SPRING EASTERN SECTIONAL SAMPLER

In this sampler, the speakers below have kindly provided introductions to their Invited Addresses for the upcoming AMS Spring Eastern Sectional Meeting **May 6–7, 2017** (Saturday–Sunday) Hunter College, City University of New York, New York, NY.

Fernando Codá Marques

James Maynard

Kavita Ramanan

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Fernando Codá Marques

Weyl's Law for Minimal Hypersurfaces

The famous Weyl's law is a beautiful formula describing the asymptotic behavior of the eigenvalues of the Laplace operator. It was discovered in 1911 by Hermann Weyl for Dirichlet eigenvalues of a bounded domain in Euclidean space, but it holds true also in the more general case of eigenvalues of the Laplace–Beltrami operator of a compact Riemannian manifold. One possible formulation states that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p \leq \cdots$ is the sequence of eigenvalues of a compact Riemannian manifold M of dimension $n+1$, then

$$\lim_{p\to\infty} \lambda_p p^{-\frac{2}{n+1}} = c(n)\mathrm{vol}\,(M)^{-\frac{2}{n+1}},$$

for some universal dimensional constant $c(n) > 0$. But what does this have to do with minimal surfaces?

Minimal hypersurfaces $\Sigma^n \subset M^{n+1}$ are critical points of the area functional, while eigenfunctions (solutions of an equation $\Delta_g \varphi + \lambda\varphi = 0$) are critical points of the Rayleigh functional $E(f) = (\int_M |\nabla f|^2 dM)/\int_M f^2 dM$. Even though the two settings seem dramatically different, they share deep topological principles whose basic implication is the existence of a Weyl law in which the eigenvalues are replaced by the areas of minimal hypersurfaces. The validity of this Weyl law was proven recently in joint work of the speaker with Liokumovich and Neves and answers a conjecture of Gromov.

The Rayleigh functional satisfies the symmetry $E(f) = E(-f)$, so it descends to the projectivization of the Sobolev space $W^{1,2}(M)$. Similarly, the space of unoriented closed hypersurfaces with the appropriate topology is weakly homotopically equivalent to $\mathbb{RP}^\infty$. One can mimic the min-max characterization of the p-th eigenvalue and define ω_p to be the infimum over all families of hypersurfaces modeled on $\mathbb{RP}^p$ of the supremum of the area functional. The sequence $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_p \leq \cdots$ is called the volume spectrum of the Riemannian manifold M and amazingly obeys a Weyl law:

$$\lim_{p\to\infty} \omega_p(M) p^{-\frac{1}{n+1}} = a(n)\mathrm{vol}\,(M)^{\frac{n}{n+1}}.$$

Yau conjectured that a compact Riemannian three-manifold should contain infinitely many closed minimal surfaces. This should be true in higher dimensions as well and the volume spectrum should play an important role. In fact the $\mathbb{RP}^\infty$ structure was used in joint work of the speaker with Neves to prove the conjecture in any dimension $(n+1) \geq 3$ for manifolds of positive Ricci curvature. This work uses the min-max theory of Almgren and Pitts, which gives that each ω_p is achieved as the area of a minimal hypersurface, possibly with integer multiplicities. We are currently improving the theory further by proving index estimates inspired by finite-dimensional Morse theory. This is not an easy thing to do, because the space of cycles lacks a Hilbert space structure and so there is no Palais–Smale condition to check. Ultimately this will lead to a program that reduces the general case of Yau's conjecture to understanding the multiplicity phenomenon.

Fernando Codá Marques is professor of mathematics at Princeton University. His e-mail address is `coda@math.princeton.edu`.

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Photo of Fernando Codá Marques is courtesy of Fernando Codá Marques.

ABOUT THE AUTHOR

Fernando Codá Marques is a differential geometer. He is a member of the Brazilian Academy of Sciences and a recipient of the Oswald Veblen Prize in Geometry.

Fernando Codá Marques

James Maynard

Large Gaps between Primes

Abstract. We discuss our recent joint proof of a conjecture of Erdős on the size of gaps between primes.

How large can gaps between primes be? This is a very basic question in the study of the distribution of primes, which could have been studied by the ancient Greeks (see Figure 1) but also has some direct relevance to the modern world. Various computer programs generate prime numbers of a given approximate size X by starting at X and then sequentially testing X, $X+1$, $X+2$, etc., in turn until one finds a prime. It is quick to test an individual number to see if it is prime, but it would take a long time to find a prime if you have to test a very large number of integers. How many numbers would you need to test before you found a prime? This is exactly the problem of how large gaps between primes are!

James Maynard is a Clay Research Fellow at Magdalen College, Oxford. His e-mail address is `james.alexander.maynard@gmail.com`.

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Figure 1. Calculations performing the sieve of Eratosthenes. This is a simple way of finding prime numbers dating back to the ancient Greeks. Generalizations of such sieves play a major role in modern work on large gaps between primes.

The prime number theorem shows that the *average* gap between prime numbers of size X is approximately $\log X$. Thus *typically* one would need to test about $\log X$ numbers for primality, which is not too many.

Our current knowledge of this problem is very limited.

Conjecture (Cramér). *Amongst primes less than X, all gaps are smaller than $C(\log X)^2$ (for some absolute constant $C > 0$).*

If Cramér's conjecture is true, then we would *never* have to test more than about $(\log X)^2$ integers. This would enable one to quickly find a prime of any given size. In fact, this would give a simple *deterministic* way of generating prime numbers of any given size very quickly, something that we don't know how to do.

Our current knowledge of this problem is very limited. The best upper bound is the following.

Theorem (Baker, Harman, Pintz). *Amongst primes less than X, all gaps are smaller than $CX^{0.525}$ (for some absolute constant $C > 0$).*

Unfortunately this doesn't rule out the possibility of there being some pairs of consecutive primes very far apart. Testing $X^{0.525}$ consecutive integers when X has a hundred digits would take much longer than a lifetime.

In the other direction, there is a very easy high school method of showing that there are arbitrarily large gaps between prime numbers. For j between 2 and n, the integer $n!+j$ is a multiple of j and so cannot be prime. This gives $n-1$ consecutive composite numbers which are all of size approximately $n!$. Put another way, this constructs gaps between primes smaller than X of size at least $c \log X/\log\log X$ (for some constant $c > 0$). This is worse than the average gap from the prime number theorem, but this argument has proven easier to generalize. A series of papers in the 1930s adapted this high school method to show that there are gaps which can be arbitrarily large compared with the average gap.

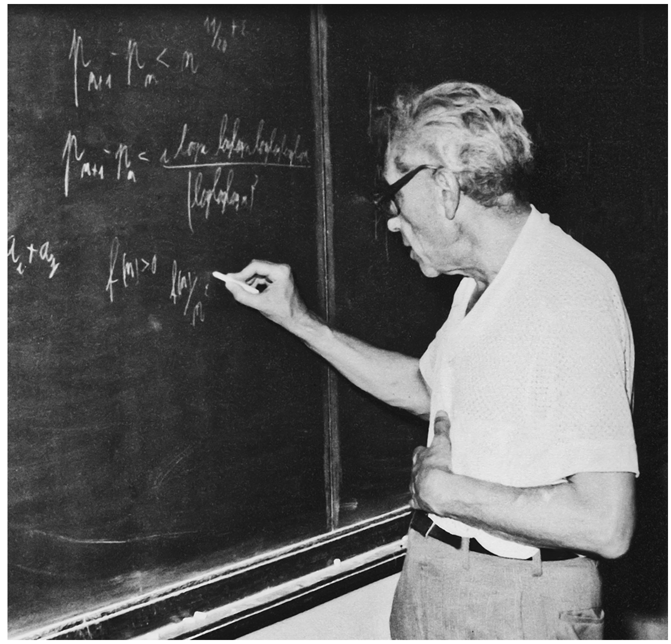
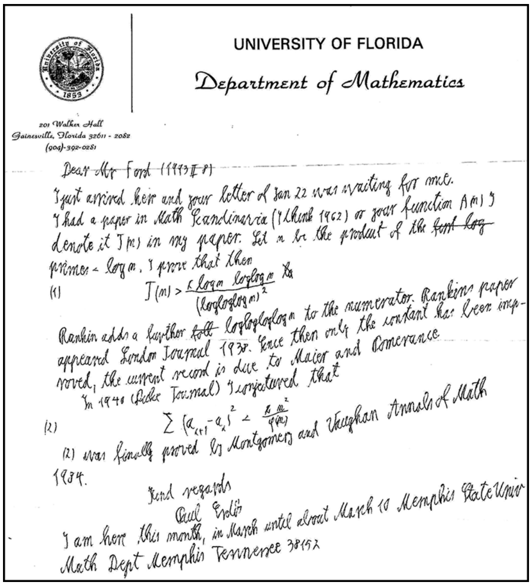


Figure 2. Paul Erdős explaining the large gaps between primes problem in a lecture in Madras, 1984.

Theorem (Erdős, Rankin, Westzynthius). *There exist consecutive primes less than X which differ by more than $c\log X \cdot \log\log X \cdot \log\log\log\log X/(\log\log\log X)^2$ (for some absolute constant $c > 0$).*

This (rather ugly) expression is only slightly larger than the average gap of size $\log X$ and well off Cramér's prediction of $(\log X)^2$, but the underlying method is the only way we currently have of showing that there are arbitrarily large gaps compared with the average size. Unfortunately, progress was slow at improving this bound; subsequent improvements over the next seventy-five years were only in the value of the constant c. Paul Erdős, who liked to offer cash prizes for math problems, offered his largest-ever cash prize for the problem of showing that the constant c above could be made arbitrarily large as $X \to \infty$ and popularized the problem by mentioning it in several lectures and letters (see Figures 2 and 3). This was because any noticeable improvement in the bound would need to use new arithmetic information about prime numbers.

In 2014 this challenge was solved independently by the author and by Ford, Green, Konyagin, and Tao using



UNIVERSITY OF FLORIDA
Department of Mathematics
201 Walker Hall
Gainesville, Florida 32611-2082
(904)-392-0281

Dear Mr Ford (1993 II 8)
I just arrived here and your letter of Jan 22 was waiting for me.
I had a paper in Math Scandinavia (I think 1962) on your function A(n) I denote it J(n) in my paper. Let n be the product of the ~~first log~~ primes < log n. I prove that then
(1) $J(n) > \frac{c \log n \log\log n}{(\log\log\log n)^2}$
Rankin adds a further ~~log~~ loglogloglog n to the numerator. Rankin's paper appeared London Journal 1938. Since then only the constant has been improved, the current record is due to Maier and Pomerance.
In 1940 (Duke Journal) I conjectured that
(2) $\sum (a_{i+1}-a_i)^2 < \frac{c\,n^2}{\varphi(n)}$
(2) was finally proved by Montgomery and Vaughan Annals of Math 1984.
Kind regards
Paul Erdős
I am here this month, in March until about March 10 Memphis State Univ
Math Dept Memphis Tennessee 38152

Figure 3. A letter from Paul Erdős to Kevin Ford when Ford was a graduate student on the large gaps problem. It includes the remark "Rankin adds a further log log log log n to the numerator." Ford later was one of the authors to solve Erdős's problem several years later.

rather different methods. Combining these approaches, the final result was

Theorem (Ford, Green, Konyagin, Maynard, Tao). *There exist consecutive primes less than X which differ by more than $c \log X \cdot \log\log X \cdot \log\log\log\log X / \log\log\log X$ (for some absolute constant $c > 0$).*

The improvement here (by a factor of $\log\log\log X$) is quantitatively quite modest, and we are still well off Cramér's conjecture. The key interest is that these approaches used stronger knowledge of the distribution of prime numbers to get these improvements of old questions. In my talk I will give an overview of this problem and recent developments, which involve a pleasing mixture of probability, combinatorics, analysis, and number theory.

Photo Credits

Figure 1 is courtesy of James Maynard.
Figure 2 is courtesy of Krishnaswami Alladi.
Figure 3 is courtesy of Kevin Ford.
Photo of James Maynard is courtesy of Eleanor Grant.}

ABOUT THE AUTHOR

James Maynard is a Clay Research Fellow at Magdalen College, Oxford. His research is in analytic number theory, particularly the distribution of prime numbers.

James Maynard

Kavita Ramanan

Random Projections of High-Dimensional Measures

The study of high-dimensional phenomena is an active research area that lies at the intersection of probability theory, statistics, and asymptotic geometric analysis. Classical theorems in probability theory concern the behavior of high-dimensional product measures. Specifically, suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables; that is, for every n, the random vector $X^{(n)} = (X_1, \ldots, X_n)$ is distributed according to the n-fold product measure $\mu^{\otimes n}$ for some Borel probability measure μ on the real line. Then a central object of study is the empirical mean $S_n = \frac{1}{n}\sum_{i=1}^n X_i$ of $X_1, \ldots, X_n$. If μ satisfies $\int_{\mathbb{R}} x^2 \mu(dx) < \infty$, assuming without loss of generality that the sample mean $\int_{\mathbb{R}} x\mu(dx)$ is zero, the celebrated central limit theorem (CLT) established at the turn of the twentieth century says that the scaled empirical mean, $\sqrt{n}S_n$, is close to a centered Gaussian distribution for large n. While the CLT describes fluctuations of S_n around the sample mean, Cramér's theorem (1938) describes the asymptotic tail behavior, or large deviations from the mean, of S_n. In particular, under an additional finite exponential moment assumption on μ, Cramér's large deviation principle (LDP) shows that the probability of S_n exceeding a value x is roughly $\exp(-nI(x))$, where the so-called rate function I that captures the exponential decay rate is nonuniversal in the sense that it depends on the distribution μ.

These questions are clearly of intrinsic interest to probabilists, [but] they are also relevant to data analysis.

Taking a geometric perspective, one can equivalently view $\sqrt{n}S_n$ as the scalar projection of the n-dimensional random vector $X^{(n)}$ along the vector $(1, 1, \ldots, 1)/\sqrt{n}$ on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Thus, the classical CLT and Cramér's LDP can be viewed as statements on the behavior of (scalar) projections of high-dimensional random vectors with a product distribution. This leads naturally to the question of whether analogous results

Kavita Ramanan is professor of applied math at Brown University. Her e-mail address is Kavita_Ramanan@brown.edu.

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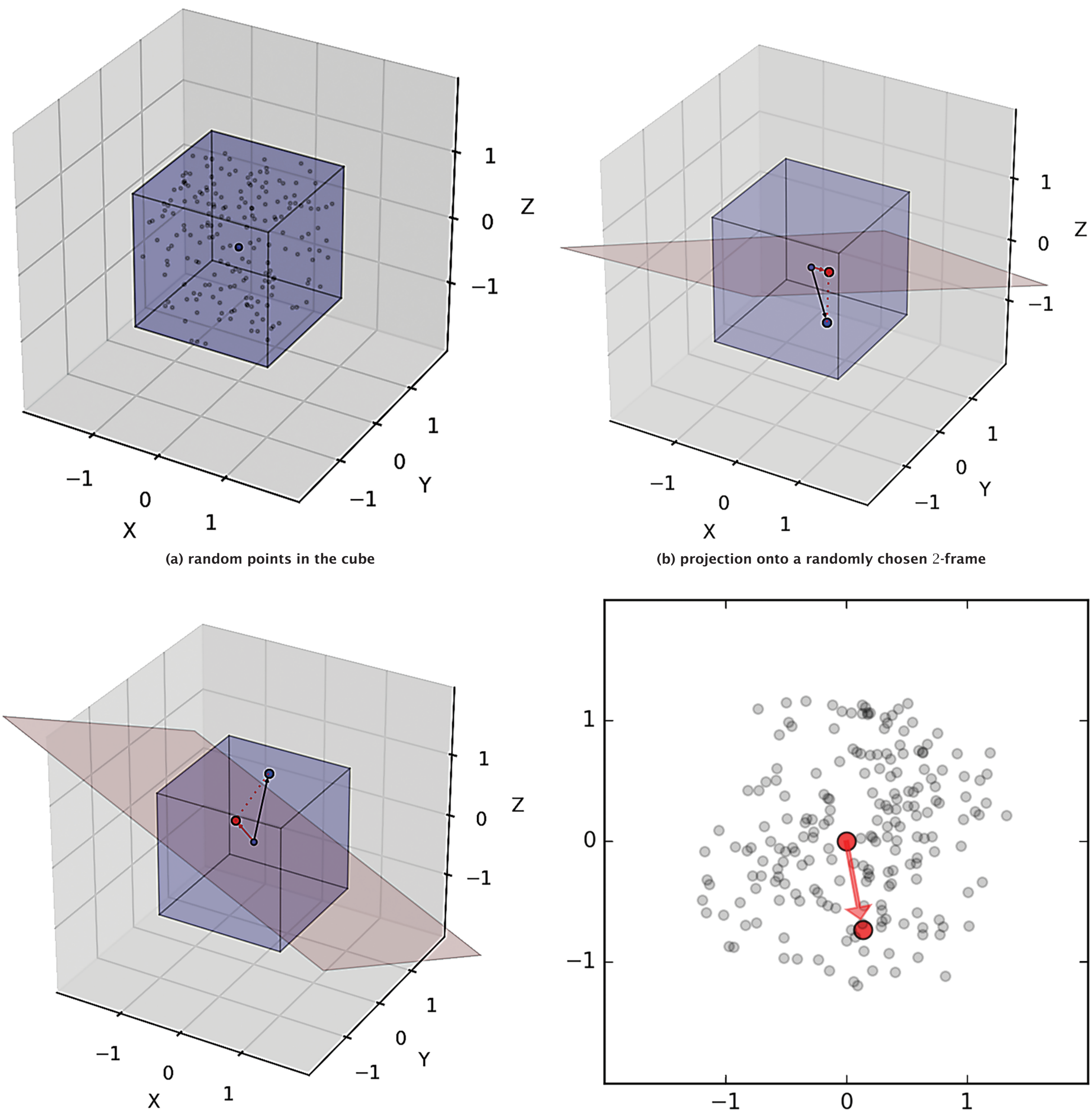

(a) random points in the cube

(b) projection onto a randomly chosen 2-frame

(c) projection onto another randomly chosen 2-frame

(d) distribution of projections onto a fixed 2-frame

Figure 1. Projections of the uniform distribution on the cube onto random 2-dimensional subspaces.

hold for (i) more general high-dimensional random vectors with a nonproduct distribution (beyond those satisfying classical weak independence conditions) and (ii) scalar projections along a random vector on the unit sphere $\mathbb{S}^{n-1}$, as well as multidimensional projections onto a random k-dimensional orthonormal basis on the Stiefel manifold. For example, Figure 1(d) depicts the distribution of coordinates of the projection of random points chosen from Lebesgue measure on the 3-dimensional cube onto a random 2-dimensional orthonormal basis.

While these questions are clearly of intrinsic interest to probabilists, they are also relevant to data analysis, where low-dimensional projections are used to study high-dimensional data and asymptotic convex geometry, where the high-dimensional distribution of interest is typically the uniform distribution on a convex set in a high-dimensional Euclidean space.

At the level of the CLT, such a line of inquiry has a long history, going back to Borel, who showed in 1906 that projections of the uniform measure on a high-dimensional sphere are approximately Gaussian. Building on subsequent work of Sudakov (1978) and Diaconis and Freedman (1984), over the last decade various generalizations have been obtained by Brehm and Voigt (2000), Attila, Ball, and Perissanski (2003), Klartag (2007), and Meckes (2012), to name a few. Essentially, these studies show that when the distribution of a high-dimensional random vector satisfies a certain geometric condition, its projections onto most k-dimensional orthonormal bases are approximately Gaussian as long as k is not too large compared to the dimension of the vector. While this is a striking universality result, it implies that fluctuations of typical low-dimensional projections do not capture much information about high-dimensional distributions.

This motivated me to look instead at the (nonuniversal) large deviation behavior of these random projections and seek corresponding geometric generalizations of Cramér's theorem, about which not much was known. In the invited address I will discuss my recent work (stemming from various collaborations with N. Gantert and graduate student S. Kim), which sheds light on the tail behavior of random projections of high-dimensional distributions. Along the way, I will describe large deviations on the Stiefel manifold and explain why the classical Cramér LDP is in a sense atypical!

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Kavita Ramanan

THE AUTHOR

Kavita Ramanan has received the IMS Medallion and the Applied Probability Society Erlang Prize and was elected Fellow of the IMS. She is founder of a math outreach group called the Math CoOp.