WHAT IS...

Perimeter

Editors

The standard notion of the area of the boundary of a region in $\mathbf{R}^3$ is too big for complicated sets. Figure 1 suggests a countable dense union R of disjoint open balls in a region of volume 100 whose surface areas sum to 1 and volumes sum to some $\epsilon < 1$. The standard topological boundary is the entire complement of R, a solid region of volume $100 - \epsilon$ and infinite two-dimensional "area." On the other hand, the boundary measure we want is just the sum of the areas of the spheres, which we've taken to be 1. This desirable answer is provided by the *perimeter* of Caccioppoli and De Giorgi.

In her article on the isoperimetric problem in this issue (see p. 980), Bandle defines this perimeter first via approximations by (finitely many) polyhedra. For large N, a good approximation in our example would be N nearly round polyhedra of N faces approximating the N largest balls. As N approaches infinity, the approximate areas would converge to the sum of the areas of the spheres, which we chose to be 1.

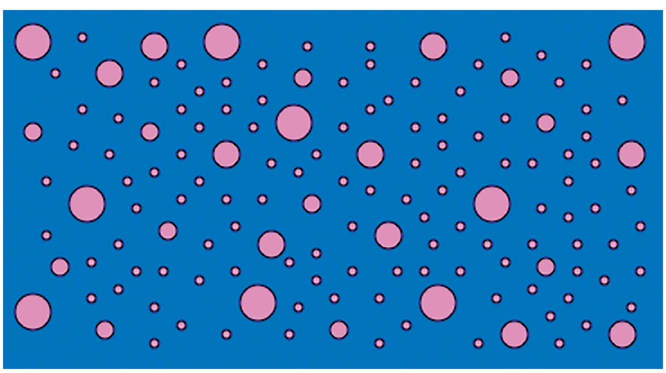

Figure 1. An infinite collection of dense disjoint balls should have finite "perimeter" even though the topological boundary can be almost the whole space.

Then Bandle provides the more standard, technical definition of the perimeter $P(R)$ of a region R as a supremum over smooth vectorfields $\mathbf{v}$:

$$P(R) = \sup_{|\mathbf{v}|\leq 1} \int_R \operatorname{div} \mathbf{v}.$$

Note that on a smooth region, by the divergence theorem,

$$\int_R \operatorname{div} \mathbf{v} = \int_{\partial R} \mathbf{v}\cdot\mathbf{n} \leq \text{area } \partial R,$$

because $|\mathbf{v}| \leq 1$. The supremum actually equals the area of ∂R (by taking the vectorfield $\mathbf{v}$ to be the unit normal $\mathbf{n}$ on the boundary).

As Bandle remarks, this perimeter is lower-semicontinuous: the perimeter of a limit is at most the limit of the perimeters. Note that this property fails for the area of the topological boundary. The infinite collection of balls suggested by Figure 1 has huge topological boundary, even though it is the limit of finite subcollections, all of which have topological boundary of area less than 1.

Such lower-semicontinuity is a key fact in general proofs of the existence of solutions to isoperimetric problems, which typically seek to minimize area among regions of prescribed volume. Such proofs begin with a sequence of regions of the prescribed volume with perimeters approaching the infimum. Certain deep compactness theorems yield a convergent subsequence. Now *by lower-semicontinuity of perimeter,* the limit has the infimum perimeter and hence solves the isoperimetric problem.

You might worry that a solution so obtained might be a strange one for which the specially defined perimeter is artificially small. The next and harder part of the theory is to prove that the resulting solution actually is nice, ideally smooth, so that perimeter means what it should.

Finally, knowing that nice solutions to the isoperimetric problem exist makes it easier to find them.

Further Reading

FRANCESCO MAGGI, *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory,* Cambridge University Press, 2012.

Image Credit

Figure 1 © Frank Morgan.

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