

Dynamical Algebraic Combinatorics: Promotion, Rowmotion, and Resonance

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Abstract. Dynamical algebraic combinatorics studies actions on objects important in algebraic combinatorics, with particular focus on actions with surprisingly nice properties. We discuss recent research in this emerging area, including actions on order ideals and tableaux that exhibit the *cyclic sieving*, *homomesy*, and *resonance* phenomena.

Dynamical Algebraic Combinatorics

Mathematical inquiry often begins with the study of *objects* (numbers, shapes, variables, matrices, ideals, metric spaces, ...) and the question, “What are the objects like?” It then moves to the study of *actions* (functions, rotations, reflections, multiplication, derivatives, shifts, ...) and the question, “How do the objects behave?” The study of actions in various mathematical contexts has been extremely fruitful; consider the study of metric spaces through the lens of dynamical systems or the study of symmetries arising from group actions. For objects and actions arising from algebraic combinatorics, we call this study *dynamical algebraic combinatorics*.

Let g be a bijective action on a finite set X . Such an action breaks the space X into *orbits*. Often, the study of the orbits of g provides insight into the structure of the objects in X , revealing hidden symmetries and connections. One typically first seeks to understand the *order* n of the action and then finds interesting properties the action exhibits. One surprisingly ubiquitous property

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is the *cyclic sieving phenomenon* [ReStWh04], which occurs when the evaluation of a *generating function* for X at the primitive n th root of unity ζ^d (where $\zeta = e^{2\pi i/n}$) counts the number of elements of X fixed under g^d .

For example, let X be the set of binary words composed of two zeros and two ones. Let g be the cyclic shift that acts by moving the first digit to the end of the word. Each word has four digits, so g is of order four. Consider the *inversion number statistic* $\text{inv}(w)$ of $w \in X$, which equals the number of pairs (i, j) with $i < j$ such that the i th digit of w is 1 and the j th digit is 0. See Figure 1 for the orbits of X under g and the inversion numbers of each binary word.

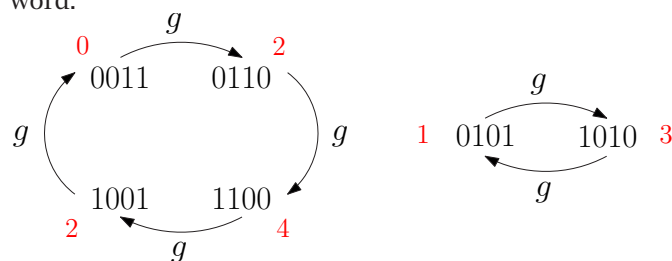


Figure 1. The orbits of binary words of length four with two ones under a cyclic shift; the inversion numbers corresponding to the binary words are shown in red.

The inversion number statistic determines the generating function

$$(1) \quad X(q) = \sum_{w \in X} q^{\text{inv}(w)} = 1 + q + 2q^2 + q^3 + q^4.$$

Since g is of order four, $\zeta = e^{2\pi i/4} = i$. One may check using Figure 1 and (1) that $X(i^d)$ equals the number of elements of X fixed by g^d , thus this is an instance of the cyclic sieving phenomenon. For example, $X(i^1) = 1 + i + 2i^2 + i^3 + i^4 = 1 + i + 2(-1) + (-i) + 1 = 0$, and zero

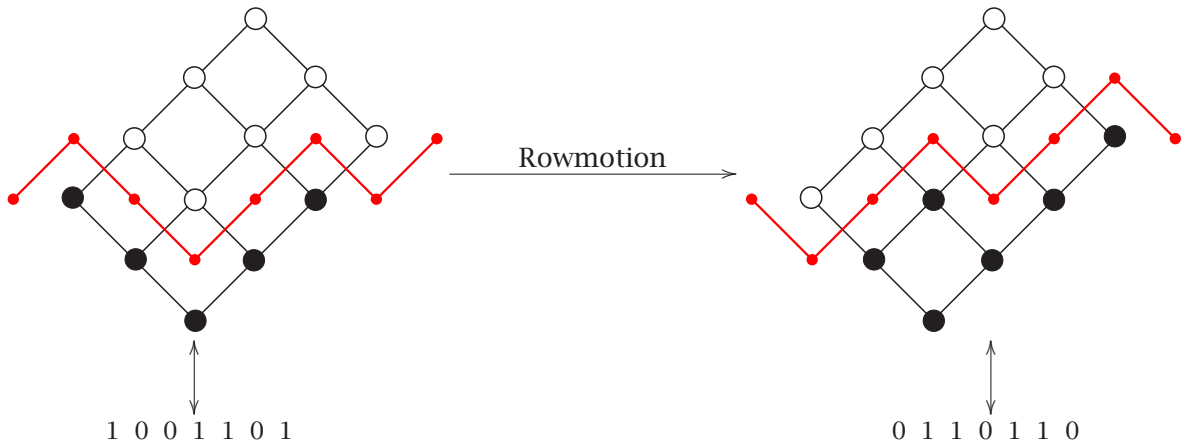


Figure 2. An order ideal in the product of chains poset 3×4 and its image under rowmotion. The red paths are *boundary paths* that divide the order ideal from the rest of the poset; the binary words corresponding to these paths are shown beneath.

elements of X are fixed under g^1 . $X(i^2) = X(-1) = 2$, and two elements are fixed under g^2 .

Another interesting property actions in dynamical algebraic combinatorics often exhibit is the *homomesy phenomenon* [PrRo15], in which the average value of a statistic on every g -orbit equals the global average value of that statistic. Each orbit in Figure 1 has an average inversion number of 2. Thus, the inversion number statistic on X is *homomesic* with respect to g . Note that both cyclic sieving and homomesy still hold in the more general case in which X is the set of binary words of length n with k ones [ReStWh04], [PrRo15].

In Figure 1, our action was visibly cyclic, so its order was clear from the definition. In the coming sections, we discuss more complicated combinatorial actions whose orders are difficult to predict. We will see examples in which an action g on a large combinatorial set X has a relatively small order; this indicates the elements of X have hidden cyclic symmetry such that g is a rotation in disguise. We will also see examples in which g is not of small order, but rather exhibits *resonance*, meaning that g maps to an underlying cyclic action with small order. We discuss examples of such actions on *order ideals* and *tableaux* and then give a surprising relation between them, which we found via this study of dynamics.

Rowmotion on Order Ideals

As our first example, let our set X be the *order ideals* of a poset.

Definition 1. A poset is a partially ordered set. The set of *order ideals* $J(P)$ of a poset P is the set of all subsets $I \subseteq P$ such that if $y \in I$ and $z \leq y$, then $z \in I$.

Specifically, we will take X to be the set of order ideals $J(P)$ for P the product of chains poset $\mathbf{a} \times \mathbf{b}$. That is, for a natural number a , $\mathbf{a} = \{1, 2, \dots, a\}$ and the product partial order on $\mathbf{a} \times \mathbf{b}$ is $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. See Figure 2 for an example. An easy

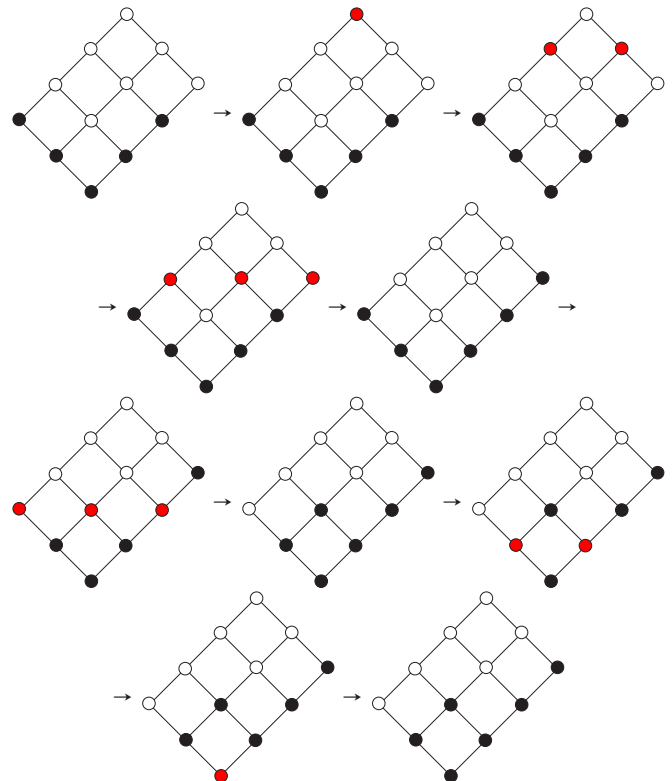


Figure 3. Rowmotion from Figure 2 computed by toggling rows from top to bottom. The elements that are being toggled at each step are shown in red, and at any step in which the toggles act nontrivially, the order ideal resulting from those toggles is shown.

counting argument shows the number of order ideals in $J(\mathbf{a} \times \mathbf{b})$ is given by the binomial coefficient $\binom{a+b}{a}$, since these order ideals are in bijection with binary words with a zeros and b ones via the *boundary path* that separates

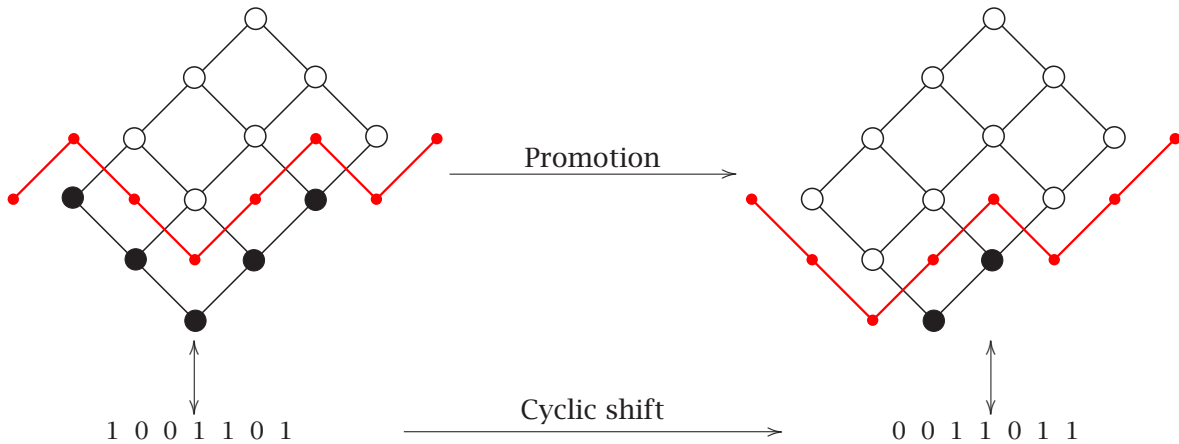


Figure 4. An order ideal in $J(3 \times 4)$ and its image under promotion. Promotion acts as a cyclic shift on the binary word corresponding to the boundary path.

the order ideal from the rest of the poset (where a one indicates an up-step and a zero indicates a down-step); see Figure 2.

Our action g will be the following; see Figure 2.

Definition 2. Let P be a poset, and let $I \in J(P)$. Then **rowmotion** on I is the order ideal generated by the minimal elements of $P \setminus I$.

The order of rowmotion on general posets is neither well behaved nor predictable. But on $J(\mathbf{a} \times \mathbf{b})$ its order is, surprisingly, much smaller than $\binom{a+b}{a}$.

Theorem 3 ([BrSc74]). *The order of rowmotion on $J(\mathbf{a} \times \mathbf{b})$ is $a + b$.*

To see why the order of a global action such as rowmotion is surprisingly small, it often helps to interpret the action as a composition of local involutions and work within the group generated by those involutions. In the case of rowmotion, we call this group the *toggle group*.

Definition 4. For each element $e \in P$ define its **toggle** $t_e : J(P) \rightarrow J(P)$ as follows.

$$t_e(I) = \begin{cases} I \cup \{e\} & \text{if } e \notin I \text{ and } I \cup \{e\} \in J(P) \\ I \setminus \{e\} & \text{if } e \in I \text{ and } I \setminus \{e\} \in J(P) \\ I & \text{otherwise} \end{cases}$$

The **toggle group** $T(J(P))$ is the subgroup of the symmetric group $S_{J(P)}$ generated by $\{t_e\}_{e \in P}$.

Theorem 5 ([CaFo95]). *Given any poset P , rowmotion is the toggle group element that toggles the elements of P from top to bottom (in the reverse order of any linear extension).*

For example, in Figure 3, we recompute rowmotion on the order ideal from Figure 2 by toggling from top to bottom by rows.

In joint work with Nathan Williams, we found a toggle group action conjugate to rowmotion, which we called *promotion*, defined for $J(\mathbf{a} \times \mathbf{b})$ as toggling the elements from left to right (we will soon make this more precise). We

showed rowmotion and promotion are conjugate actions in the toggle group of *any* ranked poset; this implies they are *equivariant*, or have the same orbit structure.

Theorem 6 ([StWi12]). *In any ranked poset, there is an equivariant bijection between the order ideals under rowmotion (toggle top to bottom by rows) and promotion (toggle left to right by columns).*

By the bijection to binary words, promotion on $J(\mathbf{a} \times \mathbf{b})$ is a cyclic shift of a binary word of length $a + b$; each toggle swaps adjacent letters, resulting in the first letter swapping all the way through the word to the end; see Figure 4. Therefore, by the above theorem, the cyclic nature of promotion on $J(\mathbf{a} \times \mathbf{b})$ gives a satisfying explanation for the order of rowmotion on $J(\mathbf{a} \times \mathbf{b})$. Furthermore, since binary words under a cyclic shift exhibit the cyclic sieving phenomenon (as discussed in Section 1), so does rowmotion on $J(\mathbf{a} \times \mathbf{b})$.

Corollary 7 ([StWi12]). *There is an equivariant bijection between the order ideals of $\mathbf{a} \times \mathbf{b}$ under rowmotion and binary words of length $a + b$ with b ones under rotation. The cyclic sieving phenomenon follows.*

Rowmotion on these order ideals also exhibits the homomesy phenomenon.

Theorem 8 ([PrRo15]). *The order ideal cardinality statistic in $J(\mathbf{a} \times \mathbf{b})$ exhibits homomesy (orbit-average = global-average) with respect to rowmotion or promotion.*

It is natural to ask whether similar results hold on posets constructed as products of more than two chains.

Theorem 9 ([CaFo95]). *The order of rowmotion on $J(\mathbf{a} \times \mathbf{b} \times 2)$ is $a + b + 1$.*

Again, as a corollary of Theorem 6, we used the toggle group to explain this result by showing promotion on $J(\mathbf{a} \times \mathbf{b} \times 2)$ is a cyclic action on combinatorial objects in bijection with these order ideals, which also exhibit the cyclic sieving phenomenon; see Figure 5.

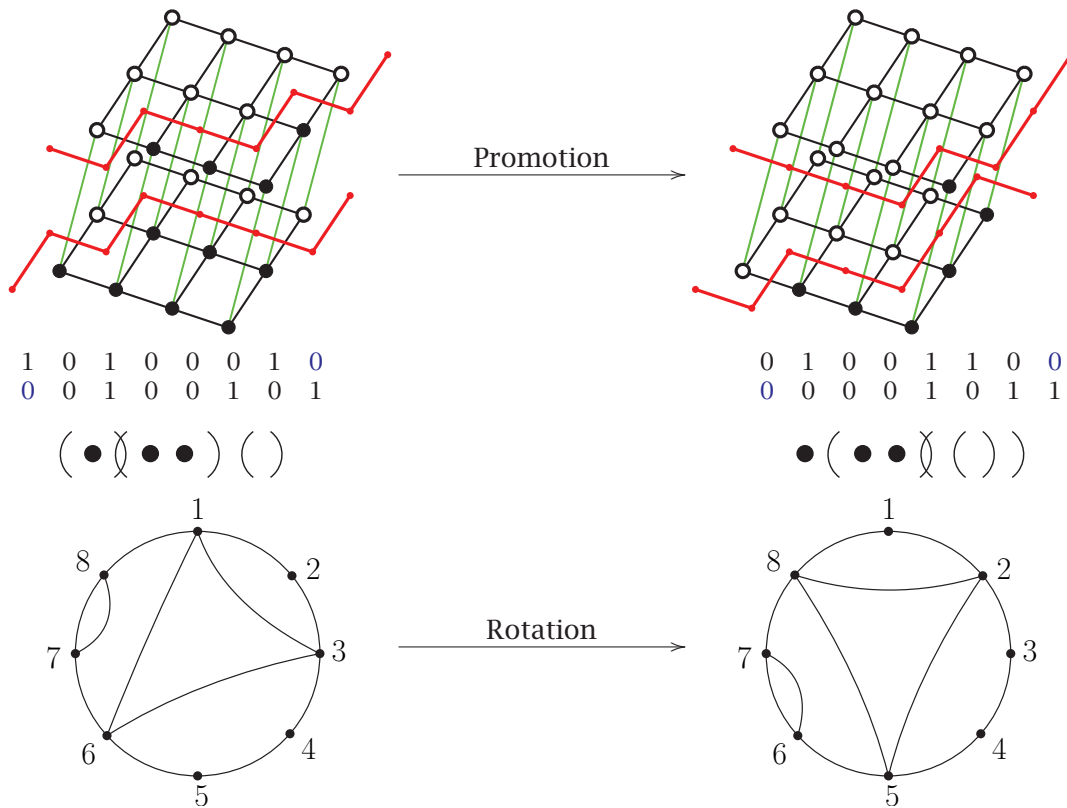


Figure 5. An order ideal in $J(4 \times 3 \times 2)$ and its image under promotion, along with the boundary paths for each layer and the corresponding *boundary path matrices*, which transform via the parenthesizations shown to the given noncrossing partitions. Promotion on $J(a \times b \times 2)$ rotates the corresponding noncrossing partition.

Theorem 10 ([StWi12]). *There is an equivariant bijection between $J(a \times b \times 2)$ under rowmotion and noncrossing partitions of $a + b + 1$ into $b + 1$ blocks under rotation. The cyclic sieving phenomenon follows.*

Homomesy also holds in this case.

Theorem 11 ([Vo17]). *The order ideal cardinality statistic in $J(a \times b \times 2)$ exhibits homomesy with respect to rowmotion or promotion.*

The above theorems may lead one to guess that rowmotion on $J(a \times b \times c)$ is of order $a + b + c - 1$ and exhibits the cyclic sieving and homomesy phenomena. This is not true in general. For example, when $a = b = c = 3$, the order of rowmotion is 8, but the generating function no longer exhibits the cyclic sieving phenomenon. For $a = b = 3, c = 4$, the cardinality statistic is no longer homomesic. When $a = b = c = 4$, the order of rowmotion is not $a + b + c - 1 = 11$, but rather 33. Similar computations for many other values of a, b, c show that the orbits of rowmotion on $J(a \times b \times c)$ are each of cardinality a *multiple of a divisor of $a + b + c - 1$* . This is an admittedly fuzzy notion, which we will make more precise in the last section of this article. But first, we move to our second example of an action on *standard Young tableaux*.

Promotion on Standard Young Tableaux

As our second example, let our set X be composed of the following well-loved objects in algebraic combinatorics and representation theory.

Definition 12. A **standard Young tableau** of partition shape λ is a bijective filling of λ with the numbers $\{1, 2, \dots, n\}$, where n is the number of boxes in λ , such that labels strictly increase from left to right across rows and from top to bottom down columns. Let $SYT(\lambda)$ denote the set of standard Young tableaux of shape λ .

See Figures 6 and 7 for examples of tableaux in $SYT(\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}) = SYT(2 \times 3)$. $SYT(\lambda)$ is enumerated by the famous Frame–Robinson–Thrall hook formula: $\prod_{x \in \lambda} \frac{n!}{h(x)}$, where x ranges over all boxes in the diagram of λ . $h(x)$ is the *hook number* of x , that is, the number of boxes in λ in the same row as x and to its right, plus the number of boxes in the same column as x and below, plus one (for x itself).

Our action g will be M.-P. Schützenberger’s *promotion* operator (see Figure 6). In general, this is a different action from the toggle-promotion defined in the previous section. A relation between these actions will be made clear later in this section and the next.

Definition 13. Given a partition λ , **promotion** on $T \in SYT(\lambda)$ is the product $\rho_{n-1} \cdots \rho_2 \rho_1(T)$ of the

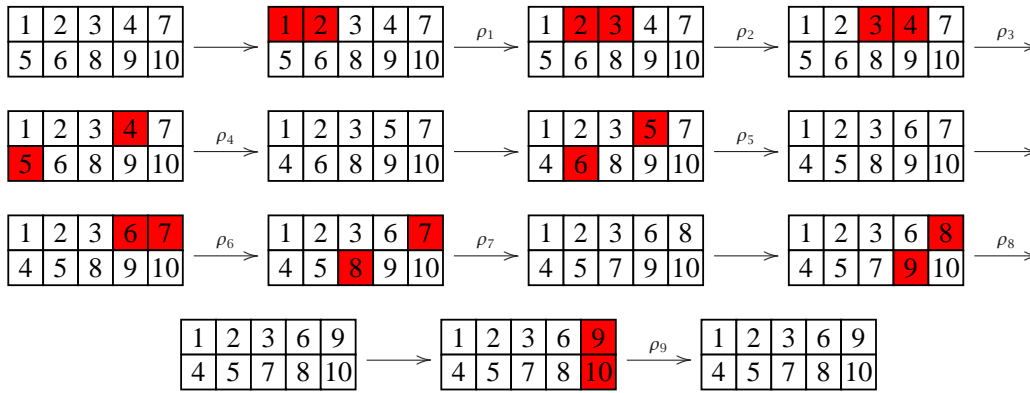


Figure 6. Promotion computed as the product $\rho_9 \cdots \rho_2 \rho_1(T)$ of Bender–Knuth involutions. The entries that are being acted on at each step are shown in red, and at any step in which the involution acts nontrivially, the resulting tableau is shown.

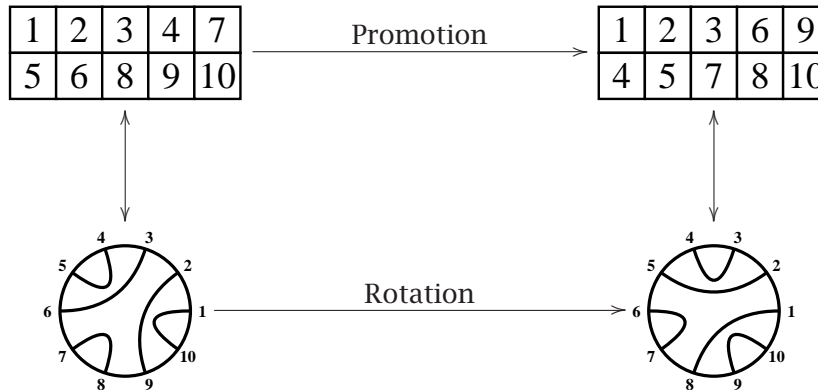


Figure 7. Promotion on $SYT(2 \times b)$ is in equivariant bijection with rotation on noncrossing matchings of $2b$ points. Each top row tableau entry is the smaller number in its matching pair.

Bender–Knuth involutions ρ_i , where each ρ_i swaps i and $i + 1$ if possible.

Promotion on standard Young tableaux in an $a \times b$ rectangle has the unexpected property that all the orbit sizes are *divisors* of the maximum value ab .

Theorem 14 ([Hai92]). *Promotion on $SYT(a \times b)$ is of order ab .*

For example, by the hook formula, $|SYT(5 \times 7)| = \frac{35!}{1^1 2^2 3^3 4^4 5^5 6^5 7^8 9^3 10^2 11^1} = 278,607,172,289,160$, but promotion is of order $7 \cdot 5 = 35$. Rectangles are among a few special shapes for which promotion is of a nice order. Promotion on tableaux of general shape does not have a nice order; for example, promotion on the partition $\begin{matrix} \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square \end{matrix}$ has order $7,554,844,752$.

Even more surprisingly, $SYT(a \times b)$ under promotion exhibits the cyclic sieving phenomenon, which means the evaluation of the q -analogue of the hook length formula for $SYT(a \times b)$ at the (ab) th root of unity $(e^{2\pi i/ab})^d$ counts the number of $T \in SYT(a \times b)$ such that $\text{Promotion}^d(T) = T$.

Theorem 15 ([Rh10]). *Promotion on $SYT(a \times b)$ exhibits the cyclic sieving phenomenon.*

Proofs of this result illuminated connections to representation theory and geometry. In the cases $a = 2, 3$, there are proofs that proceed by giving a bijection to other combinatorial objects (*noncrossing matchings* and *webs*, respectively) that sends promotion to *rotation* [PePyRh09]; see Figure 7. There are also homomesy results in this and more general settings; see [BLPeSa16].

Promotion on two-row rectangular tableaux $SYT(2 \times b)$ is equivalent to toggle-promotion on the *type A positive root poset* [StWi12], which is why we gave the name promotion to the action of toggling from left to right. A few years later, this name was further validated by the result that *hyperplane-toggle promotion* on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ is equivalent to *K-theoretic promotion* on *increasing tableaux* [DiPeSt17]; we explain this in the next section.

Resonance on Orbits of Increasing Tableaux and Plane Partitions

For our final example, we take our set X to be *increasing tableaux*, objects that have appeared in various contexts

within algebraic combinatorics, in particular, in relation to K -theoretic Schubert calculus of Grassmannians.

Definition 16. An **increasing tableau** of partition shape λ is a filling of λ with positive integers such that labels strictly increase from left to right across rows and from top to bottom down columns. Let $\text{Inc}^q(\lambda)$ denote the set of all increasing tableaux of shape λ with entries at most q .

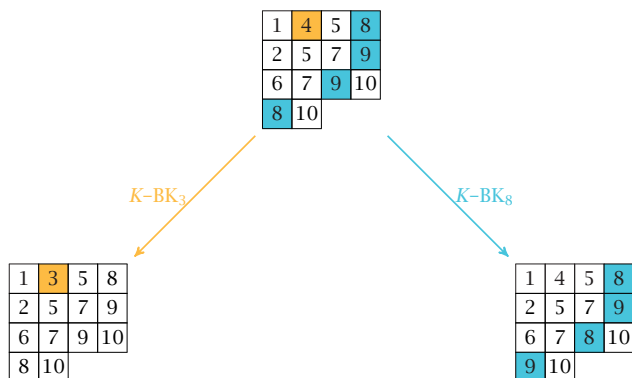


Figure 8. The action of some K -Bender-Knuth involutions on an increasing tableau in $\text{Inc}^{10}(4, 4, 4, 2)$.

See Figures 8 and 9 for examples. Note that $\text{Inc}^n(\lambda)$ (where n is the number of boxes in λ) contains $\text{SYT}(\lambda)$ as a subset; increasing tableaux differ from standard Young tableaux in that there may be repeated and/or missing numbers in the filling.

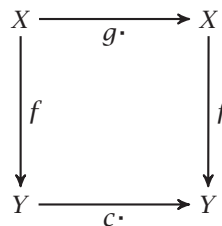
Our action g will be K -promotion (see Figure 8), which was originally defined globally using K -jeu-de-taquin; we give here our equivalent definition from [DiPeSt17]. Note that if an increasing tableau is a standard Young tableau, K -Bender-Knuth involutions are equivalent to Bender-Knuth involutions.

Definition 17. Given a partition λ and a natural number q , **K -promotion** on $T \in \text{Inc}^q(\lambda)$ is the product $K\text{-BK}_{q-1} \cdots K\text{-BK}_2 K\text{-BK}_1(T)$ of the K -Bender-Knuth involutions $K\text{-BK}_i$, where each $K\text{-BK}_i$ increments i and/or decrements $i + 1$ wherever possible.

For $\text{Inc}^q(a \times b)$ under K -promotion, it is no longer true that orbit sizes are divisors of the maximum value q , but rather, they are multiples of divisors of q . For example, $\text{Inc}^{11}(4 \times 4)$ has orbits of size 11 and 33. With Kevin Dilks and Oliver Pechenik, we made this observation more precise and gave this phenomenon the name *resonance*.

Definition 18 ([DiPeSt17]). Suppose g is a cyclic group action on a set X , c a cyclic group action of order ω acting nontrivially on a set Y , and $f : X \rightarrow Y$ a surjection. We say the triple $(X, (g), f)$ exhibits **resonance with frequency**

ω if, for all $x \in X$, $c \cdot f(x) = f(g \cdot x)$, that is, the following diagram commutes:



Resonance is a first step in understanding dynamics of more complicated combinatorial objects, since it is an analogue of an action having a nice order. Resonance is a *pseudo-periodicity* property, in that the resonant frequency ω is generally less than the order of g . (If ω equals the order of g , this is an instance of *trivial resonance*.) For non-trivial resonance, we also need a surjective map f to a simpler underlying set Y on which g corresponds to a cyclic action c with smaller order ω . In the second theorem below, this map is the **binary content vector**, defined on an increasing tableau $T \in \text{Inc}^q(\lambda)$ as the sequence $\text{Con}(T) = (a_1, a_2, \dots, a_q)$, where $a_i = 1$ if i is an entry of T and $a_i = 0$ if it is not. See Figure 9.

Theorem 19 ([DiPeSt17]). For a certain map f , $(J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}), (\text{Rowmotion}), f)$ exhibits resonance with frequency $a + b + c - 1$.

Theorem 20 ([DiPeSt17]). $(\text{Inc}^q(\lambda), (K\text{-promotion}), \text{Con})$ exhibits resonance with frequency q .

The uncanny similarity of the objects and actions in these theorems when $\lambda = a \times b$ and $q = a + b + c - 1$ led to a nonobvious connection between them.

Theorem 21 ([DiPeSt17]). $\text{Inc}^{a+b+c-1}(a \times b)$ under K -promotion is in equivariant bijection with $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ under rowmotion.

This proof relies on the following extension of Theorem 6 from the 2- to n -dimensional lattice. For this generalization, we introduced and developed the machinery of *affine hyperplane toggles* and *n -dimensional lattice projections*. We obtained a large family of toggle group actions $\{\text{Pro}_{\pi, \mathbf{v}}\}$, whose orbit structures are equivalent to that of rowmotion. That is, given a poset P , we project the elements in a way consistent with the covering relations into the n -dimensional lattice by the map π . Then $\text{Pro}_{\pi, \mathbf{v}}$ is the toggle group action that toggles elements as it sweeps through $\pi(P)$ by an affine hyperplane in the direction determined by the vector \mathbf{v} .

Theorem 22 ([DiPeSt17]). Let P be a finite poset with an n -dimensional lattice projection π . Let \mathbf{v} and \mathbf{w} be length n vectors with entries in $\{\pm 1\}$. Then there is an equivariant bijection between $J(P)$ under $\text{Pro}_{\pi, \mathbf{v}}$ and $J(P)$ under $\text{Pro}_{\pi, \mathbf{w}}$.

It is not hard to show that $\text{Pro}_{\pi, (1, 1, \dots, 1)}$ is rowmotion, so this theorem says rowmotion and $2^n - 1$ other promotions have the same orbit structure. For π the natural three-dimensional embedding, $\text{Pro}_{\pi, (1, 1, -1)}$ on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ is equivalent to K -promotion on $\text{Inc}^{a+b+c-1}(a \times b)$, which

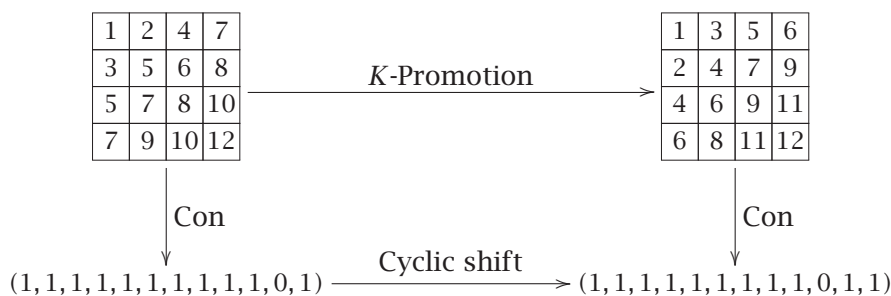


Figure 9. An increasing tableau in $\text{Inc}^{12}(4 \times 4)$ and its image under K -promotion, along with the map to the binary content of each. K -promotion cyclically shifts the binary content vector.

when combined with the above theorem gives the proof of Theorem 21, showing that toggle-promotion and tableaux-promotion coincide on a much larger set than we had previously known.

Conclusion

We have given a flavor of dynamical algebraic combinatorics through examples of combinatorial objects with actions that have small order and other nice properties, such as cyclic sieving and homomesy. Most of the objects and actions that behave nicely in these three respects are, in some sense, planar. These phenomena often still occur in objects that are only *slightly* three-dimensional, since such objects may often be mapped bijectively to planar objects, as in Theorem 10. But once such maps are no longer possible, these properties tend to no longer hold. We have seen examples of higher-dimensional combinatorial objects with natural actions that no longer have a nice order, but rather exhibit resonance. We hope that further study of resonance in such combinatorial dynamical systems will uncover more interesting properties and surprising connections.

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Figure 7 is courtesy of Nathan Williams.
 Figure 8 is courtesy of Oliver Pechenik.
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Jessica Striker is an assistant professor at North Dakota State University. A cycling enthusiast, Jessica may be spotted on a tandem or cargo bike for weekend outings with her husband and two children, on a fat tire bike for winter commuting, or on a ride share bike for sightseeing while on mathematical travels.



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Editor's Note. Related to Striker's article are two previous pieces that appeared in the *Notices*: "What is Cyclic Sieving?" by Victor Reiner, Dennis Stanton, and Dennis White, February 2014; and "What is a Young Tableau?" by Alexander Yong, February 2007.