## WHAT IS...

# a Sobolev Orthogonal Polynomial?

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Orthogonal functions such as sines and cosines and the associated Fourier series or the Legendre polynomials and series are useful in solving problems. We describe the widely used but less well-known Sobolev orthogonal polynomials, where the notion of orthogonality now involves derivatives.

Standard orthogonal polynomials $P_n$ with respect to a positive measure $\mu$ on an interval $[a,b]$ satisfy

$$(P_n, P_m) := \int_a^b P_n(x)P_m(x)\,d\mu = \begin{cases} 0, & \text{if } n \neq m, \\ \alpha_n > 0, & \text{if } n = m, \end{cases} \quad \text{with} \quad \deg(P_n) = n.$$

Thus, the classical Legendre polynomials,

$$1,\ x,\ \frac{3}{2}x^2 - \frac{1}{2},\ \frac{5}{2}x^3 - \frac{3}{2}x, \dots,$$

are orthogonal on $[-1,1]$ with respect to the Lebesgue measure.

Now we take $\mu_i$, $i = 0,1$, two finite positive Borel measures on the real line, and we construct an inner product involving derivatives

$$(f,g)_S = \int f(x)g(x)\,d\mu_0 + \int f'(x)g'(x)\,d\mu_1. \tag{1}$$

The corresponding norm

$$||f||_S^2 = (f,f)_S = \int f^2(x)\,d\mu_0 + \int (f'(x))^2\,d\mu_1$$

is the famous Sobolev norm. Thus the right normed space for this norm is a Sobolev space. The orthogonal polynomials $s_n$ with respect to (1) satisfy

$$(s_n, s_m)_S = \int s_n(x)s_m(x)\,d\mu_0 + \int s_n'(x)s_m'(x)\,d\mu_1 = \begin{cases} 0, & \text{if } n \neq m, \\ \beta_n > 0, & \text{if } n = m, \end{cases}$$

with $\deg(s_n) = n$. The presence of derivatives changes everything: most of the nice properties of the standard orthogonal polynomials no longer hold for Sobolev orthogonal polynomials. The inner product (1) can be generalized by including more derivatives.

*The presence of derivatives changes everything.*

Sobolev orthogonality originated with the problem of obtaining the polynomial that is the best least squares approximation to a function and, simultaneously, to its derivatives. That is, we need to find $p \in \mathbb{P}_n$ such that

$$||f-p||_S^2 = \inf_{q \in \mathbb{P}_n} ||f-q||_S^2,$$

where $\mathbb{P}_n$ is the linear space of the polynomials with degree at most $n$. The solution to the above extremal problem is given by

$$p(x) = \sum_{i=0}^{n} (f, s_i)_S\, s_i(x),$$

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**Figure 1. Sergei Lvovich Sobolev (1908–1989) at the International Congress of Mathematicians, Nice (France), 1970. Sobolev polynomials are orthogonal with respect to the Sobolev inner product (1).**

where $\{s_n\}_{n\geq 0}$ is a sequence of Sobolev orthonormal polynomials ($\beta_n = 1$, for all $n$) and the terms $(f, s_i)_S$ are the so-called Fourier–Sobolev coefficients.

This problem was posed by D. C. Lewis in 1947, although he did not deal with orthogonal polynomials in his paper. These were considered and studied for the first time in the 1960s, for particular measures, by several German mathematicians. The field exploded in the late 1980s when the concept of *coherent measures* was introduced, which paved the way to many algebraic, differential, and asymptotic properties of the Sobolev orthogonal polynomials. The key is to establish an algebraic relation, with a fixed number of terms, between Sobolev orthogonal polynomials and the standard ones. Actually, the concept of coherence greatly restricts the choice of measures: they are strongly connected, and all the possible coherent pairs are determined. Since then, this concept has been generalized in several ways to include more types of measures.

Sobolev orthogonal polynomials have remarkable properties. In the 1990s it was noticed that a Sobolev orthogonality holds for the famous Laguerre polynomials with nonstandard parameters. Similar connections have been made for the Gegenbauer and Jacobi polynomials with nonclassical parameters. These connections are very nice: they beautifully link classical polynomials to nonstandard orthogonality.

Another topic broadly studied corresponds to the asymptotic behavior of Sobolev orthogonal polynomials. A curious and deep look at the above Sobolev inner product suggests that the measure $\mu_1$ plays the main role in the asymptotic behavior. If we want to balance the roles of both measures, then we have to make the inner product variant; that is, one considers

$$(f, g)_S = \int f(x)g(x)d\mu_0 + \lambda_n \int f'(x)g'(x)d\mu_1.$$

When both measures have bounded support, it is enough to take $\lambda_n \sim n^{-2}$ when $n \to \infty$. When the measures have unbounded support, the situation gets more interesting. Thus, for exponential weights we should choose $n^2\lambda_n \sim a_{n+1}^2$, where $a_n$ are related to special numbers, the so-called Mhaskar–Rakhmanov–Saff numbers.

Let's see now another type of Soloblev inner product. Consider the case of an inner product involving the values of the ninth derivatives at 0:

$$(f, g)_S = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}dx + Mf^{(9)}(0)g^{(9)}(0). \tag{2}$$

When $M = 0$ we have the famous Hermite orthonormal polynomials $h_n$. Their asymptotics are given by the so-called Mehler–Heine formula:

$$\lim_{n\to\infty} (-1)^n n^{1/4} h_{2n+1}(z/\sqrt{n}) = \frac{\sin(2z)}{\sqrt{\pi}}.$$

For any arbitrary $M > 0$, the sine function in the Mehler–Heine formula for the corresponding orthonormal polynomials is replaced by a linear combination $b(z)$ of Bessel functions of the first kind. Figure 2 shows both limits.

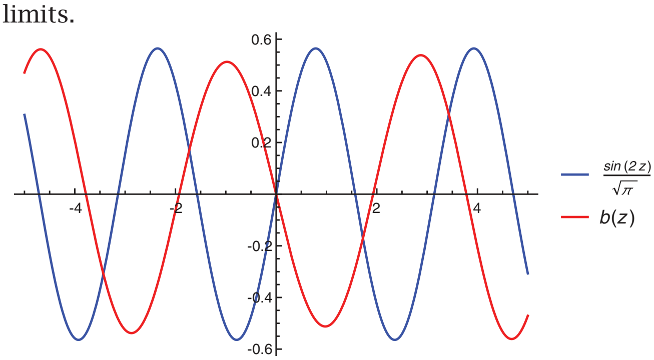


**Figure 2. When a term involving the ninth derivatives is added to the classical Hermite orthogonality, the blue sinusoidal asymptotic limit of the Hermite polynomials is replaced by the red $b(z)$, a linear combination of Bessel functions of the first kind.**

We have mentioned only some basic aspects of the theory of Sobolev orthogonal polynomials in one variable. We have said nothing here about other prominent and active topics such as Sobolev orthogonal polynomials in several variables or about the theories of Sobolev orthogonality when the derivative operator is changed by other operators. Marcellán and Xu [2] provide an updated survey tackling some of these topics. Our hope is to give readers an initial glance into the world of Sobolev orthogonality.

## References


[1] W. Gautschi, Orthogonal Polynomials: Computation and Approximation, *Numerical Mathematics and Scientific Computation*, Oxford University Press, 2004. MR 2061539

[2] F. Marcellán and Yuan Xu, On Sobolev orthogonal polynomials, *Expo. Math.* **33** (2015), 308–352. MR 3360352


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