WHAT IS...

a Generalised Mean-Curvature Flow?

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To untie a shoelace by pulling both ends of the string, shrinking the bunny ears until they disappear, gives great intellectual pleasure. Mathematically, we unknotted an unknot by decreasing the length of the curve contained in the knot. This procedure turns out to be a rather effective way to simplify geometries. It is one of the reasons why geometers like Richard Hamilton study the so-called *curve-shortening flow*, where an initial curve is continuously deformed so that its length decreases over time.

The mean-curvature flow enjoys nice properties.

The way to do this is to move or flow the curve normal to itself at a rate proportional to its curvature. There is a similar flow as in Figure 1 to reduce the area of a $(d-1)$-dimensional surface in $\mathbb{R}^d$, at a rate proportional to the *mean curvature*, the mean of the curvatures of 1-dimensional slices by orthogonal planes.

Apart from shortening curves, the mean-curvature flow enjoys other nice properties, such as *locality*, since the motion of a point depends only on an infinitesimal neighbourhood, and *rigid-motion invariance*, since "length" is invariant under rigid-motions. Other properties are more subtle. If you know that the mean curvature is the Laplacian of the function that measures distances from points to the manifold, then you can see that the motion is driven by a parabolic equation. This parabolicity gives rise to the *inclusion principle*, which says that if we deform two domains $U \subset V$ by letting their respective boundaries follow the mean-curvature flow, then these domains remain ordered. This is a consequence of the comparison principle for parabolic equations. It is also a consequence of geometry: If V failed to contain U at some time, then there would be a critical moment when U is still inside V but their boundaries touch at a certain point. Doodle a bit with curves, and you see that in this situation, to reduce "length," the boundaries are instantly pulled away from each other. Conclusion: U remains inside V.

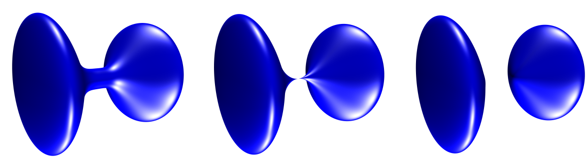

Figure 1. The surface of a dumbbell develops a singularity before turning into two disconnected spherical shapes.

All these nice properties do not make the mean-curvature flow perfect. One major flaw is that it is not well defined globally in time. After convex surfaces shrink into points, the flow is no longer defined. But this example does not capture the worst behaviour, because the object itself disappears after the critical point. Think of a dumbbell, as in Figure 1. If the plates are too big compared to the neck, the dumbbell turns into two smaller plates connected by a one-dimensional "string." Again the flow ceases to be well defined afterwards.

But what we have called a flaw is actually something that makes the mean-curvature flow analytically interesting.

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We can look for generalised/weak versions of the flow that continue the motion after the critical time. This search for generalised mean-curvature flows has become one of the central themes in analysis in the past few decades and leads to a lot of exciting mathematics.

A manifold is the zero set of the function that measures distances between any point and the manifold. This distance function satisfies a parabolic equation in the mean-curvature flow. Therefore, one natural way to get a generalised mean-curvature flow is to study generalised solutions to such equations, and then define their zero level sets to be solutions to the generalised flow. This so-called *level-set method*, as described in the December 2016 *Notices* cover story by Colding and Minicozzi, reduces the notion of a weak geometric flow to the notion of a weak solution to parabolic equations. Luckily, the viscosity solution provides exactly what we need. In fact, much of the theory of viscosity solutions is inspired by this approach to a generalised mean-curvature flow.

Another approach takes advantage of the inclusion principle. Suppose our manifold is the boundary of some domain U. Then the inclusion principle dictates that any domain initially containing U continues to do so in the future. Although U might be a very rough set where the smooth mean-curvature flow does not exist, we can always find nice sets containing U. Starting from these nice sets we do have flows, and we might simply define a generalised mean-curvature flow of U to be the intersection of all flows starting from these nice sets. This method of *minimal barriers* might remind the reader of the method of Perron, whereby one constructs solutions to elliptic equations as least supersolutions.

> *The search for generalised mean-curvature flows leads to a lot of exciting mathematics.*

A third approach, which goes back to the curve-shortening property, begins by defining a weak notion of the length of curves or the area of surfaces. The hope is that, using this generalised length/area, we would be able to consider the flow for much rougher objects. Such generalised length/area is provided by the theory of sets of finite perimeter from geometric measure theory. At each instant of time, we look for a set of finite perimeter that most efficiently reduces this generalised length/area and define this set to be the solution to the generalised flow. This gives rise to the notion of *flat flows*.

A related approach follows from the observation that the length/area can be seen as the limit of the Ginzburg-Landau energy functional when a certain parameter ϵ goes to 0. For each positive ϵ, we have a globally well-defined gradient flow for this energy. We simply define our generalised mean-curvature flow as the limit as ϵ goes to zero of this gradient flow. This is called the *phase field* approach.

The literature on generalised mean-curvature flows is particularly rich, containing many methods beyond the scope of a two-page article. To earn the label "generalised mean-curvature," however, a flow needs to satisfy several requirements: firstly, it should be globally defined; secondly, it should be consistent with smooth flows as long as the latter exist; and lastly, it should inherit certain key features of the smooth flows.

These naive-looking requirements sometimes lead to challenging problems that have inspired a great deal of research. Because generalised mean-curvature flows provide a paradigm for the study of all kinds of geometric flows, results in this area are of especially wide interest.

References

[1] G. Bellettini, Lecture notes on mean curvature flow, barriers and singular perturbations, *Publications of the Scuola Normale Superiore*, Scuola Normale Superiore **12** (2013).

[2] H. Yu, *Motion of sets by curvature and capacity potential*, submitted. Available at `https://arxiv.org/abs/1701.02837`.

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ABOUT THE AUTHOR

Hui Yu is interested in the theory of elliptic partial differential equations and calculus of variations as well as geometric flows. When he is not busy with moving surfaces, he likes to play with curvatures of the strings of the classical guitar.

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