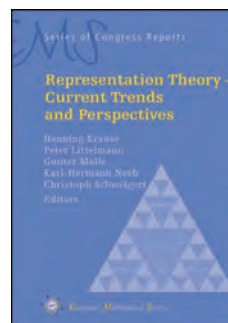
Another application of this topology is that it allows the authors to functorially endow regular holonomic D-modules with filtrations (in the derived sense).

In the course of the text, the authors also obtain some results on subanalytic geometry and make a detailed study of the derived category of filtered objects in symmetric monoidal categories.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Astérisque**, Number 383

October 2016, 120 pages, Softcover, ISBN: 978-2-85629-844-2, 2010 *Mathematics Subject Classification*: 16E35, 16W70, 18A25, 18D10, 18D35, 18F20, 32B20, 32C05, 32C38, 32S60, 46E35, 58A03, **AMS members US\$41.60**, List US\$52, Order code AST/383



### Representation Theory—Current Trends and Perspectives

**Henning Krause**, *University of Bielefeld, Germany*, **Peter Littelmann**, *University of Cologne, Germany*, **Gunter Malle**, *University of Kaiserslautern, Germany*, **Karl-Hermann Neeb**, *University of Erlangen-Nuernberg, Germany*, and **Christoph Schweigert**, *University of Hamburg, Germany*, Editors

From April 2009 until March 2016, the German Science Foundation generously supported the Priority Program SPP 1388 in Representation Theory. The core principles of the projects realized in the framework of the priority program have been categorification and geometrization, which are also reflected in the contributions to this volume.

Apart from the articles by former postdocs supported by the priority program, the volume contains a number of invited research and survey articles. This volume covers current research topics from the representation theory of finite groups, of algebraic groups, of Lie superalgebras, of finite dimensional algebras, and of infinite dimensional Lie groups.

Graduate students and researchers in mathematics interested in representation theory will find this volume inspiring. It contains many stimulating contributions to the development of this broad and extremely diverse subject.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**EMS Series of Congress Reports**, Volume 11

January 2017, 773 pages, Hardcover, ISBN: 978-3-03719-171-2, 2010 *Mathematics Subject Classification*: 14Mxx, 16Gxx, 17Bxx, 18Exx, 20Gxx, 22Exx; 58Cxx, 81Txx, **AMS members US\$94.40**, List US\$118, Order code EMSSCR/11

## Analysis



### Arithmétique $p$ -adique des Formes de Hilbert

**Fabrizio Andreatta**, *Università di Milano, Italy*, **Stéphane Bijakowski**, *Université Paris 13, Villetaneuse, France*, **Adrian Iovita**, *Concordia University, Montreal, Canada*, **Payman L. Kassaei**, *McGill University, Montreal, Canada*, **Vincent Pilloni**, *École Normale Supérieure de Lyon, France*, **Benoît Stroth**, *Université Paris 13, Villetaneuse, France*, **Yichao Tian**, *Chinese Academy of Sciences, Beijing, China*, and **Liang Xiao**, *University of Connecticut, Storrs, CT*

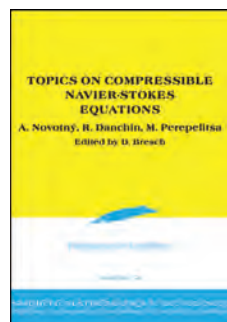
This volume is devoted to the study of Hilbert  $p$ -adic modular forms. It contains classicality theorems for overconvergent forms which generalize on the first hand Coleman criterion, which can be applied in big weights, and on the second hand Buzzard-Taylor criterion, which can be applied in weight one. The authors deduce applications to the Artin and Fontaine-Mazur conjectures. They conclude by constructing Hecke varieties for Hilbert modular forms.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Astérisque**, Number 382

October 2016, 266 pages, Softcover, ISBN: 978-2-85629-843-5, 2010 *Mathematics Subject Classification*: 37A20, 37D25, 37D30, 37A50, 37C40, **AMS members US\$65.60**, List US\$82, Order code AST/382

## Differential Equations



### Topics on Compressible Navier-Stokes Equations

**Didier Bresch**, *Université de Savoie LAMA, Le Bourget-du-Lac, France*, Editor; **Antonin Novotný**, *Université du Sud Toulon-Var, La Garde, France*, **Raphaël Danchin**, *Université Paris-est, France*, and **Misha Perepelitsa**, *University of Houston, Texas*

This issue includes contributions from the session États de la Recherche: Topics on Compressible Navier-Stokes Equations that was held from May 21–25, 2012 at the Laboratoire de Mathématiques in Le Bourget du Lac, France.

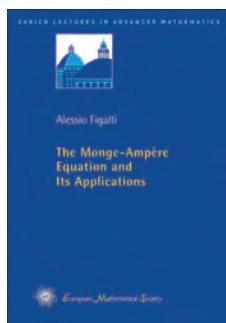
This national training session provided the opportunity to gather four internationally renowned specialists (D. Bresch, A. Novotný, R. Danchin, and M. Perepelitsa) and allow them to present the major actual mathematical developments related to the well-posedness character problem for the compressible Navier-Stokes equations to non-subject specialists.

For the sake of unity, this special issue includes only the contributions dedicated to the non-degenerate viscosities case, aiming to present a self-contained contribution on the subject: global weak-solutions à la Leray, intermediate solutions à la Hoff and strong solutions in critical spaces à la Fujita-Kato.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Panoramas et Synthèses**, Number 50

November 2016, 135 pages, Softcover, ISBN: 978-2-85629-847-3, 2010 *Mathematics Subject Classification*: 35Q30, 76N10, 35Q35, **AMS members US\$48**, List US\$60, Order code PASY/50



## The Monge-Ampère Equation and Its Applications

Alessio Figalli, *ETH Zürich, Switzerland*

The Monge-Ampère equation is one of the most important partial differential equations, appearing in many problems in analysis and geometry. This monograph is a comprehensive introduction to the

existence and regularity theory of the Monge-Ampère equation and some selected applications; the main goal is to provide the reader with a wealth of results and techniques he or she can draw from to understand current research related to this beautiful equation.

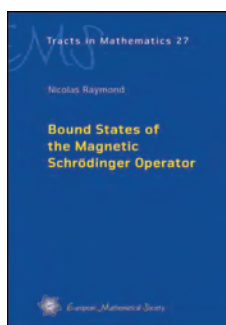
The presentation is essentially self-contained, with an appendix that contains precise statements of all the results used from different areas (linear algebra, convex geometry, measure theory, nonlinear analysis, and PDEs).

This book is intended for graduate students and researchers interested in nonlinear PDEs: explanatory figures, detailed proofs, and heuristic arguments make this book suitable for self-study and also as a reference.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**Zurich Lectures in Advanced Mathematics**, Volume 22

January 2017, 210 pages, Softcover, ISBN: 978-3-03719-170-5, 2010 *Mathematics Subject Classification*: 35J96; 35B65, 35J60, 35J66, 35B45, 35B50, 35D05, 35D10, 35J65, 53A15, 53C45, **AMS members US\$33.60**, List US\$42, Order code EMSZLEC/22



## Bound States of the Magnetic Schrödinger Operator

Nicolas Raymond, *Université de Rennes, France*

This book is a synthesis of recent advances in the spectral theory of the magnetic Schrödinger operator. It can be considered a catalog of concrete

examples of magnetic spectral asymptotics.

Since the presentation involves many notions of spectral theory and semiclassical analysis, it begins with a concise account of concepts and methods used in the book and is illustrated by many elementary examples. Assuming various points of view (power series expansions, Feshbach-Grushin reductions, WKB constructions, coherent states decompositions, normal forms) a theory of magnetic harmonic approximation is then established which allows, in particular, accurate descriptions of the magnetic eigenvalues and eigenfunctions.

Some parts of this theory, such as those related to spectral reductions or waveguides, are still accessible to advanced students while others (e.g., the discussion of the Birkhoff normal form and its spectral consequences or the results related to boundary

magnetic wells in dimension three) are intended for seasoned researchers.

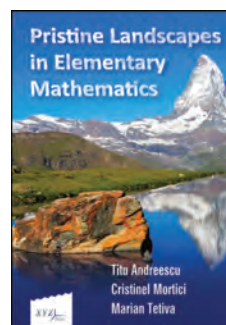
*This item will also be of interest to those working in analysis.*

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**EMS Tracts in Mathematics**, Volume 27

January 2017, 394 pages, Hardcover, ISBN: 978-3-03719-169-9, 2010 *Mathematics Subject Classification*: 35P15, 35P20, 49R05, 81Q10, 81Q20, **AMS members US\$62.40**, List US\$78, Order code EMSTM/27

## General Interest



## Pristine Landscapes in Elementary Mathematics

Titu Andreescu, *University of Texas at Dallas*, Cristinel Mortici, *Valahia University, Targoviste, Romania*, and Marian Tetiva, *Gh. Rosca Codreanu National College, Barlad, Romania*

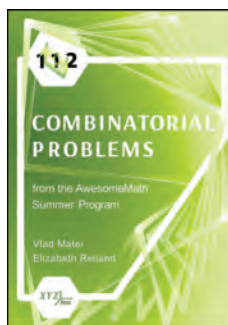
This book takes familiar ideas and extends them to a rich variety of problems. The intended audience is the ambitious high school or college student. The topics covered span algebra, geometry, number theory, and even a few elements of mathematical analysis. Each chapter explores specific themes and ideas that underlie the aforementioned subject areas. The “landscapes” presented provide a “view” into areas that are not typically encountered in great depth in standard coursework but nonetheless have profound implications.

*This item will also be of interest to those working in math education.*

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

**XYZ Series**, Volume 22

December 2016, 280 pages, Hardcover, ISBN: 978-0-9968745-7-1, 2010 *Mathematics Subject Classification*: 00A05, 00A07, 97U40, 97D50, **AMS members US\$47.96**, List US\$59.95, Order code XYZ/22



## 112 Combinatorial Problems from the AwesomeMath Summer Program

**Vlad Matei**, *University of Wisconsin, Madison*, and  
**Elizabeth Reiland**, *Johns Hopkins University, Baltimore, MD*

This book aims to give students a chance to begin exploring some introductory to intermediate topics in combinatorics, a fascinating and accessible branch of mathematics centered around (among other things) counting various objects and sets.

The book includes chapters featuring tools for solving counting problems, proof techniques, and more to give students a broad foundation to build on. The only prerequisites are a solid background in arithmetic, some basic algebra, and a love for learning mathematics.

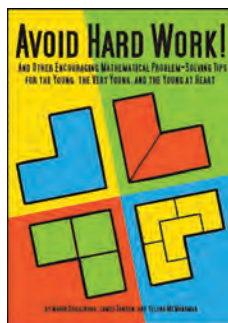
*This item will also be of interest to those working in math education.*

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

**XYZ Series**, Volume 21

December 2016, 196 pages, Hardcover, ISBN: 978-0-9968745-2-6, 2010 *Mathematics Subject Classification*: 00A05, 00A07, 97U40, 97D50, **AMS members US\$47.96**, List US\$59.95, Order code XYZ/21

## Math Education



## Avoid Hard Work!

... And Other Encouraging Mathematical Problem-Solving Tips for the Young, the Very Young, and the Young at Heart

**Maria Droujkova**, **James Tanton**, and **Yelena McManaman**

*Avoid Hard Work* gives a playful view on ten powerful problem-solving techniques. These techniques were first published by the Mathematical Association of America to help high school students with advanced math courses. These techniques and sample problems have been adapted for much younger children.

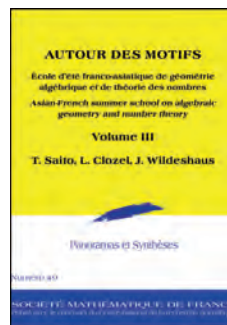
*This item will also be of interest to those working in general interest.*

A publication of Delta Stream Media, an imprint of Natural Math. Distributed in North America by the American Mathematical Society.

**Natural Math Series**, Volume 6

December 2016, 96 pages, Softcover, ISBN: 978-1-945899-01-0, **AMS members US\$12**, List US\$15, Order code NMATH/6

## Number Theory



## Autour des Motifs

Asian-French Summer School on Algebraic Geometry and Number Theory: Volume III

**Takeshi Saito**, *University of Tokyo, Japan*, **Laurent Clozel**, *Université Paris-Sud 11, Orsay, France*, and **Jörg Wildeshaus**, *Université Paris 13, Villetaneuse, France*

This volume contains the third part of the lecture notes of the Asian-French Summer School on Algebraic Geometry and Number Theory, which was held at the Institut des Hautes Études Scientifiques (Bures-sur-Yvette) and the Université Paris-Sud XI (Orsay) in July 2006. This summer school was devoted to the theory of motives and its recent developments and to related topics, notably Shimura varieties and automorphic representations.

*This item will also be of interest to those working in algebra and algebraic geometry.*

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Panoramas et Synthèses**, Number 49

November 2016, 131 pages, Softcover, ISBN: 978-2-85629-846-6, 2010 *Mathematics Subject Classification*: 14F42, 14C35, 14D10, 19E15, 19F27, **AMS members US\$41.60**, List US\$52, Order code PASY/49



**This section** contains new announcements of worldwide meetings and conferences of interest to the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. New announcements only are published in the print Mathematics Calendar featured in each *Notices* issue.

**An announcement** will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable). A second announcement will be published only if there are changes or necessary additional information. Asterisks (\*) mark those announcements containing revised information.

**In general**, print announcements of meetings and conferences carry only the date, title and location of the event.

**The complete listing** of the Mathematics Calendar is available at: [www.ams.org/meetings/calendar/mathcal](http://www.ams.org/meetings/calendar/mathcal)

**All submissions** to the Mathematics Calendar should be done online via: [www.ams.org/cgi-bin/mathcal/mathcal-submit.pl](http://www.ams.org/cgi-bin/mathcal/mathcal-submit.pl)

**Any questions** or difficulties may be directed to [mathcal@ams.org](mailto:mathcal@ams.org).

## February 2017

27 – March 10 **KK-theory, Gauge Theory, and Topological Phases**

**Location:** Lorentz Center, Leiden, NL.

**URL:** [www.lorentzcenter.nl/lc/web/2017/858/info.php?wsid=858&venue=Oort](http://www.lorentzcenter.nl/lc/web/2017/858/info.php?wsid=858&venue=Oort)

## March 2017

6 – 7 **ModCompShock: Spring School in Multiscale Modelling**

**Location:** RWTH Aachen University, Department of Mathematics, Pontdriesch 14/16, 52062 Aachen, Germany.

**URL:** <https://www.modcompshock.de>

9 – 11 **42nd Annual Arkansas Spring Lecture Series in the Mathematical Sciences: Geometry and the Equations Defining Projective Varieties**

**Location:** University of Arkansas, Fayetteville, Arkansas, USA.

**URL:** [fulbright.uark.edu/departments/math/research/spring-lecture-series/index.php](http://fulbright.uark.edu/departments/math/research/spring-lecture-series/index.php)

10 – 11 **The Topology of Real Algebraic Varieties: Deterministic and Random Aspects—Shanks Workshop on Real Algebraic Geometry**

**Location:** Vanderbilt University, Nashville, Tennessee, USA.

**URL:** [my.vanderbilt.edu/ragworkshop](http://my.vanderbilt.edu/ragworkshop)

18 – 18 **Analytic Methods in Algebraic Geometry Day**

**Location:** Northwestern University, Mathematics Department, 2033 Sheridan Rd, Evanston, IL.

**URL:** [www.math.northwestern.edu/~tosatti/gaga.html](http://www.math.northwestern.edu/~tosatti/gaga.html)

21 – 24 **Computational Issues in Oil Field Applications Tutorials**

**Location:** Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, CA.

**URL:** [www.ipam.ucla.edu/programs/workshops/computational-issues-in-oil-field-applications-tutorials](http://www.ipam.ucla.edu/programs/workshops/computational-issues-in-oil-field-applications-tutorials)

23 – April 7 **Noncommutative Geometry Festival in Shanghai**

**Location:** Fudan University, Shanghai, China.

**URL:** [www.fdias.fudan.edu.cn/en/non-commutative-geometry](http://www.fdias.fudan.edu.cn/en/non-commutative-geometry)

24 – 25 **8th Western Conference on Mathematical Finance**

**Location:** University of Washington, Seattle, WA.

**URL:** [depts.washington.edu/amath/wcmf](http://depts.washington.edu/amath/wcmf)

## April 2017

\*8 – 8 **Midwest Mini-conference on Stochastic Processes and Mathematical Finance**

**Location:** North Dakota State University, Fargo, North Dakota.

**URL:** [www.ndsu.edu/pubweb/~isengupt/mini-conf.html](http://www.ndsu.edu/pubweb/~isengupt/mini-conf.html)

10 – 11 **2-Representation Theory Workshop**

**Location:** Uppsala University, Sweden.

**URL:** [www2.math.uu.se/%7Emazor/workshop.html](http://www2.math.uu.se/%7Emazor/workshop.html)

18 – October 14 **Thematic Semester: Dynamics and Geometry**

**Location:** Universities of Angers, Brest, Nantes, Rennes (FRANCE).

**URL:** <https://www.lebesgue.fr/content/sem2017>

24 – May 3 **Simons Lecture Series in Mathematics**

**Location:** Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA.

**URL:** [math.mit.edu/news/seminars/simons.php](http://math.mit.edu/news/seminars/simons.php)

27 – 29 **Redbud Topology Conference**

**Location:** University of Arkansas, Fayetteville, Arkansas.

**URL:** [comp.uark.edu/~mattclay/Redbud](http://comp.uark.edu/~mattclay/Redbud)

## May 2017

1 – 5 **Ninth Discrete Geometry and Algebraic Combinatorics Conference**

**Location:** South Padre Island, Texas.

**URL:** [www.utrgv.edu/discgeo](http://www.utrgv.edu/discgeo)

3 – 7 **4th International Intuitionistic Fuzzy Sets and Contemporary Mathematics Conference—IFSCOM2017**

**Location:** Mersin, Turkey.

**URL:** [ifscm.com](http://ifscm.com)

## 4 – 8 4th International Workshop on Nonlinear and Modern Mathematical Physics

**Location:** *University Putra Malaysia, Kuala Lumpur, Malaysia.*  
**URL:** [einspem.upm.edu.my/nmmp2017](http://einspem.upm.edu.my/nmmp2017)

## 11 – 13 International Conference on Mathematics and Mathematics Education (ICMME-2017)

**Location:** *Harran University, Sanliurfa, Turkey.*  
**URL:** [theicmme.com](http://theicmme.com)

## 16 – 18 Geometric and Combinatorial Methods in Group Theory

**Location:** *University of Illinois at Urbana-Champaign, IL, USA.*  
**URL:** [math.uiuc.edu/~jsapir2/Sapir60](http://math.uiuc.edu/~jsapir2/Sapir60)

## 22 – 24 2nd Petra International Conference on Mathematics

**Location:** *Al-Hussein Bin Talal University Ma'an, Jordan.*  
**URL:** [ahu.edu.jo/mathconf/Location.aspx](http://ahu.edu.jo/mathconf/Location.aspx)

## 22 – 26 International Conference on Elliptic and Parabolic Problems

**Location:** *Hotel Serapo, Gaeta, Italy.*  
**URL:** [www.math.uzh.ch/gaeta2017](http://www.math.uzh.ch/gaeta2017)

## 22 – 27 Constructive Nonsmooth Analysis and Related Topics

**Location:** *Euler Institute, Saint Petersburg, Russia.*  
**URL:** [www.pdmi.ras.ru/EIMI/2017/CNSA/index.html](http://www.pdmi.ras.ru/EIMI/2017/CNSA/index.html)

## 30 – June 2 10th Chaotic Modeling and Simulation International Conference

**Location:** *Casa Convalescència of the Universitat Autònoma, Barcelona, Spain.*  
**URL:** [www.cmsim.org](http://www.cmsim.org)

## 31 – June 4 ET'NA 2017—Encounter in Topology 'n Algebra

**Location:** *Scuola Superiore di Catania, Catania, Italy.*  
**URL:** [ivanmartino.com/ETNA2017.php](http://ivanmartino.com/ETNA2017.php)

## June 2017

### 6 – 10 Geometry and Algebra of PDEs

**Location:** *UiT the Arctic University of Norway, Tromsø, Norway.*  
**URL:** [serre.mat-stat.uit.no/pdes2017/index.html](http://serre.mat-stat.uit.no/pdes2017/index.html)

### 8 – 10 Geometry and Physics, a conference dedicated to the memory of William Thurston

**Location:** *Institut de Recherche Mathématique Avancée, University of Strasbourg (France).*  
**URL:** [www-irma.u-strasbg.fr/article1576.html](http://www-irma.u-strasbg.fr/article1576.html)

### 19 – July 28 Math-to-Industry Boot Camp II

**Location:** *University of Minnesota, Minneapolis, MN.*  
**URL:** [www.ima.umn.edu/2016-2017/SW6.19-7.28.17](http://www.ima.umn.edu/2016-2017/SW6.19-7.28.17)

### 26 – 28 Conference on Classical and Geophysical Fluid Dynamics: Modeling, Reduction, and Simulation

**Location:** *Virginia Tech, Blacksburg, Virginia, USA.*  
**URL:** [www.math.vt.edu/GFD\\_conference2017/index.html](http://www.math.vt.edu/GFD_conference2017/index.html)

### 26 – 29 The Sixth Biennial Conference of the Society for Mathematics and Computation in Music.

**Location:** *Faculty of Sciences, Universidad Nacional Autónoma de México, Mexico City, Mexico.*  
**URL:** [www.mcm2017.org](http://www.mcm2017.org)

## 26 – 30 Nonlinear Analysis in Rome

**Location:** *Rome, Italy.*  
**URL:** [www3.nd.edu/~conf/nar17](http://www3.nd.edu/~conf/nar17)

## 28–30 XVIII Congress of the Portuguese Association of Operational Research

**Location:** *Polytechnic Institute of Viana do Castelo, School of Business Sciences (ESCE), based in Valença, Viana do Castelo district, Portugal.*  
**URL:** <http://apdio.pt/en/web/io2017/home>

## July 2017

### 3 – 7 12th Biennial International Conference on Sampling Theory and Applications (SampTA 2017)

**Location:** *Tallinn University of Technology/Tallinn University, Tallinn, Estonia.*  
**URL:** [www.sampTA2017.ee](http://www.sampTA2017.ee)

### 5 – 7 The 2017 International Conference of Applied and Engineering Mathematics

**Location:** *Imperial College London, London, UK.*  
**URL:** [www.iaeng.org/WCE2017/ICAEM2017.html](http://www.iaeng.org/WCE2017/ICAEM2017.html)

### 17 – 21 ACA 2017—The 23rd International Conference on Applications of Computer Algebra

**Location:** *Jerusalem College of Technology, Jerusalem, Israel.*  
**URL:** [www.aca2017.jct.ac.il](http://www.aca2017.jct.ac.il)

### 22 – 30 The International Conference and PhD-Master Summer School "Groups and Graphs, Metrics and Manifolds"

**Location:** *Yekaterinburg, Russia.*  
**URL:** [g2.imm.uran.ru/g2m2](http://g2.imm.uran.ru/g2m2)

### 23 – 29 7th PhD Summer School in Discrete Mathematics

**Location:** *Rogla, Slovenia.*  
**URL:** <https://conferences.famnit.upr.si/event/2>

### 24 – 27 All Kinds of Mathematics Remind me of You

**Location:** *Faculdade de Ciências Universidade de Lisboa, Lisbon, Portugal.*  
**URL:** [cameron17.campus.ciencias.ulisboa.pt](http://cameron17.campus.ciencias.ulisboa.pt)

### 24 – 28 International Conference on Mathematical Modelling in Applied Sciences, ICMMAS'17

**Location:** *SPbPU, Saint Petersburg, Russia.*  
**URL:** [icmmas.alpha-publishing.net](http://icmmas.alpha-publishing.net)

### 24 – August 11 Summer Northwestern Analysis Program

**Location:** *Northwestern University, Evanston, IL.*  
**URL:** [www.math.northwestern.edu/SNAP2017](http://www.math.northwestern.edu/SNAP2017)

## September 2017

### 7 – 9 Geometry, Dynamics and Physics


**Location:** *Institut de Recherche Mathématique Avancée (University of Strasbourg, France).*  
**URL:** [www-irma.u-strasbg.fr/article1609.html](http://www-irma.u-strasbg.fr/article1609.html)

## October 2017

4 – 6 **Optimization of Infinite Dimensional Non-Smooth Distributed Parameter Systems****Location:** TU-Darmstadt, Darmstadt, Germany.**URL:** [www3.mathematik.tu-darmstadt.de/index.php?id=3150](http://www3.mathematik.tu-darmstadt.de/index.php?id=3150)6 – 8 **10th International Symposium on Biomathematics and Ecology Education and Research (BEER)****Location:** Illinois State University, Normal, IL USA.**URL:** [symposium.beer](http://symposium.beer)7 – 8 **The 37th Southeastern-Atlantic Regional Conference on Differential Equations (SEARCADE)****Location:** Kennesaw State University, Kennesaw, Georgia.**URL:** [conference.kennesaw.edu/searcde](http://conference.kennesaw.edu/searcde)

## December 2017

3 – 8 **73rd Annual Deming Conference on Applied Statistics****Location:** Tropicana Casino Resort Atlantic City, New Jersey, USA.**URL:** [www.demingconference.com](http://www.demingconference.com)9 – 12 **SIAM Conference on Analysis of Partial Differential Equations (PD17)****Location:** Hyatt Regency Baltimore Inner Harbor, Baltimore, Maryland, USA.**URL:** [www.siam.org/meetings/pd17](http://www.siam.org/meetings/pd17)



香港科技大學  
THE HONG KONG UNIVERSITY OF  
SCIENCE AND TECHNOLOGY

**THE HONG KONG UNIVERSITY OF  
SCIENCE AND TECHNOLOGY**

**School of Science**

**Head of the Department of Mathematics**

The School of Science of the Hong Kong University of Science and Technology (HKUST) is seeking applications from outstanding academicians to lead the Department of Mathematics. Opened in October 1991, HKUST is a research-intensive university dedicated to the advancement of learning and scholarship, with special emphasis on postgraduate education, and close collaboration with business and industry. The School of Science, in which the Department of Mathematics is located, is also home to world-class Departments of Physics, Chemistry and Life Science. Its faculty is international in background and the official language of both administration and instruction at HKUST is English.


Reporting to the Dean of Science, the Head of the Department is expected to provide leadership for the Department, oversee faculty recruitment activities, guide and monitor resource allocation, and be responsible for the Department's academic advancement in both teaching and research. He/she is also expected to devise strategies to promote and facilitate collaborative, interdisciplinary research with individuals in other Departments within the School of Science as well as in the Schools of Engineering, Business and Humanities and Social Science.

Applicants should have an outstanding record of scholarship achievement, consistent with an appointment as Full Professor with tenure. They should have proven leadership abilities, experience leading collaborative research programs and demonstrated managerial skills. Qualified individuals should also have a broad appreciation of the research and educational opportunities in modern mathematics and possess outstanding communication and interpersonal skills.

HKUST salaries are highly competitive in the world market; within this context, the level of compensation will be commensurate with qualifications and experience. Generous fringe benefits will also be provided.

Application packages, including a curriculum vitae, a vision statement as well as the names, addresses, phone numbers and email addresses of at least three referees should be sent to: Office of the Dean of Science (Re: Head of the Department of Mathematics), The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong (or by email: [dsci@ust.hk](mailto:dsci@ust.hk)). Review of applications will begin immediately and will continue until the position is filled.

For further information about HKUST, the School of Science and the Department of Mathematics, please visit the following websites:  
 HKUST - <http://www.ust.hk>  
 School of Science - <http://science.ust.hk>  
 Department of Mathematics - <http://www.math.ust.hk>



香港中文大學  
The Chinese University of Hong Kong

**Department of Mathematics**

Founded in 1963, The Chinese University of Hong Kong (CUHK) is a forward-looking comprehensive research university with a global vision and a mission to combine tradition with modernity, and to bring together China and the West.

The Department of Mathematics in CUHK has developed a strong reputation in teaching and research. Many faculty members are internationally renowned and are recipients of prestigious awards and honors. The graduates are successful in both academia and industry. The Department is highly ranked internationally. According to the latest rankings, the Department is 39th in the Academic Ranking of World Universities, 27th in the QS World University Rankings and 28th in the US News Rankings.

**(1) Associate Professor / Assistant Professor**  
 (Ref. 16000267) (Closing date: June 30, 2017)  
 Applications are invited for a substantive-track faculty position at the Associate Professor / Assistant Professor level. Candidates with strong evidence of outstanding research accomplishments and promise in both research and teaching in Optimization or related fields in Applied Mathematics are encouraged to apply.

Appointment will normally be made on contract basis for up to three years initially commencing August 2017, which, subject to mutual agreement, may lead to longer-term appointment or substantiation later.

**(2) Research Assistant Professor**  
 (Ref. 1600027V) (Closing date: June 30, 2017)  
 Applications are invited for a position of Research Assistant Professor in all areas of Mathematics. Applicants should have a relevant PhD degree and good potential for research and teaching.

Appointment will initially be made on contract basis for up to three years commencing August 2017, renewable subject to mutual agreement.

**For posts (1) and (2):** The applications will be considered on a continuing basis but candidates are encouraged to apply by January 31, 2017.

**Application Procedure**  
 The University only accepts and considers applications submitted online for the posts above. For more information and to apply online, please visit <http://career.cuhk.edu.hk>.

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# Classified Advertising

*Positions available, items for sale, services available, and more*

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## KOREA

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### KOREA INSTITUTE FOR ADVANCED STUDY (KIAS) Assistant Professor & Research Fellow in Pure and Applied Mathematics

The School of Mathematics at the Korea Institute for Advanced Study (KIAS) invites applicants for the positions at the level of KIAS Assistant Professor and Postdoctoral Research Fellow in pure and applied mathematics.

KIAS, founded in 1996, is committed to the excellence of research in basic sciences (mathematics, theoretical physics, and computational sciences) through high-quality research programs and a strong faculty body consisting of distinguished scientists and visiting scholars.

Applicants are expected to have demonstrated exceptional research potential, including major contributions beyond or through the doctoral dissertation.

The annual salary starts from 49,500,000 Korean Won (approximately US\$40,000 at current exchange rate) for Research Fellows, and 56,500,000 Korean Won for KIAS Assistant Professors, respectively. In addition, individual research funds of 10,000,000–13,000,000 Korean Won are available per year. The initial appointment for the position is for two years and is renewable for up to two additional years, depending on research performance and the needs of the research program at KIAS.

Applications will be reviewed twice a year, May 20 and November 20, and selected applicants will be notified in a month after the review. In exceptional cases, applications can be reviewed

other times based on the availability of positions. The starting date of the appointment is negotiable. Applications must include a complete vitae with a cover letter, a list of publications, a research plan, and three letters of recommendation. (All documents should be in English.)

All should be sent by post or e-mail to: Ms. Sojung Bae (mathkias@kias.re.kr) School of Mathematics Korea Institute for Advanced Study (KIAS) 85 Hoegiro (Cheongnyangni-dong 207-43), Dongdaemun-gu, Seoul 02455, Republic of Korea.

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**Suggested uses** for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services. The publisher reserves the right to reject any advertising not in keeping with the publication's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any advertising.

**The 2017 rate is** \$3.50 per word with a minimum two-line headline. No discounts for multiple ads or the same ad in consecutive issues. For an additional \$10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

**Upcoming deadlines** for classified advertising are as follows: April 2017—February 3, 2017; May 2017—March 9, 2017; June/July 2017—May 9, 2017; August 2017—June 6, 2017; September 2017—July 7, 2017; October 2017—August 4, 2017; November 2017—September 5, 2017; December 2017—September 28, 2017.

**US laws prohibit** discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws. Details and specific wording may be found on page 1373 (vol. 44).

**Situations wanted advertisements** from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the US and Canada or 401-455-4084 worldwide for further information.

**Submission:** Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02904; or via fax: 401-331-3842; or send email to [classads@ams.org](mailto:classads@ams.org). AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 02904. Advertisers will be billed upon publication.

# MEETINGS & CONFERENCES OF THE AMS

## MARCH TABLE OF CONTENTS

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: [www.ams.org/meetings/](http://www.ams.org/meetings/).

**Important Information About AMS Meetings:** Potential organizers, speakers, and hosts should refer to page 75 in the January 2017 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

**Abstracts:** Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of  $\text{\LaTeX}$  is

necessary to submit an electronic form, although those who use  $\text{\LaTeX}$  may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in  $\text{\LaTeX}$ . Visit [www.ams.org/cgi-bin/abstracts/abstract.pl/](http://www.ams.org/cgi-bin/abstracts/abstract.pl/). Questions about abstracts may be sent to [abs-info@ams.org](mailto:abs-info@ams.org). Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

## MEETINGS IN THIS ISSUE

2017		
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2019		
January 16–19	Baltimore, Maryland	p. 303
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January 15–18	Denver, Colorado	p. 303

2021		
January 6–9	Washington, DC	p. 303

See [www.ams.org/meetings/](http://www.ams.org/meetings/) for the most up-to-date information on these conferences.

## ASSOCIATE SECRETARIES OF THE AMS

**Central Section:** Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: [benkart@math.wisc.edu](mailto:benkart@math.wisc.edu); telephone: 608-263-4283.

**Eastern Section:** Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; e-mail: [steve.weintraub@lehigh.edu](mailto:steve.weintraub@lehigh.edu); telephone: 610-758-3717.

**Southeastern Section:** Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, e-mail: [brian@math.uga.edu](mailto:brian@math.uga.edu); telephone: 706-542-2547.

**Western Section:** Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: [lapidus@math.ucr.edu](mailto:lapidus@math.ucr.edu); telephone: 951-827-5910.

# Meetings & Conferences of the AMS

**IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS:** AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See [www.ams.org/meetings/](http://www.ams.org/meetings/).

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

## Charleston, South Carolina

*College of Charleston*

**March 10–12, 2017**

*Friday – Sunday*

### Meeting #1126

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: January 2017

Program first available on AMS website: January 2017

Issue of *Abstracts*: Volume 38, Issue 2

### Deadlines

For organizers: Expired

For abstracts: Expired

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

### Invited Addresses

**Pramod N. Achar**, Louisiana State University, *Representations of algebraic groups via algebraic topology*.

**Hubert L. Bray**, Duke University, *The Geometry of Special and General Relativity*.

**Alina Chertock**, North Carolina State University, *Numerical Methods for Chemotaxis and Related Models*.

### Special Sessions

*Active Learning in Undergraduate Mathematics*, **Draga Vidakovic**, Georgia State University, **Harrison Stalvey**, University of Colorado, Boulder, and **Darryl**

**Chamberlain Jr.**, **Aubrey Kemp**, and **Leslie Meadows**, Georgia State University.

*Advances in Long-term Behavior of Nonlinear Dispersive Equations*, **Brian Pigott**, Wofford College, and **Sarah Raynor**, Wake Forest University.

*Advances in Nonlinear Waves: Theory and Applications*, **Constance M. Schober**, University of Central Florida, and **Andrei Ludu**, Embry Riddle University.

*Algebras, Lattices, Varieties*, **George F. McNulty**, University of South Carolina, and **Kate S. Owens**, College of Charleston.

*Analysis and Control of Fluid-Structure Interactions and Fluid-Solid Mixtures*, **Justin T. Webster**, College of Charleston, and **Daniel Toundykov**, University of Nebraska-Lincoln.

*Analysis, Control and Stabilization of PDE's*, **George Avalos**, University of Nebraska-Lincoln, and **Scott Hansen**, Iowa State University.

*Bicycle Track Mathematics*, **Ron Perline**, Drexel University.

*Coding Theory, Cryptography, and Number Theory*, **Jim Brown**, **Shuhong Gao**, **Kevin James**, **Felice Manganiello**, and **Gretchen Matthews**, Clemson University.

*Commutative Algebra*, **Bethany Kubik**, University of Minnesota Duluth, **Saeed Nasseh**, Georgia Southern University, and **Sean Sather-Wagstaff**, Clemson University.

*Computability in Algebra and Number Theory*, **Valentina Harizanov**, The George Washington University, **Russell Miller**, Queens College and Graduate Center - City University of New York, and **Alexandra Shlapentokh**, East Carolina University.

*Data Analytics and Applications*, **Scott C. Batson**, **Lucas A. Overbey**, and **Bryan Williams**, Space and Naval Warfare Systems Center Atlantic.

*Factorization and Multiplicative Ideal Theory*, **Jim Coykendall**, Clemson University, and **Evan Houston** and **Thomas G. Lucas**, University of North Carolina, Charlotte.

*Fluid-Boundary Interactions*, **M. Nick Moore**, Florida State University.

*Frame Theory*, **Dustin Mixon**, Air Force Institute of Technology, **John Jasper**, University of Cincinnati, and **James Solazzo**, Coastal Carolina University.

*Free-boundary Fluid Models and Related Problems*, **Marcelo Disconzi**, Vanderbilt University, and **Lorena Bociu**, North Carolina State University.

*Geometric Analysis and General Relativity*, **Hubert L. Bray**, Duke University, **Otis Chodosh**, Princeton University, and **Greg Galloway** and **Pengzi Miao**, University of Miami.

*Geometric Methods in Representation Theory*, **Pramod N. Achar**, Louisiana State University, and **Amber Russell**, Butler University.

*Geometry and Symmetry in Integrable Systems*, **Analisa Calini**, **Alex Kasman**, and **Thomas Ivey**, College of Charleston.

*Graph Theory*, **Colton Magnant**, Georgia Southern University, and **Zixia Song**, University of Central Florida.

*Knot Theory and its Applications*, **Elizabeth Denne**, Washington & Lee University, and **Jason Parsley**, Wake Forest University.

*Nonlinear Waves: Analysis and Numerics*, **Anna Ghazaryan**, Miami University, **St é phane Lafortune**, College of Charleston, and **Vahagn Manukian**, Miami University.

*Numerical Methods for Coupled Problems in Computational Fluid Dynamics*, **Vincent J. Ervin** and **Hyesuk Lee**, Clemson University.

*Oscillator Chain and Lattice Models in Optics, the Power Grid, Biology, and Polymer Science*, **Alejandro Aceves**, Southern Methodist University, and **Brenton LeMesurier**, College of Charleston.

*Recent Trends in Finite Element Methods*, **Michael Neilan**, University of Pittsburgh, and **Leo Rebholz**, Clemson University.

*Representation Theory and Algebraic Mathematical Physics*, **Iana I. Anguelova**, **Ben Cox**, and **Elizabeth Jurisch**, College of Charleston.

*Riemann-Hilbert Problem Approach to Asymptotic Problems in Integrable Systems, Orthogonal Polynomials and Other Areas*, **Alexander Tovbis**, University of Central Florida, and **Robert Jenkins**, University of Arizona.

*Rigidity Theory and Inversive Distance Circle Packings*, **John C. Bowers**, James Madison University, and **Philip L. Bowers**, The Florida State University.

# Bloomington, Indiana

*Indiana University*

**April 1–2, 2017**

*Saturday – Sunday*

## Meeting #1127

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: February 2017

Program first available on AMS website: February 23, 2017

Issue of *Abstracts*: Volume 38, Issue 2

## Deadlines

For organizers: Expired

For abstracts: Expired

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgsectional.html](http://www.ams.org/amsmtgsectional.html).*

## Invited Addresses

**Ciprian Demeter**, Indiana University, *Decouplings and applications: a journey from continuous to discrete*.

**Sarah Koch**, University of Michigan, *Title to be announced*.

**Richard Evan Schwartz**, Brown University, *Modern scratch paper: Graphical explorations in geometry and dynamics* (Einstein Public Lecture in Mathematics).

## Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Algebraic and Enumerative Combinatorics with Applications*, **Saúl A. Blanco**, Indiana University, and **Kyle Peterson**, DePaul University.

*Analysis and Numerical Computations of PDEs in Fluid Mechanics*, **Gung-Min Gie**, University of Louisville, and **Makram Hamouda** and **Roger Temam**, Indiana University.

*Analysis of Variational Problems and Nonlinear Partial Differential Equations*, **Nam Q. Le** and **Peter Sternberg**, Indiana University.

*Automorphic Forms and Algebraic Number Theory*, **Patrick B. Allen**, University of Illinois at Urbana-Champaign, and **Matthias Strauch**, Indiana University Bloomington.

*Commutative Algebra*, **Ela Celikbas** and **Olgur Celikbas**, West Virginia University.

*Computability and Inductive Definability over Structures*, **Siddharth Bhaskar**, **Lawrence Valby**, and **Alex Kruckman**, Indiana University.

*Dependence in Probability and Statistics*, **Richard C. Bradley** and **Lanh T. Tran**, Indiana University.

*Differential Equations and Their Applications to Biology*, **Changbing Hu**, **Bingtuan Li**, and **Jiaxu Li**, University of Louisville.

*Discrete Structures in Conformal Dynamics and Geometry*, **Sarah Koch**, University of Michigan, and **Kevin Pilgrim** and **Dylan Thurston**, Indiana University.

*Extremal Problems in Graphs, Hypergraphs and Other Combinatorial Structures*, **Amin Bahmanian**, Illinois State University, and **Theodore Molla**, University of Illinois Urbana-Champaign.

*Financial Mathematics and Statistics*, **Ryan Gill**, University of Louisville, **Rasitha Jayasekera**, Butler University, and **Kiseop Lee**, Purdue University.

*Fusion Categories and Applications*, **Paul Bruillard**, Pacific Northwest National Laboratory, and **Julia Plavnik** and **Eric Rowell**, Texas A&M University.

*Harmonic Analysis and Partial Differential Equations*, **Lucas Chaffee**, Western Washington University, **William Green**, Rose-Hulman Institute of Technology, and **Jarod Hart**, University of Kansas.

*Homotopy Theory*, **David Gepner**, Purdue University, **Ayelet Lindenstrauss** and **Michael Mandell**, Indiana University, and **Daniel Ramras**, Indiana University-Purdue University Indianapolis.

*Model Theory*, **Gabriel Conant**, University of Notre Dame, and **Philipp Hieronymi**, University of Illinois Urbana-Champaign.

*Multivariate Operator Theory and Function Theory*, **Hari Bercovici**, Indiana University, **Kelly Bickel**, Bucknell University, **Constanze Liaw**, Baylor University, and **Alan Sola**, Stockholm University.

*Network Theory*, **Jeremy Alm** and **Keenan M.L. Mack**, Illinois College.

*Nonlinear Elliptic and Parabolic Partial Differential Equations and Their Various Applications*, **Changyou Wang**, Purdue University, and **Yifeng Yu**, University of California, Irvine.

*Probabilistic Methods in Combinatorics*, **Patrick Bennett** and **Andrzej Dudek**, Western Michigan University.

*Probability and Applications*, **Russell Lyons** and **Nick Travers**, Indiana University.

*Randomness in Complex Geometry*, **Turgay Bayraktar**, Syracuse University, and **Norman Levenberg**, Indiana University.

*Representation Stability and its Applications*, **Patricia Hersh**, North Carolina State University, **Jeremy Miller**, Purdue University, and **Andrew Putman**, University of Notre Dame.

*Representation Theory and Integrable Systems*, **Eugene Mukhin**, Indiana University, Purdue University Indianapolis, and **Vitaly Tarasov**, Indiana University, Purdue University Indianapolis.

*Self-similarity and Long-range Dependence in Stochastic Processes*, **Takashi Owada**, Purdue University, **Yi Shen**, University of Waterloo, and **Yizao Wang**, University of Cincinnati.

*Spectrum of the Laplacian on Domains and Manifolds*, **Chris Judge** and **Sugata Mondal**, Indiana University.

*Topics in Extremal, Probabilistic and Structural Graph Theory*, **John Engbers**, Marquette University, and **David Galvin**, University of Notre Dame.

*Topological Mathematical Physics*, **E. Birgit Kaufmann** and **Ralph M. Kaufmann**, Purdue University, and **Emil Prodan**, Yeshiva University.

## Pullman, Washington

Washington State University

April 22–23, 2017

Saturday – Sunday

### Meeting #1128

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: February 2017

Program first available on AMS website: March 9, 2017

Issue of *Abstracts*: Volume 38, Issue 2

### Deadlines

For organizers: Expired

For abstracts: February 28, 2017

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtg/sectional.html](http://www.ams.org/amsmtg/sectional.html).*

### Invited Addresses

**Michael Hitrik**, University of California, Los Angeles, *Spectra for non-self-adjoint operators and integrable dynamics*.

**Andrew S. Raich**, University of Arkansas, *Title to be announced*.

**Daniel Rogalski**, University of California, San Diego, *Title to be announced*.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Analysis on the Navier-Stokes equations and related PDEs* (Code: SS 9A), **Kazuo Yamazaki**, University of Rochester, and **Litzheng Tao**, University of California, Riverside.

*Clustering of Graphs: Theory and Practice* (Code: SS 18A), **Stephen J. Young** and **Jennifer Webster**, Pacific Northwest National Laboratory.

*Combinatorial and Algebraic Structures in Knot Theory* (Code: SS 5A), **Sam Nelson**, Claremont McKenna College, and **Allison Henrich**, Seattle University.

*Combinatorial and Computational Commutative Algebra and Algebraic Geometry* (Code: SS 21A), **Hirotschi Abo**, **Stefan Tohaneanu**, and **Alexander Woo**, University of Idaho.

*Commutative Algebra* (Code: SS 3A), **Jason Lutz** and **Katharine Shultis**, Gonzaga University.

*Fixed Point Methods in Differential and Integral Equations* (Code: SS 1A), **Theodore A. Burton**, Southern Illinois University in Carbondale.

*Geometric Measure Theory and its Applications* (Code: SS 20A), **Kevin R. Vixie**, Washington State University.

*Geometry and Optimization in Computer Vision* (Code: SS 15A), **Bala Krishnamoorthy**, Washington State University, and **Sudipta Sinha**, Microsoft Research, Redmond, WA.

*Inverse Problems* (Code: SS 2A), **Hanna Makaruk**, Los Alamos National Laboratory (LANL), and **Robert Owczarek**, University of New Mexico, Albuquerque & Los Alamos.

*Mathematical & Computational Neuroscience* (Code: SS 12A), **Alexander Dimitrov**, Washington State University Vancouver, **Andrew Oster**, Eastern Washington University, and **Predrag Tasic**, Washington State University.

*Mathematical Modeling of Forest and Landscape Change* (Code: SS 11A), **Demetrios Gatzliolis**, US Forest Service, and **Nikolay Strigul**, Washington State University Vancouver.

*Microlocal Analysis and Spectral Theory* (Code: SS 17A), **Michael Hitrik**, University of California, Los Angeles, and **Semyon Dyatlov**, Massachusetts Institute of Technology.

*Noncommutative algebraic geometry and related topics* (Code: SS 16A), **Daniel Rogalski**, University of California, San Diego, and **James Zhang**, University of Washington.

*Partial Differential Equations and Applications* (Code: SS 8A), **V. S. Manoranjan**, **C. Moore**, **Lynn Schreyer**, and **Hong-Ming Yin**, Washington State University.

*Recent Advances in Applied Algebraic Topology* (Code: SS 14A), **Henry Adams**, Colorado State University, and **Bala Krishnamoorthy**, Washington State University.

*Recent Advances in Optimization and Statistical Learning* (Code: SS 19A), **Hongbo Dong**, **Bala Krishnamoorthy**, **Haijun Li**, and **Robert Mifflin**, Washington State University.

*Recent Advances on Mathematical Biology and Their Applications* (Code: SS 7A), **Robert Dillon** and **Xueying Wang**, Washington State University.

*Several Complex Variables and PDEs* (Code: SS 10A), **Andrew Raich** and **Phillip Harrington**, University of Arkansas.

*Analytic Number Theory and Automorphic Forms* (Code: SS 6A), **Steven J. Miller**, Williams College, and **Sheng-Chi Liu**, Washington State University.

*Theory and Applications of Linear Algebra* (Code: SS 4A), **Judi McDonald** and **Michael Tsatsomeros**, Washington State University.

*Undergraduate Research Experiences in the Classroom* (Code: SS 13A), **Heather Moon**, Lewis-Clark State College.

## New York, New York

*Hunter College, City University of New York*

**May 6–7, 2017**

*Saturday – Sunday*

### Meeting #1129

Eastern Section

Associate secretary: **Steven H. Weintraub**

Announcement issue of *Notices*: March 2017

Program first available on AMS website: March 29, 2017

Issue of *Abstracts*: Volume 38, Issue 2

### Deadlines

For organizers: Expired

For abstracts: March 21, 2017

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgsectional.html](http://www.ams.org/amsmtgsectional.html).*

### Invited Addresses

**Jeremy Kahn**, City University of New York, *Title to be announced.*

**Fernando Coda Marques**, Princeton University, *Title to be announced.*

**James Maynard**, Magdalen College, University of Oxford, *Title to be announced* (Erdős Memorial Lecture).

**Kavita Ramanan**, Brown University, *Title to be announced.*

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Analysis and Numerics on Liquid Crystals and Soft Matter* (Code: SS 16A), **Xiang Xu**, Old Dominion University, and **Wujun Zhang**, Rutgers University.

*Applications of Modular Forms* (Code: SS 35A), **Cormac O'Sullivan** and **Karen Taylor**, Bronx Community College, City University of New York.

*Applications of Network Analysis, in Honor of Charlie Suffel's 75th Birthday* (Code: SS 18A), **Michael Yatauro**, Pennsylvania State University-Brandywine.

*Asymptotic Properties of Discrete Dynamical Systems* (Code: SS 29A), **Ann Brett**, Johnson and Wales University, and **M. R.S. Kulenović** and **O. Merino**, University of Rhode Island.

*Automorphic Forms and Arithmetic* (Code: SS 22A), **Jim Brown**, Clemson University, and **Krzysztof Klosin**, Queens College, City University of New York.

*Banach Space Theory and Metric Embeddings* (Code: SS 10A), **Mikhail Ostrovskii**, St John's University, and **Beata Randrianantoanina**, Miami University of Ohio.

*Bases in Hilbert Function Spaces* (Code: SS 30A), **Azita Mayeli**, City University of New York, and **Shahaf Nitzan**, Georgia Institute of Technology.

*Cluster Algebras in Representation Theory and Combinatorics* (Code: SS 6A), **Alexander Garver**, Université du Québec à Montréal and Sherbrooke, and **Khrystyna Serhiyenko**, University of California at Berkeley.

*Cohomologies and Combinatorics* (Code: SS 15A), **Rebecca Patrias**, Université du Québec à Montréal, and **Oliver Pechenik**, Rutgers University.

*Common Threads to Nonlinear Elliptic Equations and Systems* (Code: SS 14A), **Florin Catrina**, St. John's University, and **Wenxiong Chen**, Yeshiva University.

*Commutative Algebra* (Code: SS 1A), **Laura Ghezzi**, New York City College of Technology-CUNY, and **Jooyoun Hong**, Southern Connecticut State University.

*Computability Theory: Pushing the Boundaries* (Code: SS 9A), **Johanna Franklin**, Hofstra University, and **Russell Miller**, Queens College and Graduate Center, City University of New York.

*Computational and Algorithmic Group Theory* (Code: SS 7A), **Denis Serbin** and **Alexander Ushakov**, Stevens Institute of Technology.

*Cryptography* (Code: SS 3A), **Xiaowen Zhang**, College of Staten Island and Graduate Center-CUNY.

*Current Trends in Function Spaces and Nonlinear Analysis* (Code: SS 2A), **David Cruz-Urbe**, University of Alabama, **Jan Lang**, The Ohio State University, and **Osvaldo Mendez**, University of Texas at El Paso.

*Differential and Difference Algebra: Recent Developments, Applications, and Interactions* (Code: SS 12A), **Omár León-Sánchez**, McMaster University, and **Alexander Levin**, The Catholic University of America.

*Dynamical Systems* (Code: SS 31A), **Marian Gidea**, Yeshiva University, **W. Patrick Hooper**, City College of New York and the City University of New York, and **Anatole Katok**, Pennsylvania State University.

*Ehrhart Theory and its Applications* (Code: SS 25A), **Dan Cristofaro-Gardiner**, Harvard University, **Quang-Nhat Le**, Brown University, and **Sinai Robins**, University of São Paulo.

*Euler and Related PDEs: Geometric and Harmonic Methods* (Code: SS 27A), **Stephen C. Preston**, Brooklyn College, City University of New York, and **Kazuo Yamazaki**, Kazuo Yamazaki, University of Rochester.

*Finite Fields and their Applications* (Code: SS 23A), **Ricardo Conceicao** and **Darren Glass**, Gettysburg College, and **Ariane Masuda**, New York City College of Technology, City University of New York.

*Geometric Function Theory and Related Topics* (Code: SS 19A), **Sudeb Mitra**, Queens College and Graduate Center-CUNY, and **Zhe Wang**, Bronx Community College-CUNY.

*Geometry and Topology of Ball Quotients and Related Topics* (Code: SS 5A), **Luca F. Di Cerbo**, Max Planck Institute, Bonn, and **Matthew Stover**, Temple University.

*Hydrodynamic and Wave Turbulence* (Code: SS 11A), **Tristan Buckmaster**, Courant Institute of Mathematical Sciences, New York University, and **Vlad Vicol**, Princeton University.

*Infinite Permutation Groups, Totally Disconnected Locally Compact Groups, and Geometric Group Theory* (Code: SS 4A), **Delaram Kahrobaei**, New York City College of Technology and Graduate Center-CUNY, and **Simon Smith**, New York City College of Technology-CUNY.

*Invariants in Low-dimensional Topology* (Code: SS 24A), **Abhijit Champanerkar**, College of Staten Island and The Graduate Center, City University of New York, and **Anastasiia Tsvietkova**, Rutgers University, Newark.

*Invariants of Knots, Links and 3-manifolds* (Code: SS 26A), **Moshe Cohen**, Vassar College, **Ilya Kofman Kofman**, College of Staten Island and The Graduate Center, City University of New York, and **Adam Lowrance**, Vassar College.

*Mathematical Phylogenetics* (Code: SS 34A), **Megan Owen**, City University of New York, and **Katherine St. John**, Lehman College, City University of New York, and American Museum of Natural History.

*Model Theory: Algebraic Structures in "Tame" Model Theoretic Contexts* (Code: SS 28A), **Alfred Dolich**, Kingsborough Community College and The Graduate Center, City University of New York, **Michael Laskowski**, University of Maryland, and **Mahmood Sohrabi**, Stevens Institute of Technology.

*Nonlinear and Stochastic Partial Differential Equations: Theory and Applications in Turbulence and Geophysical Flows* (Code: SS 8A), **Nathan Glatt-Holtz**, Tulane University, **Geordie Richards**, Utah State University, and **Xiaoming Wang**, Florida State University.

*Numerical Analysis and Mathematical Modeling* (Code: SS 32A), **Vera Babenko**, Ithaca College.

*Operator Algebras and Ergodic Theory* (Code: SS 17A), **Genady Grabarnik** and **Alexander Katz**, St John's University.

*Qualitative and Quantitative Properties of Solutions to Partial Differential Equations* (Code: SS 20A), **Blair Davey**, The City College of New York-CUNY, and **Nguyen Cong Phuc** and **Jiuyi Zhu**, Louisiana State University.

*Recent Developments in Automorphic Forms and Representation Theory* (Code: SS 21A), **Moshe Adrian**, Queens College-CUNY, and **Shuichiro Takeda**, University of Missouri.

*Representation Spaces and Toric Topology* (Code: SS 13A), **Anthony Bahri**, Rider University, and **Daniel Ramras** and **Mentor Stafa**, Indiana University-Purdue University Indianapolis.

*Topological Dynamics* (Code: SS 33A), **Alica Miller**, University of Louisville.

## Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include the New York City hotel tax (14.75 percent) and a US\$3.50 per room, per night occupancy fee.

Participants must state that they are with the **American Mathematical Society (AMS) Meeting at Hunter College** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details.**

**The Benjamin**, 125 East 50th Street, New York NY, 10022; 866.222.2365; [www.thebenjamin.com/](http://www.thebenjamin.com/) Rates are US\$279 per night for a guest room with one queen bed, this rate is applicable for single or double occupancy; each additional person US\$40. To book call 1-866-AFFINIA. A US\$30 Facility Fee has been waived and internet access has been provided complimentary in guest rooms. Amenities include a kitchenette with a mini fridge in-room, fitness center, *Rest and Renew* program for better sleep, room service, and on property James Beard Award-winning restaurant, *The National*, serving breakfast, lunch, and dinner. Valet parking is available at a daily rate of US\$65 for a sedan and US\$70 for an SUV, per day. This property is located approximately 1 mile from campus. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 7, 2017**.

**Fifty NYC**, 155 East 50th Street At Third Avenue New York, NY 10022; 212.751.5710; [www.affinia.com/fifty](http://www.affinia.com/fifty). Rates are US\$235 per night for a guest room with one queen bed or two double beds, this rate is applicable for single or double occupancy; each additional person US\$40. To book call 1-866-AFFINIA. A US\$25 Facility Fee has been waived and internet access has been provided complimentary in guest rooms. Amenities include a coffee maker and a mini refrigerator in-room, the Club Room lounge, fitness center, and room service is available. Valet Parking is available for US\$55 for cars, and US\$65 for SUVs and minivans, per day. In-and-out privileges are not included. This property is located approximately 1 mile from the campus. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 7, 2017**.

**Hyatt Herald Square**, 30 W 31st St, New York, NY 1000, 212-330-1234; <https://newyorkheraldsquare.hyatt.com/en/hotel/home.html>. Rates are US\$219 for a room with one king bed and US\$249 for a room with two queen beds, for single or double occupancy. To book please call 1-800-233-1234 or visit [Hyatt.com](http://Hyatt.com) and reference AM MATHEMATICAL SOC. Additional charges of US\$10 per person will apply for triple or quad occupancy. Amenities include complimentary Wi-Fi, a coffee maker and a mini refrigerator in-room, fitness center, room service, and on-property dining at *the Den*, *Espresso Bar*, and *Up on 20*. Valet parking is available at a daily rate of US\$60 for a standard car and US\$70 for an SUV, per day. This property is located approximately 2.2 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make

your reservation. The deadline for reservations at this rate is **April 3, 2017**.

**Hyatt Place New York/Midtown-South**, 52 West 36th Street, New York, NY 10018; 212-239-9100; <https://newyorkmidtown.place.hyatt.com/en/hotel/home.html>. Rates are US\$199 per night for a room with one king bed and US\$229 for a room with two queen beds, these rates are applicable for single or double occupancy. Queen rooms also include sofa sleeper. To book please call 1-888-492-8847 or visit [Hyatt.com](http://Hyatt.com) and use G-MATH as the group/corporate number. Amenities include complimentary Wi-Fi, refrigerator and single serve coffee maker in room, fitness room, and complimentary hot breakfast. Also located on property are *Gallery Menu*, serving made to order fresh food, and *Coffee to Cocktails*, serving Starbucks® specialty coffees & teas, premium beers, wines & cocktails. Self-parking is located at Meyers Parking Garage at 12 West 36th Street. This property is located approximately 2.3 miles from campus. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 3, 2017**.

**Hyatt Centric Times Square**, 135 West 45th Street, New York, NY 10036; 646-364-1234; <https://timesquare.centric.hyatt.com/en/hotel/home.html>. Rates are US\$319 per night for a room with one king bed or for a room with two queen beds, these rates are applicable for single or double occupancy. Amenities include complimentary Wi-Fi, fitness room, business center, and Timeless: A Marilyn Monroe Spa. Also located on property are *T45*, a bistro-style restaurant, and *Bar 54*, a rooftop bar located on the 54th level. Valet parking is available, inclusive of SUVs and oversized vehicles, for a flat fee of US\$65 per day. This property is located approximately 1.7 miles from campus. Check-in is at 4:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 5, 2017**.

**Lexington New York City**, 511 Lexington Avenue (at 48th Street), New York, NY, 10017, 1-800-448-4471 or 212-755-4400; <https://www.lexingtonhotelnyc.com/>. Rates are **US\$179** per night for a single or double occupancy room with a king bed and complimentary Wi-Fi. Amenities include in-room laptop safe, fitness center, business center, Starbucks coffeehouse and on-site bar, *the Mixing Room*, and bistro *Raffles* serving breakfast, lunch, and dinner. Valet parking or off-site parking is available for US\$65 per day, oversized vehicles will be US\$75 per day. This property is located approximately 1.2 miles from the campus. Check-in is at 4:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at a reduced rate is **April 5, 2017**.

**Millennium Broadway Hotel New York**, 145 West 44th Street, New York, NY; 212-768-4400;

<https://www.millenniumhotels.com/en/new-york/millennium-broadway-hotel-new-york/#about>. Rates are US\$279 per night for a Superior guest room with a king or two double beds or US\$319 for a Premier guest room with a king or queen bed. Either rate is for single or double occupancy. Premier rooms offer complimentary continental breakfast, and afternoon hors d'oeuvres with wine or beer. Complimentary internet access in guest rooms has been included in this rate. Amenities include fitness center, business center, and on-site restaurant, *Charlotte*, serving breakfast. Valet parking is available, inquire about rates when making a reservation. This property is located approximately 1.8 miles from the campus. Check-in is at 4:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 5, 2017**.

**NYLO**, 2178 Broadway (at W 77th Street), New York, NY 10024; 212-362-1100; <http://www.nylohotels.com/nyc>. Rates are US\$269 on 5/5 for a standard room with a queen or king bed and US\$289 on 5/6 for a standard room with a queen or king bed this rate is applicable for single or double occupancy. To book please call 1-800-509-7598. Amenities include complimentary Wi-Fi, fitness center, in room coffee maker, and complimentary bottled water in guest rooms. Also located on property are *Serafina*, serving Northern Italian cuisine for breakfast, lunch, or dinner, for dine in or take-out; *Red Farm* serving modern Chinese cuisine and dim sum for lunch, brunch, or dinner; and *LOCL Bar*, a lounge. This property is located approximately 1.8 miles from campus. Check-in is at 4:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 5, 2017**.

**Park Central**, 870 7th Avenue New York, NY 10019; 212-247-8000; [www.parkcentralny.com/hotel](http://www.parkcentralny.com/hotel). Rates for classic guest rooms (one king bed or two double beds) at this property will be discounted by 15 percent off of the best available rate at the time of booking. A Facility Fee of US\$31 has been waived for this property and allows guests Wi-Fi access through the hotel's guest's rooms and public spaces for up to five devices, unlimited local and domestic long distance calls, and use of the fitness center. Amenities include on-site business center, fitness center, *Park Lounge* cocktail lounge, and "grab-and-go" *Central Market Cafe*. The full service restaurant, *Redeye Grill*, is located just steps outside this property, serving lunch and dinner. Valet parking or off-site parking is available for US\$65 per day, oversized vehicles will be US\$75 per day. This property is located approximately 1.2 miles from the campus. Check-in is at 4:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 10, 2017**.

**Renaissance New York Hotel 57**, 130 East 57th St. New York, NY 10022; 207-443-9741; [www.marriott.com/hotels/travel/nycbr-renaissance-new-york-hotel-57/](http://www.marriott.com/hotels/travel/nycbr-renaissance-new-york-hotel-57/). To book a room at this rate directly call Marriott Reservations at 1-800-468-3571 and identify the room block as *the American Mathematical Society Spring Meeting at Hunter College at the Renaissance*. Rates are US\$199 per night for a cozy guest room with one double bed, single/double occupancy. Complimentary internet access in guest rooms is included in this rate. Amenities include on-site business center, fitness center, on-site bar, *Opia Lounge and Bar*, and *Opia Restaurant* offering French cuisine for breakfast, lunch, and dinner. Valet parking or off-site parking is available for US\$55 per day, oversized vehicles will be US\$65 per day. This property is located approximately 0.5 miles from the campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **April 5, 2017**.

### Food Services

**On Campus:** Most campus dining will be closed during the dates of this meeting. There is a Starbucks located in the West Lobby which will be open on Saturday between 7:30 am–2:30 pm.

**Off Campus:** New York City offers an endless array of dining options throughout the city. For more information on dining throughout NYC and for more information on visiting New York in general please visit, [www.nycgo.com/](http://www.nycgo.com/). Some dining options nearby to the 68th Street Campus include:

**Burger King**, 226 E 86th Street (3rd & 2nd Avenue), (212) 452-4675, Saturday 6:00 am–10:00 pm and Sunday, 7:00 am–10:00 pm; fast-food burgers.

**Dunkin Donuts/Baskin Robbins**, 1225 1st Avenue (66th & 67th Street), (212) 734-5465, open 24 hours; coffee, donuts, sandwiches, ice cream.

**Fig and Olive**, 808 Lexington Ave. (Between 62nd & 63rd Streets), (212) 207-4555, Saturday 11:00 am–11:00 pm and Sunday, 11:00 am–10:00 pm; French/Spanish fusion.

**Gourmet Bagel**, 874 Lexington Avenue (65th & 66th Street), (212) 879-5200, Saturday and Sunday, 6:00 am–6:00 pm; bagels, sandwiches, deli fare.

**Hale & Hearty Soups**, 849 Lexington Ave (64th & 65th Street), (212) 517-7600, Saturday 11:00 am–5:30 pm and Sunday, 11:00 am–4:30 pm; Brooklyn based chain with from-scratch soups, sandwiches, and salads.

**Hunter Delicatessen**, 966 Lexington Ave (70th & 71st Street), (212) 439-7758, Saturday and Sunday, 7:00 am–5:00 pm; salad bar and made-to-order sandwiches.

**J G Melon**, 1291 3rd Ave. (at 74th St.), (212) 744-0585, Saturday 11:30 am–4:00 am and Sunday, 11:30 am–midnight, *cash only*; burgers, traditional American fare.

**Lenwich/Lenny's Sandwiches**, 1269 1st Avenue (68th & 69th Street), (212) 288-5171, Saturday and Sunday, 8:00 am–7:00 pm; sandwiches and salads.

**Mariella Pizza**, 965 Lexington Ave (70th & 71st Street), (212) 249-2065, Saturday and Sunday, 11:00 am–9:00 pm; pizza by the slice and pie, pasta and sandwiches.

**McDonald's**, 1286 1st Avenue (69th & 70th Street), (212) 249-3551, open 24 hours; fast-food burgers.

**PJ Bernstein Deli**, 1215 3rd Ave (70th & 71st Street), (212) 879-0914, Saturday and Sunday, 8:00 am–9:00 pm; kosher-style deli restaurant.

**Starbuck's Coffee**, 1128 3rd Avenue (65th & 66th Street), (212) 472-6535, Saturday and Sunday, 5:30 am–9:30 pm; coffee and light fare.

**Subway**, 1269 1st Avenue (68th & 69th Street), (212) 628-6358, Saturday 8:00 am–11:30 pm and Sunday, 9:00 am–4:30 pm; fast-food sandwich shop.

**Tasti D-Lite**, 1221 3rd Avenue (70th & 71st Street), (212) 288-7088, Saturday and Sunday, 11:00 am–11:00 pm; soft serve frozen dessert.

## Registration and Meeting Information

**Advance Registration:** Advance registration for this meeting opened on **January 19th**. Advance registration fees are US\$59 for AMS members, US\$85 for nonmembers, and US\$10 for students, unemployed mathematicians, and emeritus members. Participants may cancel registrations made in advance by e-mailing [mmsb@ams.org](mailto:mmsb@ams.org). The deadline to cancel is the first day of the meeting.

**On-site Information and Registration:** The registration desk, AMS book exhibit, and coffee service will be located in the West Lobby, in the atrium of the main entrance on 68th Street. The Invited Addresses and Erdős Memorial Lecture will be held in W615, on the 6th Floor in the Hunter West Building. Special Sessions and Contributed Paper Sessions will take place in several buildings on campus, within walking distance of each other and connected by indoor skybridges. These buildings will include Hunter West Building, Hunter East Building, and Hunter North Building. Please look for additional information about specific session room locations on the web and in the printed program. For further information on building locations, a campus map is available at [www.hunter.cuny.edu/visitorscenter/repository/images/68thStreet.jpg](http://www.hunter.cuny.edu/visitorscenter/repository/images/68thStreet.jpg). The registration desk will be open on Saturday, May 6, 7:30 am–4:00 pm and Sunday, May 7, 8:00 am–12:00 pm. The same fees apply for on-site registration, as for advance registration. Fees are payable on-site via cash, check, or credit card.

## Other Activities

**Book Sales:** Stop by the on-site AMS bookstore to review the newest publications and take advantage of exhibit discounts and free shipping on all on-site orders! AMS members receive 40 percent off list price. Nonmembers receive a 25 percent discount. Not a member? Ask a representative about the benefits of AMS membership. **Complimentary coffee** will be served courtesy of AMS Membership Services.

**AMS Editorial Activity:** An acquisitions editor from the AMS book program will be present to speak with

prospective authors. If you have a book project that you wish to discuss with the AMS, please stop by the book exhibit.

## Erdős Memorial Lecture

There will be an Erdős Memorial Lecture given by James Maynard, Magdalen College. This lecture will take place in W615, in the Hunter West Building at 5:10 pm on Saturday.

There will be a **reception** for participants on Saturday evening immediately following the Erdős lecture, in the Faculty Dining Room and adjacent terrace, located on the 8th Floor of the Hunter West Building. Please watch for more details on this event on the AMS website and at the registration desk on-site at the meeting. The AMS thanks our hosts for their gracious hospitality.

## Special Needs

It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:

- Registering early for the meeting
- Checking the appropriate box on the registration form, and
- Sending an e-mail request to the AMS Meetings Department at [mmsb@ams.org](mailto:mmsb@ams.org) or [meet@ams.org](mailto:meet@ams.org).

## AMS Policy on a Welcoming Environment

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity, or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

## Local Information and Maps

This meeting will take place on the 68th Street Campus of Hunter College. A campus map can be found at [www.hunter.cuny.edu/visitorscenter/repository/images/68thStreet.jpg](http://www.hunter.cuny.edu/visitorscenter/repository/images/68thStreet.jpg). Information about the Hunter Department of Mathematics and Statistics can be found at [math.hunter.cuny.edu/](http://math.hunter.cuny.edu/). Please visit the Hunter College website at [www.hunter.cuny.edu](http://www.hunter.cuny.edu). Please watch the AMS website at [www.ams.org/meetings/sectional/sectional.html](http://www.ams.org/meetings/sectional/sectional.html) for additional information on this meeting.

## Parking

There are many parking garages nearby to the Hunter campus. Participants are encouraged to use mobile apps and Web searches to determine parking availability and the most economical pricing for the day of their trip.

The College did indicate at the time of publication that the following garages are close in proximity to the main campus: 254 East 68th Street (3rd & 2nd Avenue), 1315 2nd Avenue (69th & 70th Street), and 700 Park Avenue (69th & 70th Street).

### Travel

The main campus of Hunter College is located in the heart of Manhattan on 68th Street.

**By Air:** There are three major airports servicing the NYC area: John F. Kennedy International Airport (JFK) or LaGuardia Airport (LGA), both in Queens, or Newark Liberty International Airport (EWR) in neighboring New Jersey.

*John F. Kennedy International Airport (JFK)*, New York's largest airport, is located in Jamaica, in the borough of Queens, approximately 15 miles from midtown Manhattan and offers service from over 80 carriers.

To travel from JFK to the Hunter campus there are many ground transportation options to choose from. Taxi fare is a US\$52.50 flat fare (non metered) plus bridge and tunnel tolls and gratuity, and should take between 30 minutes and 60 minutes travel time depending on traffic. For more information, call 212-NYC-TAXI. AirTrain JFK links the airport to the subway for a fare of US\$5. Connecting from the JFK AirTrain at a JFK Airport MTA Station, expect that journey should take 60 to 75 minutes to arrive in midtown Manhattan. For more information on the subway as well as city bus options, please visit [tripplanner.mta.info](http://tripplanner.mta.info). Shuttle buses are also available from the following vendors: NYC Airporter (<https://www.nycairporter.com/>), Go Airlink NYC (<https://www.goairlinkshuttle.com/>) and SuperShuttle (<https://www.supershuttle.com/locations/newyorkcity-jfk-lga/>). For more information on JFK International Airport and other ground transportation options, please visit [www.panynj.gov/airports/JFK.html](http://www.panynj.gov/airports/JFK.html).

*LaGuardia Airport (LGA)*, New York's 2nd largest airport, is located in Jackson Heights in the borough of Queens, approximately 8 miles from midtown Manhattan and offers service from over 20 carriers.

To travel from LGA to the Hunter campus there are many ground transportation options to choose from. Taxi fare is approximately a US\$29-US\$37 metered fare plus bridge and tunnel tolls and gratuity, and should take between 20-25 minutes travel time depending on traffic. For more information, call 212-NYC-TAXI. Two express city buses serve LaGuardia; the M60 and Q70. The Q70 goes nonstop to Jackson Heights/Roosevelt Avenue, a major subway hub in Queens with five lines. The M60 runs to Harlem and connects to all the major subway lines in Manhattan. For more information on city bus options, please visit [tripplanner.mta.info](http://tripplanner.mta.info). Shuttle buses are also available from the following vendors: NYC Airporter (<https://www.nycairporter.com/>), Go Airlink NYC (<https://www.goairlinkshuttle.com/>) and SuperShuttle (<https://www.supershuttle.com/locations/newyorkcity-jfk-lga/>). For more information on LaGuardia Airport and other ground transportation options, please visit [laguardiaairport.com/](http://laguardiaairport.com/).

*Newark Liberty International Airport (EWR)*, is located in Newark, New Jersey across the Hudson River and approximately 16 miles from midtown Manhattan and offers service from over 23 carriers.

To travel from EWR to the Hunter campus there are many ground transportation options to choose from. Taxi fare is approximately a US\$50-US\$75 metered fare plus bridge and tunnel tolls and gratuity, and should take between 45-60 minutes travel time depending on traffic. There may also be a US\$5 surcharge for travel into New York state, during certain days and times. Please note, when traveling to the airport from Midtown Manhattan, service is via New York City's regulated yellow taxis. Metered fares range US\$69-US\$75, plus a US\$17.50 surcharge in addition to tolls and gratuity. To travel by train, AirTrain links to the airport via NJ Transit and Amtrak's Newark (or EWR) train station. Travel by train will take approximately 45 to 90 minutes to Midtown Manhattan, requiring a transfer from the AirTrain line to the NJ Transit line or Amtrak. Shuttle buses are also available from the following vendors: NYC Airporter (<https://www.nycairporter.com/>), Go Airlink NYC (<https://www.goairlinkshuttle.com/>) and SuperShuttle (<https://www.supershuttle.com/locations/newyorkcity-jfk-lga/>). For more information on JFK International Airport and other ground transportation options, please visit [www.panynj.gov/airports/JFK.html](http://www.panynj.gov/airports/JFK.html).

**By Train:** New York City has two main rail stations in Midtown: Grand Central Terminal (on the east side) and Penn Station (on the west side). Grand Central Terminal is the main terminal for Metro-North Railroad services ([www.mta.info/mnr](http://www.mta.info/mnr)) serving NYC and Connecticut suburbs, as well as multiple subway lines. Penn Station is served by the Long Island Railroad ([www.mta.info/lirr](http://www.mta.info/lirr)), Amtrak (<https://www.amtrak.com/home>), and NJ Transit ([www.njtransit.com](http://www.njtransit.com)), as well as multiple subway lines.

**By Bus:** There are a number bus lines that travel to New York City from around the United States and parts of Canada. Options include BoltBus (<https://www.boltbus.com/>), Megabus ([us.megabus.com/](http://us.megabus.com/)) and Greyhound. For more information on Greyhound please call 800-231-2222 or visit <https://www.greyhound.com/>.

**By Car:** Please note that driving directions are constantly updated due to construction and road closures.

**From Queens/ Long Island:** Take the Long Island Expressway west to the Queens Midtown Tunnel. Make a right onto Third Avenue and proceed north to 69th Street. Make a left turn on 69th Street. Proceed to Lexington Avenue and make a left. The College is located at the intersection of East 68th Street and Lexington Avenue.

**From Bronx / Westchester:** Take the New York Thruway (I-87) to the Major Deegan Expressway. Continue to the 3rd Avenue Bridge, then to the FDR Drive, then to the 71st Street exit. Continue straight to Lexington Avenue and make a left. Continue to East 68th Street. The College is located at the intersection of East 68th Street and Lexington Avenue.

**Via the Lincoln Tunnel:** Take the Lincoln Tunnel. After exiting the tunnel, take 10th Avenue north to 65th Street. Turn right and cross Central Park. After exiting Central Park, continue along East 65th Street to Park Avenue. Turn left on Park Avenue. Go three blocks and make a right turn on 68th Street. The College will be at the end of the block.

**Via the George Washington Bridge:** Take the George Washington Bridge to the Harlem River Drive. Continue to the FDR Drive, then to the 71st Street exit. Continue straight to Second Avenue. Make a left turn and travel on Second Avenue to 69th Street. Make a right on 69th Street and travel to Lexington Avenue. Make a left turn on Lexington. The College is located at the intersection of East 68th Street and Lexington Avenue.

**Car Rental:** Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at [www.hertz.com](http://www.hertz.com), click on the box "Enter a discount or promo code," and type in our convention number (CV): CV#04N30007. You can also call Hertz directly at 800-654-2240 (US and Canada) or 1-405-749-4434 (other countries). At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

For directions to campus, inquire at your rental car counter.

## Local Transportation

New York City has excellent public transportation in the forms of buses and subways. For more information on the MTA or to plan your trip please visit [www.mta.info/](http://www.mta.info/). Taxis and other car services are also available throughout the city.

**Subway Directions:** The **6 train** stops directly under the College at the *68th Street Station*. There is an entrance to the College in the Subway station. Turn right upon exiting the turnstile and the entrance will be directly in front of you.

The **F train** is another option, using the *East 63rd Street and Lexington Avenue* stop. After exiting the station, walk north on Lexington Avenue to East 68th Street. The College is located at the intersection of East 68th Street and Lexington Avenue.

**Bus Directions:** The crosstown **M66** bus goes east on 68th Street, and west on 67th Street. Hunter College is located at the intersection of East 68th Street and Lexington Avenue. The **M98**, **M101**, **M102**, and **M103** go south on Lexington Avenue and north on 3rd Avenue. Hunter College is located at the intersection of East 68th Street and Lexington Avenue.

## Weather

Weather in May in New York City is typically pleasant and mild. The average high temperature is approximately 75

degrees Fahrenheit and the average low is approximately 60 degrees Fahrenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

## Social Networking

Attendees and speakers are encouraged to tweet about the meeting using the hashtags #AMSmtg.

## Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the US found at <https://travel.state.gov/content/travel/en.html>. If you need a preliminary conference invitation in order to secure a visa, please send your request to [mac@ams.org](mailto:mac@ams.org).

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

- \* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

- \* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

- \* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

- \* Include a letter of invitation from the meeting organizer or the US host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

- \* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

- \* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

# Montréal, Quebec Canada

McGill University

July 24–28, 2017

Monday – Friday

## Meeting #1130

*The second Mathematical Congress of the Americas (MCA 2017) is being hosted by the Canadian Mathematical Society (CMS) in collaboration with the Pacific Institute for the Mathematical Sciences (PIMS), the Fields Institute (FIELDS), Le Centre de Recherches Mathématiques (CRM), and the Atlantic Association for Research in the Mathematical Sciences (AARMS).*

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: Expired

For abstracts: March 31, 2017

# Denton, Texas

University of North Texas

September 9–10, 2017

Saturday – Sunday

## Meeting #1131

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: June 2017

Program first available on AMS website: July 27, 2017

Issue of *Abstracts*: Volume 38, Issue 3

## Deadlines

For organizers: Expired

For abstracts: July 18, 2017

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtg/sectional.html](http://www.ams.org/amsmtg/sectional.html).*

## Invited Addresses

**Mirela Çiperiani**, University of Texas at Austin, *Title to be announced.*

**Adrianna Gillman**, Rice University, *Title to be announced.*

**Kevin Pilgrim**, Indiana University, *Title to be announced.*

## Special Sessions

*If you are volunteering to speak in a Special Session, you*

*should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Algebraic Combinatorics of Flag Varieties* (Code: SS 11A), **Martha Precup**, Northwestern University, and **Edward Richmond**, Oklahoma State University.

*Banach Spaces and Applications* (Code: SS 9A), **Pavlos Motakis**, Texas A&M University, and **Bönyamin Sari**, University of North Texas.

*Combinatorics and Representation Theory of Reflection Groups: Real and Complex* (Code: SS 14A), **Elizabeth Drellich**, Swarthmore College, and **Drew Tomlin**, Hendrix College.

*Commutative Algebra* (Code: SS 10A), **Jonathan Montano**, University of Kansas, and **Alessio Sammartano**, Purdue University.

*Differential Equation Modeling and Analysis for Complex Bio-systems* (Code: SS 8A), **Pengcheng Xiao**, University of Evansville, and **Honghui Zhang**, Northwestern Polytechnical University.

*Dynamics, Geometry and Number Theory* (Code: SS 1A), **Lior Fishman** and **Mariusz Urbanski**, University of North Texas.

*Fractal Geometry and Ergodic Theory* (Code: SS 5A), **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

*Homological Methods in Commutative Algebra* (Code: SS 15A), **Peder Thompson**, Texas Tech University, and **Ashley Wheeler**, University of Arkansas.

*Invariants of Links and 3-Manifolds* (Code: SS 7A), **Mieczysław K. Dabkowski** and **Anh T. Tran**, The University of Texas at Dallas.

*Lie algebras, Superalgebras, and Applications* (Code: SS 3A), **Charles H. Conley**, University of North Texas, **Dimitar Grantcharov**, University of Texas at Arlington, and **Natalia Rozhkovskaya**, Kansas State University.

*Noncommutative and Homological Algebra* (Code: SS 4A), **Anne Shepler**, University of North Texas, and **Sarah Witherspoon**, Texas A&M University.

*Nonlocal PDEs in Fluid Dynamics* (Code: SS 12A), **Changhui Tan**, Rice University, and **Xiaoqian Xu**, Carnegie Mellon University.

*Numbers, Functions, Transcendence, and Geometry* (Code: SS 6A), **William Cherry**, University of North Texas, **Mirela Çiperiani**, University of Texas Austin, **Matt Papanikolas**, Texas A&M University, and **Min Ru**, University of Houston.

*Real-Analytic Automorphic Forms* (Code: SS 2A), **Olav K Richter**, University of North Texas, and **Martin Westerholt-Raum**, Chalmers University of Technology.

*Topics Related to the Interplay of Noncommutative Algebra and Geometry* (Code: SS 13A), **Richard Chandler**, University of North Texas at Dallas, **Michaela Vanciliff**, University of Texas at Arlington, and **Padmini Veerapen**, Tennessee Technological University.

# Buffalo, New York

*State University of New York at Buffalo*

**September 16–17, 2017**

*Saturday – Sunday*

## Meeting #1132

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: June 2017

Program first available on AMS website: August 3, 2017

Issue of *Abstracts*: Volume 38, Issue 3

## Deadlines

For organizers: February 16, 2017

For abstracts: July 25, 2017

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Invited Addresses

**Inwon Kim**, University of California at Los Angeles, *Title to be announced.*

**Govind Menon**, Brown University, *Title to be announced.*

**Bruce Sagan**, Michigan State University, *Title to be announced.*

## Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Advanced Techniques in Graph Theory* (Code: SS 9A), **Sogol Jahanbekam** and **Paul Wenger**, Rochester Institute of Technology.

*CR Geometry and Partial Differential Equations in Complex Analysis* (Code: SS 5A), **Ming Xiao**, University of Illinois at Urbana-Champaign, and **Yuan Yuan**, Syracuse University.

*Cohomology, Deformations, and Quantum Groups: A Session Dedicated to the Memory of Samuel D. Schack* (Code: SS 6A), **Miodrag Iovanov**, University of Iowa, **Mihai D. Staic**, Bowling Green State University, and **Alin Stancu**, Columbus State University.

*Geometric Group Theory* (Code: SS 4A), **Joel Louwsma**, Niagara University, and **Johanna Mangahas**, University at Buffalo-SUNY.

*High Order Numerical Methods for Hyperbolic PDEs and Applications* (Code: SS 2A), **Jae-Hun Jung**, University at Buffalo-SUNY, **Fengyan Li**, Rensselaer Polytechnic Institute, and **Li Wang**, University at Buffalo-SUNY.

*Infinite Groups and Geometric Structures: A Session in Honor of the Sixtieth Birthday of Andrew Nicas* (Code: SS 7A), **Hans Boden**, McMaster University, and **David Rosenthal**, St. John's University.

*Nonlinear Partial Differential Equations Arising from Life Science* (Code: SS 8A), **Junping Shi**, College of William and Mary, and **Xingfu Zou**, University of Western Ontario.

*Nonlinear Wave Equations, Inverse Scattering and Applications*. (Code: SS 1A), **Gino Biondini**, University at Buffalo-SUNY.

*Structural and Chromatic Graph Theory* (Code: SS 10A), **Hong-Jian Lai**, **Rong Luo**, and **Cun-Quan Zhang**, West Virginia University, and **Yue Zhao**, University of Central Florida.

*p-adic Aspects of Arithmetic Geometry* (Code: SS 3A), **Ling Xiao**, University of Connecticut, and **Hui June Zhu**, University at Buffalo-SUNY.

# Orlando, Florida

*University of Central Florida, Orlando*

**September 23–24, 2017**

*Saturday – Sunday*

## Meeting #1133

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: June 2017

Program first available on AMS website: August 10, 2017

Issue of *Abstracts*: Volume 38, Issue 4

## Deadlines

For organizers: February 23, 2017

For abstracts: August 1, 2017

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Invited Addresses

**Christine Heitsch**, Georgia Institute of Technology, *Title to be announced.*

**Jonathan Kujawa**, University of Oklahoma, *Title to be announced.*

**Christopher D Sogge**, Johns Hopkins University, *Title to be announced.*

## Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Algebraic Curves and their Applications* (Code: SS 3A), **Lubjana Beshaj**, The University of Texas at Austin.

*Applied Harmonic Analysis: Frames, Samplings and Applications* (Code: SS 6A), **Dorin Dutkay**, **Deguang Han**, and **Qiyu Sun**, University of Central Florida.

*Commutative Algebra: Interactions with Algebraic Geometry and Algebraic Topology* (Code: SS 1A), **Joseph Brennan**, University of Central Florida, and **Alina Iacob** and **Saeed Nasseh**, Georgia Southern University.

*Fractal Geometry, Dynamical Systems, and Their Applications* (Code: SS 4A), **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

*Graph Connectivity and Edge Coloring* (Code: SS 5A), **Colton Magnant**, Georgia Southern University.

*Nonlinear Waves and Dispersive Equations* (Code: SS 7A), **Benjamin Harrop-Griffiths**, New York University, **Jonas Lührmann**, Johns Hopkins University, and **Dana Mendelson**, University of Chicago.

*Structural Graph Theory* (Code: SS 2A), **Martin Rolek**, **Zixia Song**, and **Yue Zhao**, University of Central Florida.

## Riverside, California

University of California, Riverside

November 4–5, 2017

Saturday – Sunday

### Meeting #1134

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: September 2017

Program first available on AMS website: September 21, 2017

Issue of *Abstracts*: Volume 38, Issue 4

### Deadlines

For organizers: April 14, 2017

For abstracts: September 12, 2017

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtg/sectional.html](http://www.ams.org/amsmtg/sectional.html).*

### Invited Addresses

**Paul Balmer**, University of California, Los Angeles, *Title to be announced.*

**Pavel Etingof**, Massachusetts Institute of Technology, *Title to be announced.*

**Monica Vazirani**, University of California, Davis, *Combinatorics, Categorification, and Crystals.*

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Algebraic and Combinatorial Structures in Knot Theory* (Code: SS 3A), **Patricia Cahn**, Smith College, and **Sam Nelson**, Claremont McKenna College.

*Analysis and Geometry of Fractals* (Code: SS 6A), **Erin Pearse**, California Polytechnic State University, and **Goran**

**Radunovic**, University of California, Riverside.

*Applied Category Theory* (Code: SS 4A), **John Baez**, University of California, Riverside.

*Combinatorial Representation Theory* (Code: SS 5A), **Vyjayanthi Chari**, University of California, Riverside, and **Maria Monks Gillespie** and **Monica Vazirani**, University of California, Davis.

*Combinatorial aspects of the polynomial ring* (Code: SS 1A), **Sami Assaf** and **Dominic Searles**, University of Southern California.

*Ring Theory and Related Topics (Celebrating the 75th Birthday of Lance W. Small)* (Code: SS 2A), **Jason Bell**, University of Waterloo, **Ellen Kirkman**, Wake Forest University, and **Susan Montgomery**, University of Southern California.

## San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 10–13, 2018

Wednesday – Saturday

### Meeting #1135

*Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: October 2017

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 39, Issue 1

### Deadlines

For organizers: April 1, 2017

For abstracts: To be announced

### Call for Proposals

The AMS solicits proposals for AMS Special Sessions at the 2018 Joint Mathematics Meetings to be held Wednesday, January 10 through Saturday January 13, 2018, in San Diego, CA. Each proposal must include:

1. the name, affiliation, and e-mail address of each organizer, with one organizer designated as the contact person for all communication about the session;
2. the title and a brief (no longer than one or two paragraphs) description of the topic of the proposed special session;
3. the primary two-digit MSC number for the topic: See [www.ams.org/mathscinet/msc/msc2010.html](http://www.ams.org/mathscinet/msc/msc2010.html).
4. a sample list of speakers whom the organizers plan to invite. (It is not necessary to have received confirmed commitments from these potential speakers.)

Organizers are strongly encouraged to consult the AMS Manual for Special Session Organizers at: [www.ams.org/meetings/specialsessionmanual.html](http://www.ams.org/meetings/specialsessionmanual.html).

Proposals for AMS Special Sessions should be sent by e-mail to AMS Associate Secretary Georgia Benkart ([benkart@math.wisc.edu](mailto:benkart@math.wisc.edu)) by April 7, 2017. Late proposals will not be considered. No decisions will be made on Special Session proposals until after the submission deadline has passed.

Special Sessions will be allotted between 5 and 10 hours in which to schedule speakers. To enable maximum movement of participants between sessions, organizers must schedule session speakers for either (a) a 20-minute talk with 5-minute discussion and 5-minute break or (b) a 45-minute talk with 5-minute discussion and 10-minute break. Any combination of 20-minute and 45-minute talks is permitted, but all talks must begin and end at the scheduled time.

The number of Special Sessions in the AMS program at the Joint Mathematics Meetings is limited and because of the large number of high-quality proposals, not all can be accepted. Please be sure to submit as detailed a proposal as possible for review by the Program Committee. Organizers of proposals will be notified whether their proposal has been accepted by May 12, 2017. Additional instructions and the session's schedule will be sent to the contact organizers of the accepted sessions shortly after that deadline.

## Columbus, Ohio

*Ohio State University*

**March 17–18, 2018**

*Saturday – Sunday*

### Meeting #1136

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Probability in Convexity and Convexity in Probability* (Code: SS 2A), **Elizabeth Meckes**, **Mark Meckes**, and **Elisabeth Werner**, Case Western Reserve University.

*Recent Advances in Approximation Theory and Operator Theory* (Code: SS 1A), **Jan Lang** and **Paul Nevai**, The Ohio State University.

## Nashville, Tennessee

*Vanderbilt University*

**April 14–15, 2018**

*Saturday – Sunday*

### Meeting #1138

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

## Portland, Oregon

*Portland State University*

**April 14–15, 2018**

*Saturday – Sunday*

### Meeting #1137

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

### Invited Addresses

**Sándor Kovács**, University of Washington, Seattle, *Title to be announced.*

**Elena Mantovan**, California Institute of Technology, *Title to be announced.*

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Inverse Problems* (Code: SS 2A), **Hanna Makaruk**, Los Alamos National Laboratory (LANL), and **Robert Owcza-rek**, University of New Mexico, Albuquerque & Los Alamos.

*Pattern Formation in Crowds, Flocks, and Traffic* (Code: SS 1A), **J. J. P. Veerman**, Portland State University, **Alethea**

**Barbaro**, Case Western Reserve University, and **Bassam Bamieh**, UC Santa Barbara.

## Boston, Massachusetts

*Northeastern University*

**April 21–22, 2018**

*Saturday – Sunday*

### Meeting #1139

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgsectional.html](http://www.ams.org/amsmtgsectional.html).*

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl).*

*Arithmetic Dynamics* (Code: SS 1A), **Jacqueline M. Anderson**, Bridgewater State University, **Robert Benedetto**, Amherst College, and **Joseph H. Silverman**, Brown University.

## Shanghai, People's Republic of China

*Fudan University*

**June 11–14, 2018**

*Monday – Thursday*

### Meeting #1140

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

## Newark, Delaware

*University of Delaware*

**September 29–30, 2018**

*Saturday – Sunday*

### Meeting #1141

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

## Fayetteville, Arkansas

*University of Arkansas*

**October 6–7, 2018**

*Saturday – Sunday*

### Meeting #1142

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

**Deadlines**

For organizers: To be announced

For abstracts: To be announced

## Ann Arbor, Michigan

*University of Michigan, Ann Arbor*

**October 20–21, 2018**

*Saturday – Sunday*

### Meeting #1143

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

# San Francisco, California

*San Francisco State University*

**October 27–28, 2018**

*Saturday – Sunday*

**Meeting #1144**

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: August 2018

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: To be announced

For abstracts: To be announced

# Baltimore, Maryland

*Baltimore Convention Center, Hilton  
Baltimore, and Baltimore Marriott Inner  
Harbor Hotel*

**January 16–19, 2019**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 125th Annual Meeting of the AMS, 102nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: October 2018

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: April 2, 2018

For abstracts: To be announced

# Honolulu, Hawaii

*University of Hawaii at Manoa*

**March 22–24, 2019**

*Friday – Sunday*

Central Section & Western Section

Associate secretaries: Georgia Benkart and Michel L. Lapidus

Announcement issue of *Notices*: To be announced

MARCH 2017

Program first available on AMS website: To be announced  
Issue of *Abstracts*: To be announced

## Deadlines

For organizers: To be announced

For abstracts: To be announced

# Denver, Colorado

*Colorado Convention Center*

**January 15–18, 2020**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)*

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: To be announced

Program first available on AMS website: November 1, 2019

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: April 1, 2019

For abstracts: To be announced

# Washington, District of Columbia

*Walter E. Washington Convention Center*

**January 6–9, 2021**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: October 2020

Program first available on AMS website: November 1, 2020

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: April 1, 2020

For abstracts: To be announced

"I hope that seeing the excitement of solving this problem will make young mathematicians realize that there are lots and lots of other problems in mathematics which are going to be just as challenging in the future."

—Sir Andrew Wiles

## Andrew Wiles Stats:

Earliest indexed publication: 1977

Total publications: 26

Total author/related publications: 38

Total citations: 1,645

PhD University of Cambridge 1979

Advisor: John Henry Coates

According to the MathSciNet® database, as of December 9, 2016, Andrew Wiles has 21 students and 178 descendants.

## Two most cited works:

Modular elliptic curves and Fermat's last theorem, Ann. of Math. (1995).

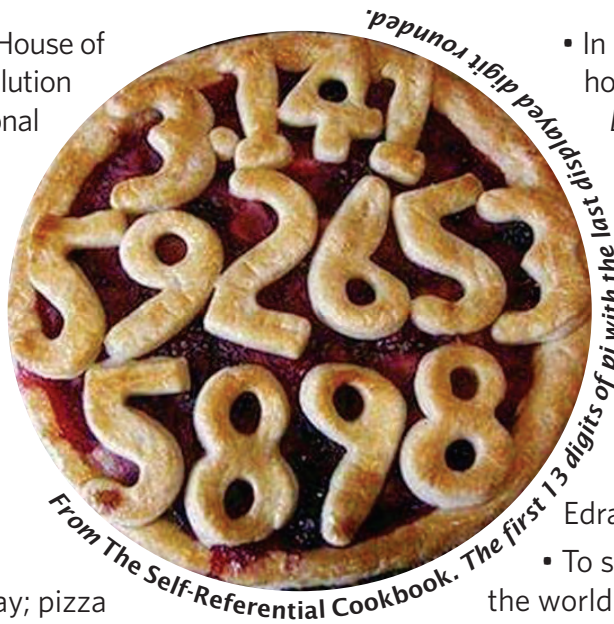
With Richard Taylor, Ring-theoretic properties of certain Hecke algebras, Ann. of Math (1995).

## Facts About Pi Day:

- On March 12, 2009, the US House of Representatives passed a resolution recognizing March 14 as National Pi Day.

- The earliest large-scale celebration of Pi Day was organized by Larry Shaw in 1988 at the San Francisco Exploratorium. The staff and public participants marched around one of the museum's circular spaces and then partook of fruit pies.

- The Exploratorium holds Pi Day celebrations to this day; pizza pie has been added to their official Pi Day menu.



- In Rhode Island, the AMS will be holding its annual *Who Wants to Be a Mathematician* game with local high school students.

- Read more about Pi Day on AMS Blogs: *The Pi Day Link Roundup of the Century*, by Evelyn Lamb; *Pi Day and Other Math Holidays*, by Maya Sharma; and *The Ubiquity of Pi Day: It's Not Just for Math Geeks*, by Edray Goins. See [blogs.ams.org](http://blogs.ams.org).

- To see celebrations from around the world and to add evidence of your own Pi Day celebration go to Twitter and use hashtag #PiDay.

## QUESTIONABLE MATHEMATICS

The *Boston Globe* (April 2003) reported on data suggesting that "15 percent of US youngsters are severely overweight or obese, defined as having a body-mass index that is greater than the index for 95 percent of their peers."

Submitted by Richard Stanley

**What crazy things happen to you?** Readers are invited to submit original short amusing stories, math jokes, cartoons, and other material to: [noti-backpage@ams.org](mailto:noti-backpage@ams.org).

# IN THE NEXT ISSUE OF NOTICES



## APRIL 2017...



**Full announcements** of the 2017 Leroy P. Steele Prizes, Ruth Lyttle Satter Prize in Mathematics, Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student, Leonard Eisenbud Prize for Mathematics and Physics, Joseph L. Doob Prize, Levi L. Conant Prize, Cole Prize in Number Theory, Bôcher Memorial Prize, which were presented at the 123rd Annual Meeting of the AMS in Atlanta, Georgia, in January 2017.



### The AWM Research Symposium 2017 Lecture Sampler

The Symposium (April 8–9, 2017 at UCLA) will showcase the research of women in the mathematical professions. Some of the plenary speakers and members of the committee have offered to share a sneak-peek of their presentations with *Notices*.

## AMERICAN MATHEMATICAL SOCIETY

### A Literary Spotlight on Women in Mathematics

#### Change Is Possible

##### Stories of Women and Minorities in Mathematics

**Patricia Clark Kenschaft**, Montclair State University, Upper Montclair, NJ

*The role of minority and women mathematicians in developing our American mathematical community is an important but previously under-told story. Pat Kenschaft, in her highly readable and entertaining style, fills this knowledge gap. This valuable book should be in your personal library!*

—Donald G. Saari, University of California, Irvine

An entertaining look at previously under-told stories of mathematicians who defied stereotypes and overcame barriers to success.

2005; 212 pages; Softcover; ISBN: 978-0-8218-3748-1; List US\$34; AMS members US\$27.20; Order code CHANGE

#### Pioneering Women in American Mathematics

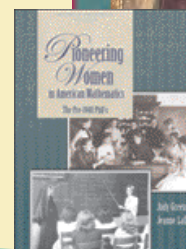
##### The Pre-1940 PhD's

**Judy Green**, Marymount University, Arlington, VA, and **Jeanne LaDuke**, DePaul University, Chicago, IL

*What a service Judy Greene and Jeanne LaDuke have done the mathematics community! Approximately thirty years of research have produced a detailed picture of graduate mathematics for women in the United States before 1940. ... The book is well-organized and well-written, and I recommend it heartily to all.*

—AWM Newsletter

History of Mathematics, Volume 34; 2009; 345 pages; Hardcover; ISBN: 978-0-8218-4376-5; List US\$86; AMS members US\$68.80; Order code HMATH/34



Women's History Month

Visit [bookstore.ams.org](http://bookstore.ams.org) to learn about these titles and more!

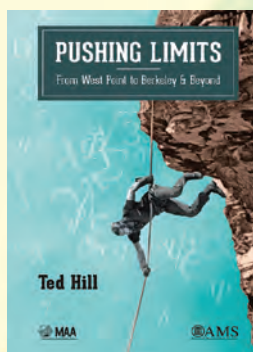
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AMERICAN MATHEMATICAL SOCIETY

# New from the AMS

## Now available for pre-order!



### Pushing Limits

From West Point to Berkeley and Beyond

Ted Hill, Georgia Tech, Atlanta and Cal Poly, San Luis Obispo

A fascinating journey from pure adventurism...through West Point and the Vietnam War to the highest intellectual accomplishments. At the center is a beautiful portrayal of the tedious, but highly rewarding road from graduate school to becoming a substantial research mathematician. A joy to read.

—David Gilat, Professor Emeritus, School of Mathematical Sciences, Tel Aviv University

Ted Hill is the Indiana Jones of mathematics...A West Point graduate, [he] served in Vietnam, swam with sharks in the Caribbean, and has resolutely defied unreasoned authority. With this same love of adventure, he has confronted the sublime challenges of mathematics. Whether it's discovering intellectual treasures or careening down jungle trails, this real life Dr. Jones has done it all.

—Michael Monticino, Professor of Mathematics and Special Assistant to the President, U. North Texas

This book is co-published with the Mathematical Association of America.

2017; approximately 334 pages; Hardcover; ISBN: 978-1-4704-3584-4; List US\$45; AMS members US\$36; Order code MBK/103

### Game Theory, Alive TEXTBOOK

Anna R. Karlin, University of Washington, Seattle and Yuval Peres, Microsoft Research, Redmond, WA

By focusing on theoretical highlights and presenting exciting connections between game theory and other fields, this broad overview emphasizes game theory's real-world applications and mathematical foundations.

2017; approximately 396 pages; Hardcover; ISBN: 978-1-4704-1982-0; List US\$75; AMS members US\$60; Order code MBK/101

### It's About Time

Elementary Mathematical Aspects of Relativity

Roger Cooke, University of Vermont, Burlington

This book explores a selection of topics from special and general relativity, accompanied by mathematical explanations accessible to those at the intermediate undergraduate level.

2017; 403 pages; Hardcover; ISBN: 978-1-4704-3483-0; List US\$75; AMS members US\$60; Order code MBK/102


### Algebra in Action TEXTBOOK

A Course in Groups, Rings, and Fields

Shahriar Shahriari, Pomona College, Claremont, CA

Aimed at undergraduates who are new to abstract algebra, this text is a readable, student-friendly, and somewhat sophisticated introduction to groups, rings, and fields.

Pure and Applied Undergraduate Texts, Volume 27; 2016; approximately 655 pages; Hardcover; ISBN: 978-1-4704-2849-5; List US\$115; AMS members US\$92; Order code AMSTEXT/27

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