

AMS SPRING SECTIONAL SAMPLER



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Spring Southeastern Sectional Meeting
March 10–12, 2017 (Friday–Sunday)
College of Charleston, Charleston, SC



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Spring Central Sectional Meeting
April 1–2, 2017 (Saturday–Sunday)
Indiana University, Bloomington, IN



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Spring Western Sectional Meeting
April 22–23, 2017 (Saturday–Sunday)
Washington State University, Pullman, WA



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***Spring Eastern Sectional Meeting**
May 6–7, 2017 (Saturday–Sunday)
Hunter College, City University of New York, New York, NY
* A sampler from this meeting will appear in the May issue of *Notices*.

Make the time to visit any of the AMS Spring Sectional Meetings listed above.

In this sampler, the speakers below have kindly provided introductions to their Invited Addresses for the upcoming AMS Spring Central Sectional (Indiana University, April 1–2) and the AMS Spring Western Sectional (Washington State University, April 22–23) Meetings.

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DOI: <http://dx.doi.org/10.1090/noti1530>

Michael Hitrik

Spectra for Non-self-adjoint Operators and Integrable Dynamics

This creates a challenge, but also provides an opportunity.

some form of damping or a possibility of escape to infinity, will be governed by a non-self-adjoint operator. From the point of view of mathematics, an essential feature making the study of non-self-adjoint operators notoriously difficult is the potential instability of their spectra. When A is a self-adjoint matrix, we know from the spectral theorem that the operator norm of the resolvent $(A - zI)^{-1}$ is equal to the reciprocal of the distance from z to the spectrum of A . In contrast, let us consider a (non-self-adjoint) Jordan block matrix

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

of size n , with n large. The spectrum of J_n is equal to $\{0\}$, while the operator norm of the resolvent $(J_n - zI)^{-1}$ grows exponentially with n , roughly like $1/|z|^n$, for z in the open unit disc in the complex plane. As one can imagine, things are even more intricate for, say, differential operators. This creates a challenge, but also provides an opportunity, accounting for some of the complex and fascinating traits in the spectral behavior of non-self-adjoint operators.

For most operators, we cannot compute the spectra exactly, and, as is natural in many physical situations, one studies *asymptotic* properties for problems depending on a parameter. To make things uniform, we consider a small parameter h and refer to the study of the limit $h \rightarrow 0$ as the semiclassical approximation. It is important to remember, however, that h may mean different things in different settings: as the name suggests, it could be the Planck

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I am most grateful to Johannes Sjöstrand, Joe Viola, and Maciej Zworski for their very helpful comments on this note.

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DOI: <http://dx.doi.org/10.1090/noti1495>

constant (quantum mechanics), but also the temperature (Fokker–Planck equation), or the reciprocal of the square root of an eigenvalue (spectral geometry), of frequency (scattering theory), of Reynolds number (fluid mechanics), or of the power of a line bundle (complex geometry). Roughly speaking, h measures the slow variation of the medium compared to the length of the waves, hence the alternative name short wave approximation. In PDE theory, this is the basis of *microlocal analysis*, where $h \sim 1/|\partial_x|$. Adopting the philosophy of the quantum-classical correspondence principle of Niels Bohr, whereby the behavior of quantum mechanical systems should reduce to classical physics in the semiclassical limit $h \rightarrow 0$, we are then led to the following basic problem: can we describe the distribution of complex eigenvalues of a non-self-adjoint operator, in the semiclassical régime, in terms of the underlying classical dynamics? The latter is given by a Hamilton flow on the classical phase space, and to implement the quantum-classical correspondence in a non-self-adjoint environment, one frequently needs to consider complex such flows on the complexification of the real phase space.

In joint work with Johannes Sjöstrand, we have carried out a detailed spectral analysis for non-self-adjoint perturbations of self-adjoint operators in dimension two under suitable assumptions of analytic and dynamical nature. Specifically, let us consider a non-self-adjoint operator of the form

$$P_\varepsilon = P_0 + i\varepsilon Q,$$

acting on a compact two-dimensional real-analytic Riemannian manifold M . Here, to fix the ideas, we may take $P_0 = -h^2\Delta$ to be the semiclassical Laplacian, and Q is a multiplication by an analytic real-valued function q on M , which we can think of as a damping coefficient or a potential. The small parameter $\varepsilon > 0$ measures the strength of the non-self-adjoint perturbation, which should not be too weak. The eigenvalues of P_ε (see Figure 1) are confined to a thin band near the real domain, and thanks to a classical result originating in work of Torsten Carleman, we know that the distribution of the real parts of the eigenvalues of P_ε near the energy level $E = 1$, say, is governed by the same Weyl law as that for the unperturbed self-adjoint operator P_0 . The distribution of the imaginary parts of the eigenvalues of P_ε is much more subtle, however, and is expected to depend on the dynamics of the geodesic flow and its relation to the non-self-adjoint perturbation.

In order to obtain a rigorous justification of the intuition above, let us assume that the geodesic flow is *completely integrable*. The phase space T^*M is then foliated by Lagrangian tori, invariant under the geodesic flow, and the flow on each such torus becomes linear when expressed in suitable canonical coordinates. Following the ideas of the method of averaging (Birkhoff normal forms), we are led to consider the long-time averages of q along the geodesic flow. The following two radically different cases occur, depending on the type of the dynamics: periodic

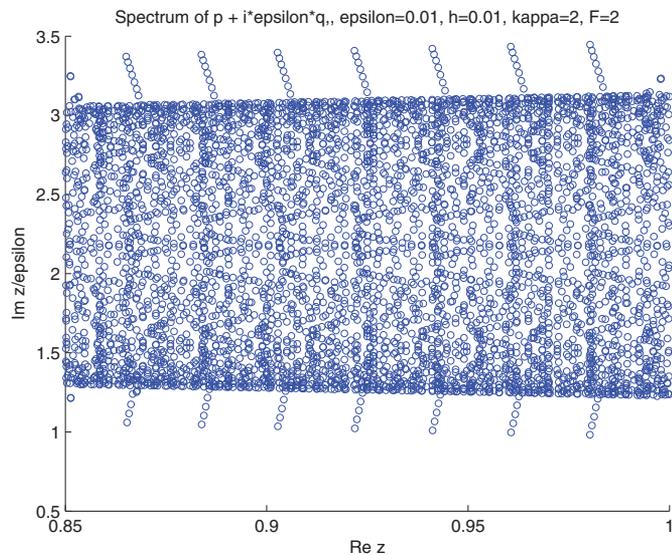


Figure 1. Numerical computation of the real and normalized imaginary parts of the eigenvalues of the non-self-adjoint operator $P_\epsilon = P_0 + i\epsilon Q$ on the two-dimensional torus, when Q is multiplication by a trigonometric polynomial q of degree 2. The legs of the spectral centipede represent the influence of rational tori for the classical flow.

(rational tori) or dense (irrational or, better, Diophantine tori). In joint work with Sjöstrand and San Vũ Ngọc, we have obtained a complete semiclassical description of the spectrum of P_ϵ near a complex energy level of the form $1 + i\epsilon F$, assuming that F is given by the average of q along a Diophantine torus and that no other torus has this property, roughly speaking. It turns out that the spectrum in this region has the structure of a distorted two-dimensional lattice, given by a quantization condition of Bohr-Sommerfeld type.

What about spectral contributions of rational tori? These have remained rather mysterious for quite some time, until recent work with Sjöstrand demonstrated that rational tori can produce eigenvalues of P_ϵ close to the edges of the spectral band, isolated away from the bulk of the spectrum, and that a complete semiclassical description of these extremal, “rational” eigenvalues can be achieved. Somewhat surprisingly perhaps, the extremal eigenvalues turn out to have the structure of the legs in a spectral “centipede” as in Figure 1, with the body of the centipede agreeing with the range of torus averages of q .

The following vague philosophical remark may perhaps clarify the reason for a detailed study of dimension two: as one knows from basic quantum mechanics texts, for problems in dimension one, the Bohr-Sommerfeld quantization rules work very well to determine spectra of self-adjoint observables, whose energy surfaces are real curves. For complex Hamiltonians in dimension two, energy surfaces are *complex curves*, and this can

sometimes be used to obtain spectral results that are more precise than those for real Hamiltonians.

The spectral structures described above are expected to occur also for other genuinely two-dimensional non-self-adjoint analytic spectral problems and notably for the distribution of scattering resonances for convex obstacles in \mathbf{R}^3 , assuming that the boundary geodesic flow is completely integrable or is a small perturbation of it. The study of these fascinating issues, which is very much at the heart of the current research, will also be touched upon in the talk.

Credits

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THE AUTHOR



Michael Hitrik

Andrew Raich

Closed Range of $\bar{\partial}$ in L^2 on Domains in \mathbb{C}^n

At the AMS Western Sectional Meeting, I will talk about my recent work on the L^2 theory of the Cauchy-Riemann operator $\bar{\partial}$ in several complex variables and the solvability/regularity of solutions u to the Cauchy-Riemann equation $\bar{\partial}u = f$ in a domain Ω in \mathbb{C}^n .

In one complex variable, we have the Cauchy kernel as an incredibly powerful tool to solve $\bar{\partial}u = f$. of the Cauchy kernel rests largely on the facts that it is holomorphic and can be used to build a variety of operators. Integrating against the Cauchy kernel on Ω produces a solution to the Cauchy-Riemann operator, while integrating against the Cauchy kernel on the boundary $\partial\Omega$ builds an operator that (re)produces holomorphic functions. Also, for dimensional reasons, the boundary of any domain $\Omega \subset \mathbb{C}$ lacks any complex structure

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DOI: <http://dx.doi.org/10.1090/noti1494>

The Riemann Mapping Theorem fails in the most spectacular way possible.

(a one real dimensional object cannot carry any complex structure), which, philosophically, allows for a result like the Riemann Mapping Theorem to hold.

In several variables, the Cauchy kernel has no higher dimensional analog. All of the replacements have serious deficiencies; for example, they are noncanonical or not holomorphic. Moreover, domains in \mathbb{C}^n

have boundaries that are $(2n - 1)$ -real dimensional manifolds, so there is an $(n - 1)$ -complex dimensional structure as part of the boundary. Not coincidentally, the Riemann Mapping Theorem fails in the most spectacular way possible—the unit ball and unit polydisk (product of real 2D disks) are not biholomorphic. We are left in a landscape where the solvability and regularity of the Cauchy-Riemann equations $\bar{\partial}u = f$ depend on subtle geometric and potential-theoretic quantities associated to the boundary $b\Omega$. This makes the analysis much more difficult than either the one complex dimensional case or the analogous real problem $du = f$ on Ω .

To discuss the $\bar{\partial}$ -problem in any depth, we must frame the problem more carefully. Given a $(0, q)$ -form f on a domain Ω , the problem is to find a $(0, q - 1)$ -form u so that $\bar{\partial}u = f$. As with d , $\bar{\partial}^2 = 0$, so solving $\bar{\partial}u = f$ has the necessary condition $\bar{\partial}f = 0$. The question then becomes to solve $\bar{\partial}u = f$ when $\bar{\partial}f = 0$. Before the 1960s, the inhomogeneous Cauchy-Riemann equation was studied in the C^∞ -topology. In the 1960s, Hörmander and Kohn pioneered the use of L^2 techniques to solve the equation. From a functional analytic viewpoint, the switch from C^∞ to L^2 is a major improvement, because the space $C^\infty(\Omega)$ is nonmetrizable, while $L^2(\Omega)$ is a Hilbert space, which allows for the use of tremendously powerful tools. Hörmander showed that $\bar{\partial}u = f$ can be solved in $L^2_{0,q}(\Omega)$ at every form level ($1 \leq q \leq n$) if and only if Ω satisfies a curvature condition called pseudoconvexity. Moreover, he also showed that there is no L^2 cohomology for $\bar{\partial}$ on pseudoconvex domains. Pseudoconvexity can be understood as the complex analysis version of convexity. For example, pseudoconvexity is invariant under biholomorphisms while convexity is not.

A common approach to solving the inhomogeneous Cauchy-Riemann equations is to establish a Hodge theory for $\bar{\partial}$. To do this, we need to introduce the L^2 adjoint of $\bar{\partial}$, $\bar{\partial}^*$. Computationally, $\bar{\partial}^*$ is obtained via integration by parts, and as every calculus student knows, integration by parts introduces a boundary term. For a form to be in the domain of $\bar{\partial}^*$ this boundary term must vanish. This vanishing condition is the source of trouble for proving regularity results. Although $\bar{\partial} \oplus \bar{\partial}^*$ provides an elliptic system, the inverse to $\bar{\partial}$ (when it exists) never has an elliptic gain. The $\bar{\partial}$ -Neumann Laplacian $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ acts diagonally

on forms, and componentwise it is nothing more than the ordinary Laplacian. However, inverting \square never gains two derivatives. Sometimes the solving operator gains one derivative, sometimes it gains a fractional derivative, and sometimes the inverse is just continuous on L^2 . It is in the domain of $\bar{\partial}^*$ that the geometric information of the boundary is encoded, and it is the geometry of the boundary that determines how nice solutions of $\bar{\partial}u = f$ can be.

In the decades since the 1960s the exploration into the inhomogeneous Cauchy-Riemann equation $\bar{\partial}u = f$ has expanded to various function spaces, complex manifolds, the tangential Cauchy-Riemann equations on CR manifolds, and, more recently, nonpseudoconvex domains and unbounded domains. This is where my colleague Phil Harrington and I enter the picture. To understand the more subtle geometry, we work on answering the question, What is the replacement condition for pseudoconvexity for optimal solvability of $\bar{\partial}u = f$ in $L^2_{0,q}(\Omega)$ for a fixed q , $1 \leq q \leq n$? In my talk, I will outline the progress that we have made on this problem and put our results in context. Time permitting, I will also discuss our progress on solving the inhomogeneous Cauchy-Riemann equation on unbounded domains and the additional complications that this entails.

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Daniel Rogalski

Noncommutative Projective Surfaces

In this note we introduce the reader to the subject of noncommutative projective geometry, which will be the focus of our talk at the AMS regional meeting in

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DOI: <http://dx.doi.org/10.1090/noti1502>

Pullman, WA, in April 2017. In the talk we will begin with an overview of noncommutative geometry and then give a survey of some results about noncommutative surfaces which have been the subject of our own research. In this short space, however, we primarily describe the main idea of the subject and then at the end briefly mention one of the interesting examples that will be explored in more detail in the talk. Given our informality here, we do not provide references. The reader can find more information in the author's more extensive course notes on the subject [BRSSW].

We assume the reader is familiar with the definition of a ring. We always assume that rings have an identity element 1, and in fact all of our examples will be algebras over the complex numbers \mathbb{C} , which means that the ring contains a copy of the field \mathbb{C} and can be thought of as a \mathbb{C} -vector space. In a first course in abstract algebra, the discussion of rings invariably focuses almost exclusively on commutative rings, where $ab = ba$ for all elements a and b . Perhaps the matrix ring $M_n(\mathbb{C})$ might be mentioned as an example of a noncommutative ring. Of course, it makes sense to get comfortable first with commutative rings, which are essential to many applications of ring theory. Plenty of interesting rings arising in nature are noncommutative, however, such as the various quantum algebras that arise in physics to describe the interaction of operators (such as position and momentum) that do not necessarily commute. One simple but already nontrivial example of a noncommutative algebra is the *quantum plane*, $A_q = \mathbb{C}\langle x, y \rangle / (yx - qxy)$, for a nonzero scalar $q \in \mathbb{C}$. Here, $\mathbb{C}\langle x, y \rangle$ is the free associative algebra in two variables, that is, the \mathbb{C} -vector space with basis all words in the (noncommuting) variables x and y , with product given by concatenation. To obtain A_q we mod out by the ideal generated by one relation, which says that the variables commute "up to scalar." It is not hard to see that the elements of A_q can be identified with the usual commutative polynomials $\sum_{i,j \geq 0} a_{ij} x^i y^j$ in the variables, but the multiplication is changed so that every time we move a y to the right of an x , a factor of q appears. We describe some properties of this algebra below.

Commutative ring theorists have long enjoyed the advantage of the tight connection of their subject with algebraic geometry. This means both that ring-theoretic results may have an application to geometry and that geometric intuition can help to prove and interpret results about rings. In a course on algebraic geometry over the complex numbers, a student learns first about the most fundamental varieties: affine n -space $\mathbb{A}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{C}\}$, and projective n -space \mathbb{P}^n , which can be identified with the set of lines through the origin in \mathbb{A}^{n+1} , with the line through the point (a_0, a_1, \dots, a_n) written as $(a_0 : a_1 : \dots : a_n)$. Both affine and projective spaces have a natural topology called the Zariski topology, and affine varieties are defined to be closed subsets of affine spaces, while projective varieties are defined to be closed subsets of projective spaces. In the more modern approach to the subject via the theory of schemes, the

connection between varieties and rings becomes more important, and one learns how to define a projective variety all at once using a graded ring. In this note, a *graded \mathbb{C} -algebra* will be a \mathbb{C} -algebra R with a \mathbb{C} -vector space decomposition $R = R_0 \oplus R_1 \oplus \dots$, where $R_n R_m \subseteq R_{n+m}$ for all n, m , and where for simplicity we always assume that $R_0 = \mathbb{C}$ and that R is generated by R_1 as an algebra over \mathbb{C} . For a commutative graded \mathbb{C} -algebra, one defines $\text{Proj } R$ to be the set of homogeneous prime ideals in R , that is, those prime ideals P such that $P = \bigoplus_{n \geq 0} (P \cap R_n)$, excluding the *irrelevant ideal* $R_{\geq 1}$ spanned by all elements of positive degree. The maximal elements of this space $\text{Proj } R$ correspond to the points of a projective variety, while the other points are called "generic" and are important to the scheme-theoretic point of view. For example, the polynomial ring $R = \mathbb{C}[x_0, x_1, \dots, x_n]$ in $n + 1$ variables over \mathbb{C} is graded, where R_n is simply the span of monomials of degree n , and in this case $\text{Proj } R = \mathbb{P}^n$. As a more explicit example, the homogeneous primes of $\mathbb{C}\langle x, y \rangle$ are (0) , $(ax + by)$

for $a, b \in \mathbb{C}$ not both 0, and (x, y) . Removing the irrelevant ideal (x, y) , the maximal elements are those ideals of the form $(ax + by)$ which correspond to points $(a : b)$ in \mathbb{P}^1 .

Noncommutative projective geometry is an attempt to understand noncommutative graded rings in a geometric way or to associate some kind of interesting geometric space to a noncommutative ring, similar to the way that the Proj construction associates a projective variety to a commutative graded ring. The development of the subject was surely delayed by the fact that the most obvious analog of the Proj construction for a noncommutative ring does not lead to an interesting theory, even for such fundamental examples as the quantum plane. There is a natural definition of prime ideal for a noncommutative ring: P is prime if whenever $IJ \subseteq P$ for ideals I and J , $I \subseteq P$ or $J \subseteq P$ (this definition reduces to the usual one for commutative rings). However, in contrast to the commutative case, the quantum plane A_q usually has very few prime ideals. In particular, if q is not a root of unity, the only homogeneous prime ideals of A_q are (0) , (x) , (y) , and (x, y) . Removing the irrelevant ideal (x, y) , one is not left with a very interesting space. On the other hand, A_q behaves a lot like a commutative polynomial ring in two variables in many other ways, so one might hope it would have a geometry more similar to $\text{Proj } \mathbb{C}\langle x, y \rangle = \mathbb{P}^1$.

As a general principle, often one has to find a particular way of formulating a concept in the commutative case, not always the most elementary or obvious way, to

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geometric way.*

get something that generalizes to a useful notion for noncommutative rings. In fact, there are two more subtle ways of thinking about how \mathbb{P}^1 is connected with the ring $\mathbb{C}[x, y]$ which *do* generalize well to the quantum plane. First, given any graded \mathbb{C} -algebra A , a *point module* over A is a graded left module $M = \bigoplus_{n \geq 0} M_n$ (that is, $A_i M_j \subseteq M_{i+j}$ for all i, j) such that $M = AM_0$ and $\dim_{\mathbb{C}} M_n = 1$ for all $n \geq 0$. The point modules for $\mathbb{C}[x, y]$ are the cyclic modules $M_{(a:b)} = \mathbb{C}[x, y]/(ax + by)$, as $(a : b)$ varies over points in \mathbb{P}^1 . One can even say that the point modules are *parametrized* by \mathbb{P}^1 in a way that is made more precise in algebraic geometry. It turns out that the point modules for A_q are also parametrized by \mathbb{P}^1 , being of the form $N_{(a:b)} = A_q/A_q(ax + by)$ where $A_q(ax + by)$ is the *left* ideal generated by $ax + by$. Thus, the space of point modules associated to A_q is always the projective line \mathbb{P}^1 .

An even more subtle approach involves generalizing the idea of sheaves. A sheaf on a projective variety can be defined by taking an open cover by rings and taking a module over each ring such that the modules “glue together” appropriately. This is the most natural definition geometrically, but it turns out that there is a purely algebraic way of getting at this notion. Given a commutative graded \mathbb{C} -algebra A , we define the *category* of finitely generated \mathbb{Z} -graded left A -modules $A\text{-gr}$ and define $A\text{-qgr}$ to be the quotient category of $A\text{-gr}$ by the subcategory of modules M which have $M_n = 0$ for $n \gg 0$. The category $A\text{-qgr}$ turns out to be the same as the category of *coherent sheaves* on $\text{Proj } A$. The category of coherent sheaves on a variety is extremely fundamental to the study of the variety, similar to the way that the representations of a finite group are fundamental to the study of the group. Fortunately, the category $A\text{-qgr}$ as defined above makes perfect sense for any noncommutative graded algebra A , and thus it intuitively represents some kind of “category of coherent sheaves” *even though there is no noncommutative space that these objects are sheaves on!* Thus the category becomes the fundamental object of study in the noncommutative case, and in fact $A\text{-qgr}$ is called the *noncommutative projective scheme* associated to the algebra A . One may prove that the noncommutative projective scheme of the quantum plane, $A_q\text{-qgr}$, is in fact equivalent to $\mathbb{C}[x, y]\text{-qgr}$, the usual category of coherent sheaves on \mathbb{P}^1 . We say that A_q is a *noncommutative coordinate ring* of \mathbb{P}^1 .

Most of the attention in noncommutative geometry has focused on analogs of surfaces and higher-dimensional varieties, where things become more complicated. Artin and Schelter defined a notion of *regular algebra* which captures those noncommutative graded algebras which behave *homologically* like commutative polynomial rings; these are the rings that correspond geometrically to noncommutative projective spaces. The noncommutative \mathbb{P}^2 s were classified by Artin, Schelter, Tate, and Van den Bergh. One of the most important such examples is the

Sklyanin algebra

$$S = \mathbb{C}\langle x, y, z \rangle / (ax^2 + byz + czy, ay^2 + bzx + cxz, az^2 + bxy + cyx)$$

for general scalars $a, b, c \in \mathbb{C}$. It turns out that noncommutative \mathbb{P}^2 s can have fewer points than one might at first expect. In particular, the point modules for S turn out to be parametrized by an *elliptic curve* E , and the beautiful theory of elliptic curves in algebraic geometry plays a big role in the analysis of the properties of the algebra S . The noncommutative projective scheme $S\text{-qgr}$ is similar in some ways to the category of coherent sheaves on \mathbb{P}^2 , but is not the same—it is a genuinely noncommutative \mathbb{P}^2 , with only an elliptic curve’s worth of points!

In the talk we plan to describe our long-term program to develop a minimal model program for noncommutative surfaces, which has driven a lot of our recent research. To close this note, we wish to mention a weird example that is particularly close to our heart. One of the joys of noncommutative algebra is the wide abundance of examples and counterexamples which constantly point out the myriad ways in which noncommutative rings are more complicated (or we could say, with obvious bias, more interesting!) than commutative rings. Because of this zoo of examples, there are still many foundational questions in noncommutative geometry about what assumptions on graded rings are necessary to get a good theory. It is natural to restrict to the study of graded rings which are *Noetherian* (after Emmy Noether), which means that they do not have infinite ascending chains of left or right ideals. This is a basic structural assumption, just as in the commutative case, and one which is satisfied by most reasonable examples. However, it is not enough to prevent certain kinds of pathologies in noncommutative geometry. In particular, there turn out to be many Noetherian graded algebras R whose space of point modules is so wild that it is not parametrized by a projective variety. One simple example of such an R can be obtained as follows. First define $B = \mathbb{C}\langle x, y, z \rangle / (zx - pxz, xy - ryx, yz - qzy)$, where $p, q, r \in \mathbb{C}$ are general scalars satisfying $pqr = 1$. This is a very nice noncommutative \mathbb{P}^2 ; in fact, B has point modules parametrized by \mathbb{P}^2 , and $B\text{-qgr}$ is equivalent to coherent sheaves on \mathbb{P}^2 . The subalgebra $R = \mathbb{C}\langle x - y, y - z \rangle \subseteq B$ generated by $x - y$ and $y - z$ inside B is Noetherian, yet has point modules corresponding to an *infinite blowup* of \mathbb{P}^2 , where each of infinitely many points in the orbit of a certain automorphism of \mathbb{P}^2 gets replaced by a whole projective line. The noncommutative projective scheme $R\text{-qgr}$ also has unusual properties. Ring theoretically, the strangeness of R shows up in the failure of R to be *strongly Noetherian*; that is, the base extension $R \otimes_{\mathbb{C}} \mathbb{C}$ is not Noetherian for some Noetherian commutative \mathbb{C} -algebra C . There is now a whole theory of rings like this, which are called *naïve blowups*. They form one of the main classes of examples in the theory of what are known as *birational commutative surfaces*, as have been classified

by the author and Stafford, and in a more general case by Sierra.

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THE AUTHOR

Daniel Rogalski

Sarah C. Koch

Postcritical Sets in Complex Dynamics

I'll begin by introducing some key objects and ideas from the field of complex dynamics, specifically looking at the dynamics of quadratic polynomials. I will then discuss the work I plan to present at the AMS meeting in April 2017.

Iterating Quadratic Polynomials

For each complex number c , consider the map

$$p_c : \mathbb{C} \rightarrow \mathbb{C} \text{ given by } p_c : z \mapsto z^2 + c.$$

Although p_c is just a quadratic polynomial, a wealth of complicated and deep behavior can emerge when it is iterated. Given $z_0 \in \mathbb{C}$, the *orbit* of z_0 is the sequence $z_n := p_c(z_{n-1})$. There is a nonempty compact subset $K_c \subseteq \mathbb{C}$ associated to iterating p_c called the *filled Julia set* of p_c ; see Figure 1. By definition K_c consists of all $z_0 \in \mathbb{C}$ such that the orbit of z_0 is bounded.

In complex dynamics, the orbits of the critical points (those points where the derivative of the map vanishes) play an important role. For example, the filled Julia set

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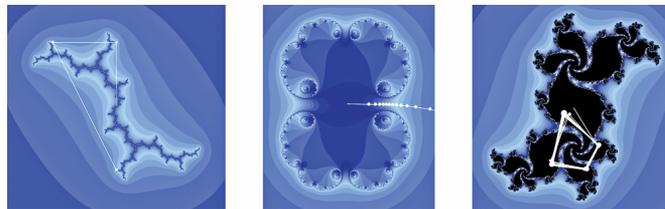


Figure 1. Filled Julia sets K_c of three quadratic polynomials p_c are drawn in black. The critical point $z_0 = 0$ is in the center of each of these pictures, and its orbit is drawn in white. LEFT: The set K_c is connected and has no interior. The critical point eventually maps to a repelling cycle of period 2. CENTER: The set K_c is a Cantor set. The critical point escapes to infinity under iteration. RIGHT: The set K_c is connected, and every interior point (including the critical point) is attracted to an attracting cycle of period 4.

K_c is connected if and only if the orbit of the critical point $z_0 = 0$ is bounded, that is, if and only if $0 \in K_c$. As another example, if p_c possesses an attracting periodic cycle, then the critical point is necessarily attracted to it under iteration.

The Mandelbrot Set

Moving from the z -plane to the c -parameter plane, one considers the *Mandelbrot set*

$$M := \{c \in \mathbb{C} \mid K_c \text{ is connected}\} = \{c \in \mathbb{C} \mid 0 \in K_c\}.$$

The Mandelbrot set (see Figure 2) is a fundamental object in the field of complex dynamics. It has been thoroughly studied over the past few decades with much success, and

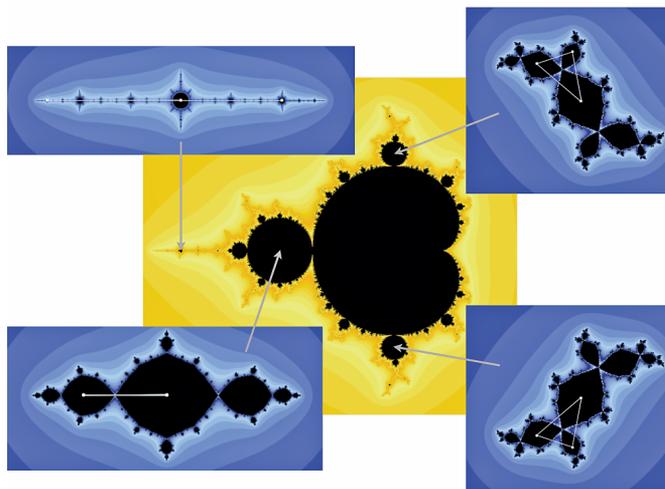


Figure 2. The picture in the center is the c -parameter plane for the family $p_c(z) = z^2 + c$; the Mandelbrot set is drawn in black. Each of the surrounding pictures contains a filled Julia set K_c , and the arrows indicate the corresponding value of c .

it continues to be a central topic of research. It is compact and connected. Among the connected components of its interior are the *hyperbolic components*, which consist of those parameters c such that the polynomial p_c possesses an attracting periodic cycle.

Postcritically Finite Quadratic Polynomials

Of particular interest are the polynomials p_c for which the orbit of the critical point $z_0 = 0$ is finite. Such a polynomial is said to be *postcritically finite*. If p_c is postcritically finite, then the critical point is either periodic or preperiodic; that is, it eventually maps to a periodic cycle under p_c . Therefore, if p_c is postcritically finite, then $0 \in K_c$, so K_c is connected. It follows that the parameters c for which p_c is postcritically finite are all contained in M . In fact, they can be found explicitly, as demonstrated in the following example.

Example. Suppose we are interested in finding all parameters $c \in M$ for which the critical point $z_0 = 0$ is periodic of period 3. The equation defined by the *critical orbit relation* $(p_c)^3(0) = 0$ is

$$c(c^3 + 2c^2 + c + 1) = 0.$$

We are interested in the three roots of $c^3 + 2c^2 + c + 1$ (there is a degenerate solution at $c = 0$ which we ignore). One root is real, and the other two are complex conjugates. Thus, there are three quadratic polynomials p_c for which 0 is periodic of period 3. The filled Julia sets of these polynomials are drawn in Figure 2: the two at the top, and the one at the lower right.

More generally, each $c \in M$ for which 0 is periodic under p_c is an algebraic integer, as it is determined by the critical orbit relation $(p_c)^n(0) = 0$. Each of these parameters is contained in a hyperbolic component of M , and every hyperbolic component contains one and only one such parameter, called its *center*. This has been incredibly fruitful for understanding the structure of M ; indeed, using the dynamics of p_c as c runs over all centers, one can essentially catalog and organize all hyperbolic components. The arrows in Figure 2 point to four centers of hyperbolic components in M .

The parameters $c \in M$ for which 0 is not periodic itself but strictly preperiodic are also algebraic integers. Indeed, each of these parameters is determined by a critical orbit relation $(p_c)^m(0) = (p_c)^n(0)$. The strictly preperiodic parameters are dense in the boundary of M .

If c is algebraic, then the critical orbit

$$0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c)^2 + c \mapsto \dots$$

is also algebraic. Therefore, if p_c is postcritically finite, then the critical orbit is a finite algebraic subset of \mathbb{C} . One question to explore is, Which finite algebraic subsets of \mathbb{C} arise as critical orbits of p_c as $c \in M$ runs over all postcritically finite parameters?

There is no reason to restrict our attention to just postcritically finite quadratic polynomials; there is a completely analogous discussion for rational maps.

Postcritically Finite Rational Maps

Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map on the Riemann sphere of degree $d \geq 2$. By the Riemann-Hurwitz formula, f has $2d - 2$ critical points, counted with multiplicity; let C_f be the set of critical points of f . The map f is *postcritically finite* if the *postcritical set*

$$P_f := \bigcup_{n>0} f^n(C_f)$$

is finite.

In My Talk

Based on joint work with L. DeMarco and C. McMullen, we study the subsets $X \subseteq \mathbb{P}^1$ that arise as P_f for some postcritically finite rational map f and also investigate the extent to which the combinatorics of $f : P_f \rightarrow P_f$ can be specified. We employ a variety of results to explore this problem, ranging from Belyi's celebrated theorem [B] to analytic techniques used in the proof of Thurston's topological characterization of rational maps [DH], one of the most central theorems in complex dynamics.

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Credits

Figures 1 and 2 are produced with FractalStream, which is written by M. Noonan and freely available online. Photo of Sarah Koch is courtesy of Sarah Koch.

ABOUT THE AUTHOR

Sarah Koch works in complex dynamics and Teichmüller theory. She is honored to have received an NSF-CAREER award in 2015 and the University of Michigan Class of 1923 Memorial Teaching Award in 2016.



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