**What is a Knot?**

A knot $k$ is a circle smoothly embedded in the 3-dimensional sphere $\mathbb{S}^3$. Two knots $k_1, k_2$ are equivalent, and hence regarded as the same, if there is an orientation-preserving homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ such that $h(k_1) = k_2$. A basic theorem of topology assures us that such a homeomorphism is isotopic to the identity. Consequently, $k_1$ and $k_2$ are the same if we can deform $k_1$, through a sequence of intermediate knots, into $k_2$.

The trivial knot, also called the unknot, is represented by a simple closed circle in the plane. Any other knot is said to be nontrivial. A collection of pairwise disjoint knots is a link, with equivalence defined in the obvious way.

Johann Benedict Listing, a student of Gauss, and the Scottish physicist Peter Guthrie Tait independently began the first sustained investigations of the subject, in the mid-nineteenth century. Tait’s interest arose from the “vortex atom theory” of Lord Kelvin, a fanciful theory in which atoms are infinitesimal knots of frictionless, invisible æther. Classifying knots then became the main goal of knot theory. Without effective tools, it remained so until the second decade of the last century, when penetrating algebraic methods became available. Today there are so many strong invariants of knots that classification is no longer a main objective. Instead the need to understand relationships among invariants has become paramount.

A knot generically projected on a plane, without triple intersection or tangent points, can be viewed as a regular 4-valent plane graph. If we add a trompe l’œil effect at each vertex, indicating how one strand passes over another, the resulting picture is called a knot diagram. A diagram of the figure-eight knot appears in Figure 1.

K. Reidemeister in 1927 (and, independently, J. W. Alexander and G. B. Briggs in 1926) showed that two diagrams represent the same knot if and only if one can be deformed into the other by planar isotopy and a finite number of applications of three types of local changes that leave diagrams unaltered outside of the prescribed regions. The three local changes are called Reidemeister moves. See Figure 2.

Reidemeister moves enable us to investigate knots combinatorially. Any quantity assignable to a diagram is a knot invariant if and only if it is unchanged by allowed moves of the diagram. Some of the most powerful knot invariants such as the knot group and the Jones polynomial can be defined and shown to be invariant in this way.

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**Figure 1.** The figure-eight knot, also known as Listing’s knot, can be drawn with four crossings.
Virtual Knots

The combinatorial perspective inspired an entirely new direction for knot theory in 1999. In that year L. H. Kauffman proposed a more general type of knot, a virtual knot, described by a decorated 4-valent graph, as before, but with a second type of crossing, a virtual crossing, indicated by encircling the vertex. A virtual knot is an equivalence class of diagrams, two diagrams being equivalent if and only if one can be deformed into the other by planar isotopy and a finite number of applications of extended Reidemeister moves. The latter include the moves of Figure 2 as well as the additional moves in Figure 3.

Not allowed are two “forbidden moves,” passing an arc of the diagram over or under a virtual crossing, as in Figure 4.

Virtual knots and links can in fact be regarded as Gauss codes (with extra symbols encoding crossing information)
modulo a suitable equivalence relation. Again knot theory becomes combinatorial!

Invariants of classical knots that can be defined combinatorially can often be defined for virtual knots. This is true of one of the most important classical knot invariants, the knot group of $k$, denoted here by $\pi(k)$. To define it, consider any diagram for $k$. Each maximal connected component, or arc, of the diagram corresponds to a generator, and each classical crossing determines a relation. We ignore virtual crossings. When the diagram has no virtual crossings, the presentation that we get this way is the well-known Wirtinger presentation of the fundamental group $\pi_1(S^3 \setminus k)$. This is illustrated in Figure 6 for the figure-eight knot. In this case $\pi(k)$ is infinite cyclic if and only if $k$ is trivial.

![Figure 6. Assigning generators to the arcs of the diagram as shown, we obtain a presentation of the group of the figure-eight knot, $\pi(k) \cong \langle a, b, c, d \mid ac = da, ba = db, ca = bc, dc = bd \rangle$.](image)

The group of a classical knot is a strong invariant. However, there are nontrivial virtual knots with infinite cyclic groups. One such knot, commonly called Kishino’s knot, appears in Figure 7. Assigning generators to the arcs of the diagram as shown, we obtain a presentation of the group of the figure-eight knot, $\pi(k) \cong \langle a, b, c, d \mid ac = da, ba = db, ca = bc, dc = bd \rangle$.

The main idea is due to N. Kamada, expanded upon by J. S. Carter, S. Kamada, and M. Saito. Neither the genus of the surface nor the embedding is, in general, unique. However, G. Kuperberg showed that when the knot or link is represented by an embedding in a surface of smallest possible genus, then the embedding is unique up to isotopy. Of course, classical knots and links are represented uniquely in the thickened sphere. From this perspective, virtual knot theory might seem a bit less mysterious.

Mathematicians have played with knots for a relatively short period of time. It is possible that one day we will understand that these tangled ropes represent deep and important relations, a vision that so far has eluded us. Relaxing our axioms, as virtual knot theory demands, might just bring that day closer. In the meantime, we will play and enjoy!

### Why Virtual Knots?

As no evidence of Kelvin’s æther was found, the vortex atom theory dissipated, allowing knot theory to step out from the fog. Henri Poincaré saw knot theory as an important paradigm of the codimension-2 placement problem, understanding how a manifold can embed in another manifold with two extra dimensions. The status of the subject climbed even higher in the early 1960s, when W. B. R. Lickorish and A. H. Wallace proved that every closed, orientable connected 3-manifold can be obtained from a link in the 3-sphere by a simple procedure called “spherical modification” or “surgery.”

What then is the significance of virtual knots? Rather than living in the 3-sphere, virtual knots or links can be regarded as simple closed curves embedded in thickened surfaces $S \times I$ modulo a suitable equivalence relation. From this point of view the classical crossings arise from projecting onto $S$ while the virtual crossings come from projecting $S$ onto the plane.

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