WHAT IS...

a Virtual Knot?

Daniel S. Silver and Susan G. Williams

Communicated by Cesar E. Silva

Virtual knots generalize classical knots. They were introduced by L. H. Kauffman in 1999.[1] In order to describe virtual knots we must back up a bit.

What is a Knot?

A *knot* k is a circle smoothly embedded in the 3-dimensional sphere $\mathbb{S}^3$. Two knots k_1, k_2 are equivalent, and hence regarded as the same, if there is an orientation-preserving homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ such that $h(k_1) = k_2$. A basic theorem of topology assures us that such a homeomorphism is isotopic to the identity. Consequently, k_1 and k_2 are the same if we can deform k_1, through a sequence of intermediate knots, into k_2.

The *trivial knot*, also called the *unknot*, is represented by a simple closed circle in the plane. Any other knot is said to be *nontrivial*. A collection of pairwise disjoint knots is a *link*, with equivalence defined in the obvious way.

Johann Benedict Listing, a student of Gauss, and the Scottish physicist Peter Guthrie Tait independently began the first sustained investigations of the subject, in the mid-nineteenth century. Tait's interest arose from the "vortex atom theory" of Lord Kelvin, a fanciful theory in which atoms are infinitesimal knots of frictionless, invisible æther. Classifying knots then became the main goal of knot theory. Without effective tools, it remained so until the second decade of the last century, when penetrating algebraic methods became available. Today there are so many strong invariants of knots that classification is no longer a main objective. Instead the need to understand relationships among invariants has become paramount.

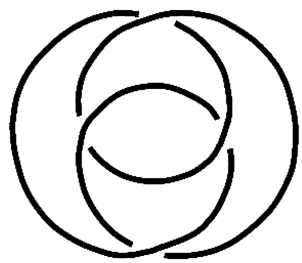

Figure 1. The figure-eight knot, also known as Listing's knot, can be drawn with four crossings.

A knot generically projected on a plane, without triple intersection or tangent points, can be viewed as a regular 4-valent plane graph. If we add a *trompe l'oeil* effect at each vertex, indicating how one strand passes over another, the resulting picture is called a *knot diagram*. A diagram of the figure-eight knot appears in Figure 1.

K. Reidemeister in 1927 (and, independently, J. W. Alexander and G. B. Briggs in 1926) showed that two diagrams represent the same knot if and only if one can be deformed into the other by planar isotopy and a finite number of applications of three types of local changes that leave diagrams unaltered outside of the prescribed regions. The three local changes are called *Reidemeister moves*. See Figure 2.

Reidemeister moves enable us to investigate knots combinatorially. Any quantity assignable to a diagram is a knot invariant if and only if it is unchanged by allowed moves of the diagram. Some of the most powerful knot invariants such as the knot group and the Jones polynomial can be defined and shown to be invariant in this way.

Daniel S. Silver is emeritus professor of mathematics at the University of South Alabama. His e-mail address is silver@southalabama.edu.

Susan G. Williams is emeritus professor of mathematics at the University of South Alabama. Her e-mail address is swilliam@southalabama.edu.

The authors are grateful for the support of the Simons Foundation.

[1] *An equivalent theory of abstract knot diagrams was formulated independently by N. Kamada.*

DOI: http://dx.doi.org/10.1090/noti1520

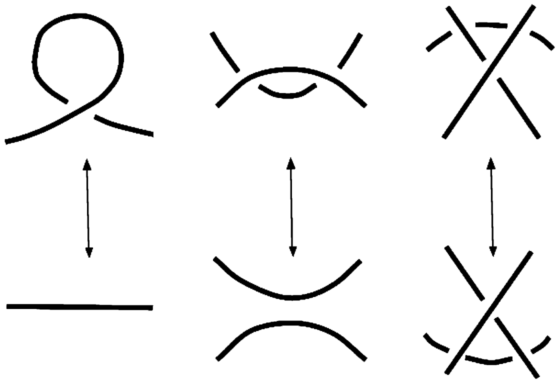

Figure 2. Pictured here are Reidemeister moves on the knot diagrams, which leave the knots themselves unchanged.

Virtual Knots

The combinatorial perspective inspired an entirely new direction for knot theory in 1999. In that year L. H. Kauffman proposed a more general type of knot, a *virtual knot*, described by a decorated 4-valent graph, as before, but with a second type of crossing, a *virtual crossing*, indicated by encircling the vertex. A *virtual knot* is an equivalence class of diagrams, two diagrams being equivalent if and only if one can be deformed into the other by planar isotopy and a finite number of applications of *extended Reidemeister moves*. The latter include the moves of Figure 2 as well as the additional moves in Figure 3.

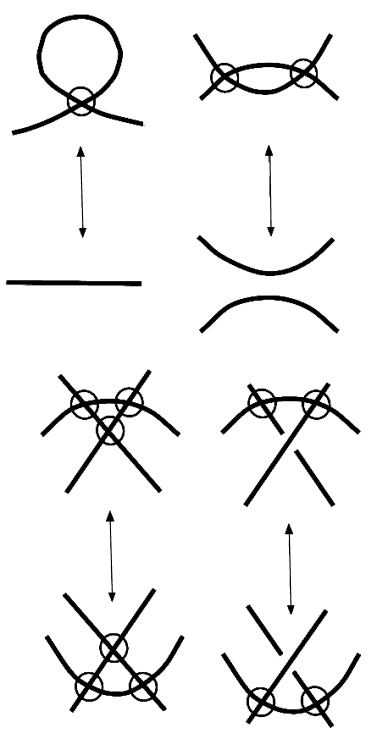

Figure 3. Virtual knots admit additional Reidemeister moves, shown here.

Not allowed are two "forbidden moves," passing an arc of the diagram over or under a virtual crossing, as in Figure 4.

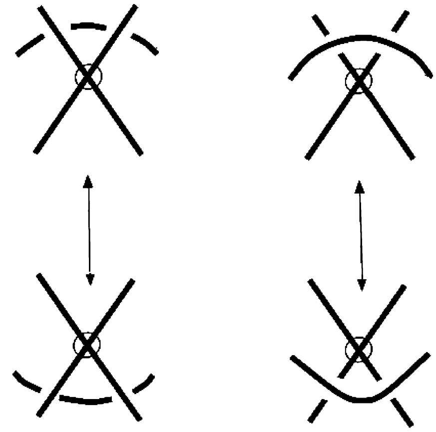

Figure 4. These "forbidden moves" do not preserve virtual knots.

In the new parlance, a *classical knot* is simply a knot that can be represented by a diagram without virtual crossings. (We complete the lexicon by calling a crossing that is not virtual a *classical crossing*.) Virtual crossings can always be introduced into a diagram and then later removed using extended Reidemeister moves. However, a theorem of M. Gourssarov, M. Polyak, and O. Viro assures us that if two classical knots are equivalent as virtual knots, then they are the same classical knots. Happily, virtual knot theory is an extension of the classical theory.

Kauffman's motivation for defining virtual knots came from Gauss's idea for encoding classical knots. A Gauss code for a classical knot diagram is obtained in the following way. First, number the crossings, say $1, \dots, m$. Then beginning at crossing 1, and moving along the diagram in a direction, record the crossings as they are encountered until arriving back at the starting point. In this way each number is recorded twice. A simple example appears in Figure 5.

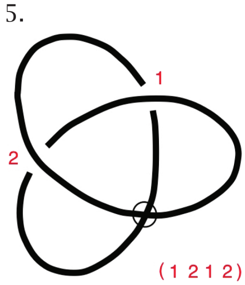

Figure 5. The virtual knot shown here realizes the Gauss code (1212)**.**

Kauffman's inspiration came from the observation that if we wish to achieve an arbitrary permutation of $1, 1, \dots, m, m$, then it is necessary, but also sufficient, that we introduce some virtual, unlabeled crossings into our diagram.

Virtual knots and links can in fact be regarded as Gauss codes (with extra symbols encoding crossing information) modulo a suitable equivalence relation. Again knot theory becomes combinatorial!

Invariants of classical knots that can be defined combinatorially can often be defined for virtual knots. This is true of one of the most important classical knot invariants, the *knot group* of k, denoted here by $\pi(k)$. To define it, consider any diagram for k. Each maximal connected component, or *arc*, of the diagram corresponds to a generator, and each classical crossing determines a relation. We ignore virtual crossings. When the diagram has no virtual crossings, the presentation that we get this way is the well-known *Wirtinger presentation* of the fundamental group $\pi_1(\mathbb{S}^3 \backslash k)$. This is illustrated in Figure 6 for the figure-eight knot. In this case $\pi(k)$ is infinite cyclic if and only if k is trivial.

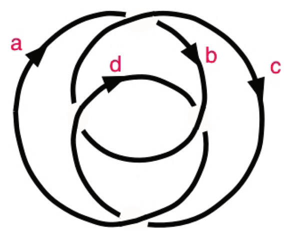

Figure 6. Assigning generators to the arcs of the diagram as shown, we obtain a presentation of the group of the figure-eight knot, $\pi(k) \cong \langle a, b, c, d \mid ac = da, ba = db, ca = bc, dc = bd \rangle$.

The group of a classical knot is a strong invariant. However, there are nontrivial virtual knots with infinite cyclic groups. One such knot, commonly called *Kishino's knot*, appears in Figure 7. Applying the second forbidden move of Figure 4 to a diagram obviously does not affect the group. Since using the move along with the allowed virtual knot moves enables us to turn Kishino's knot into the unknot, the reader can check easily that the group of the knot is infinite cyclic. Fortunately, other invariants can show that Kishino's knot is nontrivial.

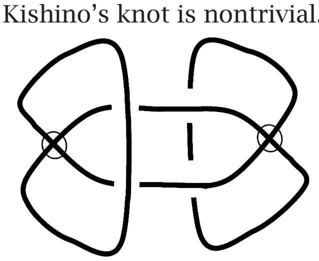

Figure 7. Kishino's knot, depicted here, is a nontrivial virtual knot with the same knot group as the trivial knot.

Pick up a diagram for a virtual knot k and turn it over. (For this it might be helpful to imagine the diagram made of rigid material.) We obtain a diagram of another virtual knot k^*. If k is classical, then k and k^* are the same. However, in general, k^* can be different. In fact their groups can be non-isomorphic. While virtual knot groups have been characterized, both algebraically and topologically, no one has yet characterized the possible pairs $(\pi(k), \pi(k^*))$.

The Jones polynomial is a powerful invariant of classical knots. Discovered by V. Jones in 1984, it sparked new interest in combinatorial knot-theoretic methods. It is an open question whether a nontrivial classical knot can have Jones polynomial equal to 1.

Using Kauffman's bracket polynomial formulation, the Jones polynomial can be defined for virtual knots. Kauffman discovered a method for constructing nontrivial virtual knots with Jones polynomial equal to 1. Could such a knot be shown to be classical, thereby answering the open question?

Why Virtual Knots?

As no evidence of Kelvin's æther was found, the vortex atom theory dissipated, allowing knot theory to step out from the fog. Henri Poincaré saw knot theory as an important paradigm of the codimension-2 placement problem, understanding how a manifold can embed in another manifold with two extra dimensions. The status of the subject climbed even higher in the early 1960s, when W. B. R. Lickorish and A. H. Wallace proved that every closed, orientable connected 3-manifold can be obtained from a link in the 3-sphere by a simple procedure called "spherical modification" or "surgery."

What then is the significance of *virtual* knots? Rather than living in the 3-sphere, virtual knots or links can be regarded as simple closed curves embedded in thickened surfaces $S \times I$ modulo a suitable equivalence relation. From this point of view the classical crossings arise from projecting onto S while the virtual crossings come from projecting S onto the plane.

The main idea is due to N. Kamada, expanded upon by J. S. Carter, S. Kamada, and M. Saito. Neither the genus of the surface nor the embedding is, in general, unique. However, G. Kuperberg showed that when the knot or link is represented by an embedding in a surface of smallest possible genus, then the embedding is unique up to isotopy. Of course, classical knots and links are represented uniquely in the thickened sphere. From this perspective, virtual knot theory might seem a bit less mysterious.

Mathematicians have played with knots for a relatively short period of time. It is possible that one day we will understand that these tangled ropes represent deep and important relations, a vision that so far has eluded us. Relaxing our axioms, as virtual knot theory demands, might just bring that day closer. In the meantime, we will play and enjoy!

Further Reading

1. L. H. KAUFFMAN, Introduction to virtual knot theory, *J. Knot Theory Ramifications* **21** (2012), no. 13, 1240007, 37 pp.
2. V. O. MANTUROV, Virtual Knots: the State of the Art, *Series on Knots and Everything: Volume 51*, World Scientific, Singapore, 2012.

Photo Credits

Photo of Daniel S. Silver is courtesy of Susan G. Williams.
Photo of Susan G. Williams is courtesy of Daniel S. Silver.

ABOUT THE AUTHORS

Daniel S. Silver

Daniel S. Silver's research explores relationships between knot theory and dynamical systems. He also publishes articles on the history of mathematics. No animals were harmed in the preparation of this article.

Susan G. Williams

Susan G. Williams has a background in ergodic theory and dynamical systems. Her work includes applications of symbolic and algebraic dynamics to knot theory, and lately to graph theory. She has an active interest in origami; when she's not writing papers, she's folding them.